Chapter 4: Techniques of Circuit Analysis

This chapter gives us many useful tools for solving and simplifying circuits. We saw a few simple tools in the last chapter (reduction of circuits via series and parallel combinations of resistances, for example) but in this chapter we take circuit simplification much farther…

4.1 Terminology

<u>Planar Circuits</u> can be drawn on a plane with no crossing branches.

<u>Non-Planar Circuits</u> cannot be drawn on a plane without branches crossing one another somewhere.

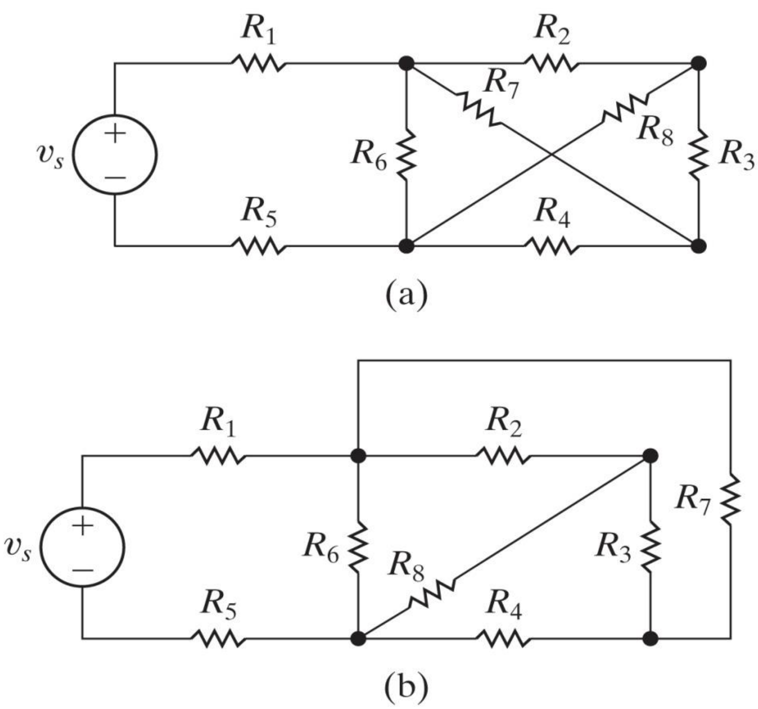


Figure: 04-01a,b

Figure 4.1 (left) shows a circuit that has crossing branches but that can be re-drawn without any branches crossing. This is a planar circuit.

Figure 4.2 (right) shows a circuit that is not planar: there is no way to draw it without branches crossing.

In *Circuit Analysis* we will consider planar circuits only. Many of the techniques we learn here cannot be applied to non-planar circuits.

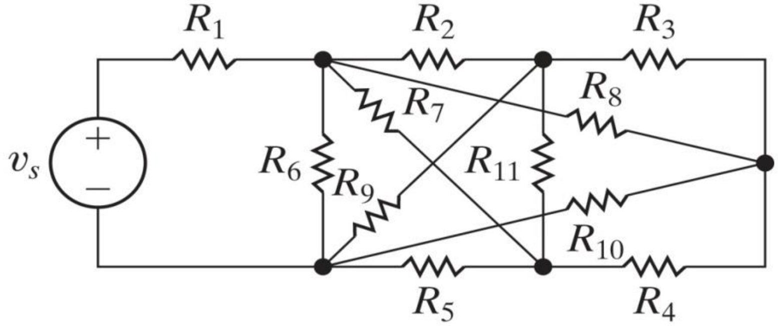


Figure: 04-02

Definitions

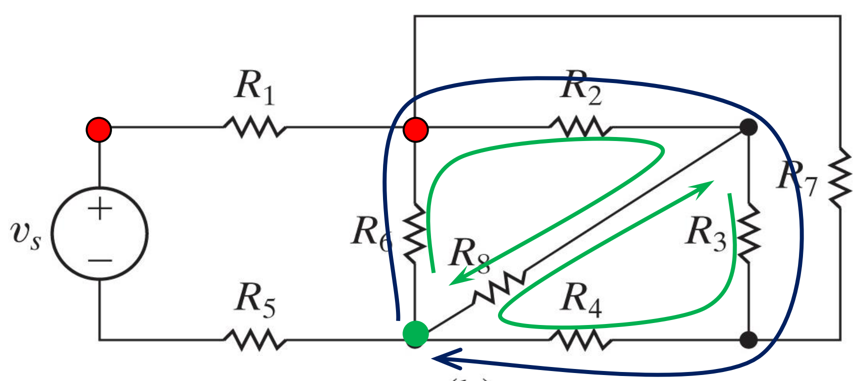
A **node** is a point in a circuit where two or more circuit elements meet. The number of nodes in a circuit is **n**.

A **path** is formed when adjoining (connected) circuit elements are traced, in order, without passing through any node more than once. A **closed path** is a path whose starting and ending points are the same.

A **branch** is a path connecting two nodes.

A **mesh** is a closed path that does not contain any other closed paths The number of meshes in a circuit is **m**.

Looking at the circuit in Figure 4.1b (shown again below), we can identify: 4 meshes; 8 closed paths; 6 nodes. Two of the four meshes are shown in green; two nodes are shown as red dots; one possible closed path which is not a mesh is shown in blue. The blue closed path contains two meshes. (The arrows on the paths are just for show: we can trace a path in either direction.)



(b)

Essential Node: A useful kind of node is the essential node, which is a node at which at least three circuit elements meet. In the circuit above there are four of these; one of them is labeled as a green dot. One of the red dots is also an essential node; the other is not.

An **essential branch** is a branch that connects two essential nodes. There are quite a few in the circuit above.

Where are we going with all this?? This terminology helps us figure out ***how many equations*** we will need to solve the circuit:

We need as many equations as there are *essential* branches with unknown currents; call this number b_e. Then we can write $(n_e – 1)$ KCL equations so we need $[b_e – (n_e – 1)]$ KVL equations. Here, n_e is the number of essential nodes.

We will shortly look at two very important and very powerful circuit analysis techniques:

- **Node Voltage Method** (NVM): allows us to solve a circuit with $(n_e – 1)$ KCL equations.

- **Mesh Current Method** (MCM): allows us to solve a circuit with $[b_e – (n_e – 1)]$ KVL equations.

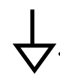
Using either method, we save quite a few equations, since without them we would need $(n_e – 1)$ KCL equations ***and*** $[b_e – (n_e – 1)]$ KVL equations.

4.2 Introduction to the Node Voltage Method

To use the *Node Voltage Method* (NVM) we define new circuit variables called the ***node voltages***. We find the node voltages by solving simultaneous equations called the ***node voltage equations***. These are generally much fewer in number than we need with basic KVL, KCL, Ohm's Law techniques. If we have the node voltages, we can find any other current or voltage in the circuit with a single equation (i.e., no more simultaneous equations will be necessary).

Node Voltage Method Algorithm

Basic steps in using NVM (we will clarify and expand on this shortly):

- Find and label all essential nodes.
  - There are n_e essential nodes, and we will need to solve $n_e - 1$ node voltage equations.
- Choose one of the essential nodes as a reference and label it with the symbol .
  - This is where the "- 1" arises in "ne – 1 equations": we do not need a node voltage equation at the reference.
  - Typically we will choose the reference node to be the one with the largest number of circuit elements connected to it. But we do not have to do this, and we will see there may be reasons for choosing a different node as the reference.
- Define the non-reference node voltages.
  - The non-reference nodes are the ones you did not choose as the reference node. The non-reference node voltages are the voltage drops from each non-reference node to the reference node (an example or two will clarify this). These are labeled with the positive sign at the node, and the negative sign at the reference node.
- Apply Kirchhoff's Current Law (KCL) to each non-reference node, writing currents at each node in terms of the node voltages and any sources present.
  - Although there is more than one way to do this, in this class we will always set the sum of the currents leaving the node equal to zero.
- Solve the node voltage equations. The resulting node voltages are considered the solution to the circuit. If you have those, you can find any current or voltage in the circuit with one equation (no more simultaneous equations needed).

To see how to express the current leaving a branch in terms of the node voltages, we look at the following example.

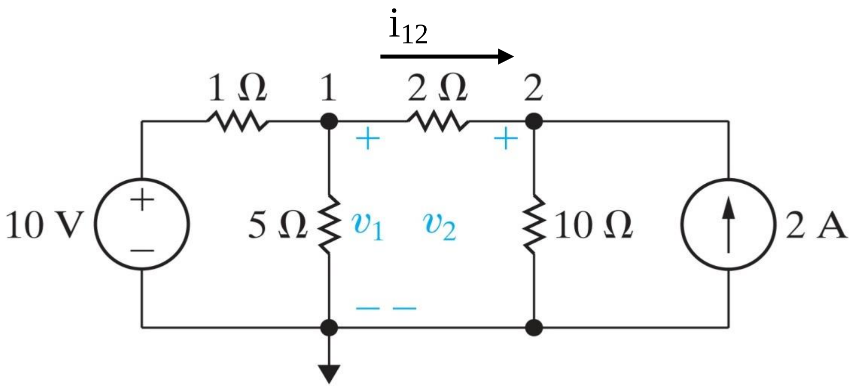


Figure: 04-06

The essential nodes are labeled 1, 2, and (reference node). The node voltages are defined as described above: positive at the non-reference node and negative at the reference. The current i_{12} is leaving node 1 and heading from node 2. It can be written in terms of node voltages as

$$i_{12} = \frac{v_1 - v_2}{2[\Omega]} .$$

Where did we get that equation for i_{12}? From KVL:

$$v_1 - v_2 - i_{12}(2[\Omega]) = 0$$

Solving this equation for i_{12} gives the previous equation. We continue in this way to set up KCL at each non-reference node. So, at node 1, the sum of the currents leaving node 1 is

$$\frac{v_1}{5} + \frac{v_1 - v_2}{2} + \frac{v_1 - 10}{1} = 0$$

The sum of the currents leaving node 2 is

$$\frac{v_2 - v_1}{2} + \frac{v_2}{10} - 2 = 0$$

If the third term of the first equation is confusing, do a KVL around the loop containing the 10 [V] source, the 1 [Ω] resistor, and the 5 [Ω] resistor. Solve this KVL for the current leaving node 1, and you will get the third term in that equation. Eventually, you will get to the point where these terms can be written down quickly without having to examine each KVL individually.

The third term in the second equation is easy, since we know the current in that branch: it's 2 [A] entering node 2, so the current leaving is – 2 [A]. In this case, we don't try to express it in terms of the node voltages. Note also that we have summed the currents leaving each node. This is an arbitrary choice, but we must choose something, and be consistent.

We have two equations for two unknowns. The solution is…

$$v_1 = 9.091\ [V] \quad v_2 = 10.91\ [V]$$

Now that we have the node voltages, we can find any other voltage or current in the circuit. To do that we probably need to solve one equation, but we will not have to solve simultaneous equations anymore. For example, the current in each branch is easy – we have already shown how to calculate i_{12} in the equation above. Also, the voltage across any individual component can be found with one KVL. As one more example, the voltage across the current source in the circuit above is nothing but v_2.

Notation Rules (you knew we were going to have this…)

All node voltages must be labeled just as they are in the diagram above. Those node voltages have a '+' sign at the non-reference node, and a '-' sign near the reference node. It is not sufficient to simply label v_1, for example, near node 1. It is certainly not acceptable to leave out the node voltages from your diagram. If you do, significant credit will be subtracted from quizzes and exams.

4.3 The Node Voltage Method and Dependent Sources

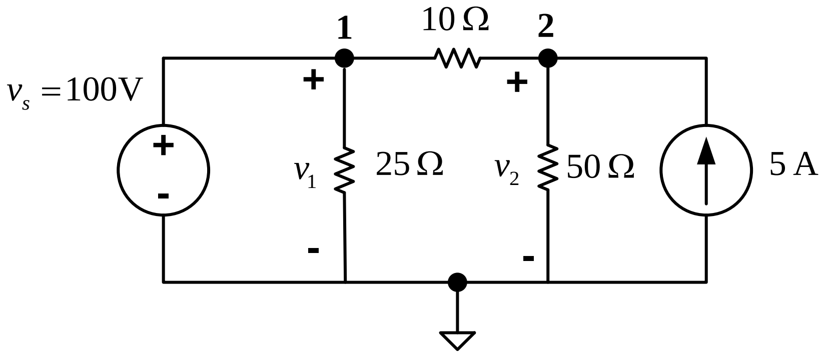
If we have dependent sources, we will need one equation for each controlling variable. We call these ***auxiliary equations***. We will need to write these equations in terms of the node voltages. This is because we are trying to find the node voltages (which are the circuit variables), and we don't want to introduce any extra variables.

4.4 The Node Voltage Method: Some Special Cases

Special cases arise when we have a voltage source connected either (1) between an essential node and the reference node, or (2) between two non-reference essential nodes.

(1) *Voltage source between an essential node and the reference node.*

In the circuit below the 100 [V] source is connected from essential node 1 to the reference node.



This is a simple case. Remember that we are trying to find the node voltages, which are the voltage values between each of the essential nodes and the reference node. But we already know that in this case !!

$$v_1 = v_s = 100\text{V}$$

This is the node voltage equation for node 1. We have still have the same number of node voltage equations, but one of them is trivial. This works for dependent sources, too, although we of course still need the auxiliary equation.

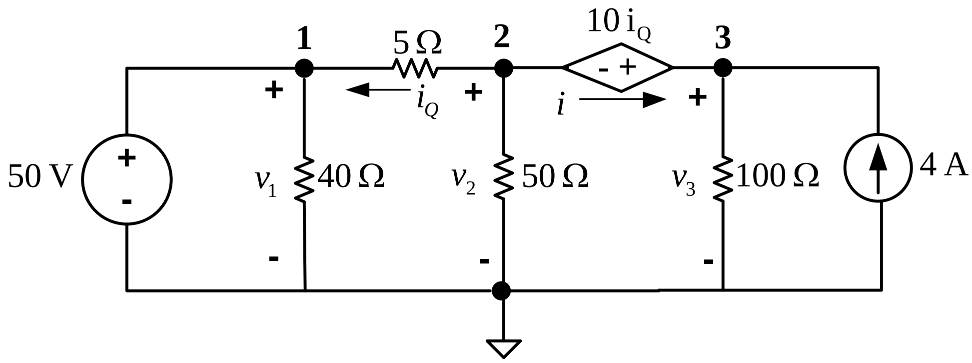
The rest of the circuit is solved as follows:

$$\frac{v_2 - v_1}{10} + \frac{v_2}{50} - 5 = 0$$

This gives v_1 = 100 [V]; v_2 = 125 [V].

(2) *Voltage source between two non-reference essential nodes.*

In the circuit below the dependent source is connected between two non-reference essential nodes. We are using a dependent source for illustration, but our remarks are valid for independent sources.



Node voltage v_1 is already known:

$$v_1 = 50\ [V]\ .$$

Let's write a node voltage equation at node 2:

$$\frac{v_2 - v_1}{5} + \frac{v_2}{50} + i = 0$$

How do we write the current in the dependent source? There is no way to write a 'v/R' term there. But there is a current there that we have to account, for so we will call it '*i*'.

Now at node 3:

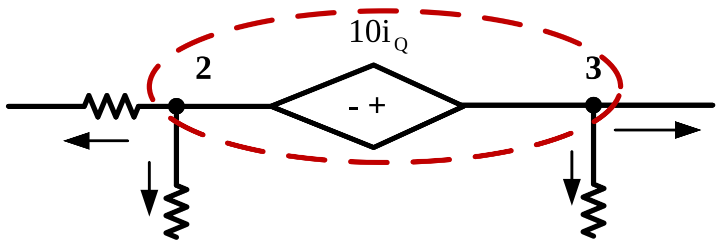
$$\frac{v_3}{100} - 4 - i = 0$$

What should we do about this unknown current '*i*'? Let's add the last two equations together:

$$\frac{v_2 - v_1}{5} + \frac{v_2}{50} + \frac{v_3}{100} - 4 = 0$$

Hmmm…this last equation is what we have got if we had treated node 2 and node 3, with the dependent source in between, as one big node…it's as if it were a … a…

Supernode!!



In the figure above, the red dashed line encircles the supernode, and the arrows show currents leaving it. Let's apply our usual node voltage technique to those currents.

At node 2:

$$\frac{v_2}{50} + \frac{v_3 - v_1}{5} + \frac{v_3}{100} - 4 = 0$$

As we found above, node voltage $v1$ is already determined:

$$v_1 = 50\ [V]$$

These are our node voltage equations. But we have four unknowns (v_1, v_2, v_3, and i_Q) and only two equations. We need two more equations. For one, note that we have not included any information about the dependent source. We include this by simply doing a KVL around a loop that includes the dependent source:

$$v_2 - v_3 + 10\, i_Q = 0$$

This equation comes from the fact that the voltage between node 2 and node 3 is "constrained" by the voltage source. This is the ***constraint equation***.

Finally, since we have a dependent source, we need an auxiliary equation. Remember it has to be written in terms of the node voltages.

$$i_Q = \frac{v_2 - v_1}{5}$$

Now we have four equations in four unknowns and we can solve for the node voltages.

Important Note The supernode equation (the first of our four equations above) contains v_2 AND v_3, so we do not have any fewer node voltages to solve for. But the kind of equations is different from the case where we do not have a supernode: Instead of two node voltage equations, we have a supernode equation and a constraint equation.

Counting Equations

It is good practice to count essential nodes and determine how many equations, and of what type, you are going to need. We will do this in the examples in class. You will need…

- …one node voltage equation for each non-reference essential node.
- …one constraint equation for each supernode
- …one auxiliary equation for each dependent source.

Choosing the Reference Node

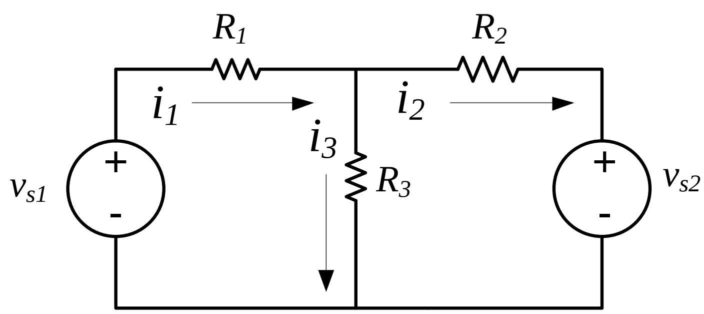
You can choose any essential node as the reference node. Typically this will be the one with the largest number of branches connected to it, since then you will avoid having to write an equation with a lot of terms.

One reason you might not want to choose the node with the largest number of branches is that you might be able to simplify the problem by choosing the reference node so that one or more of them is trivial. If you choose the reference so that a voltage source (dependent or independent) is connected to it, with the other end of the voltage source at another essential node, then that node voltage is trivial.

4.5 Introduction to the Mesh Current Method

To use the *Mesh Current Method* (MCM) we define new circuit variables called the ***mesh currents***. We find the mesh currents by solving simultaneous equations called the ***mesh current equations***. These are generally much fewer in number than we need with basic KVL, KCL techniques. If we have the mesh currents, we can find any other current or voltage in the circuit with a single equation (i.e., no more simultaneous equations will be necessary).

Consider the circuit below, where KCL is applied to the node at the top.



KCL:

$$i_1 = i_2 + i_3$$

KVL around the left and right loops gives…

$$v_{S1} = i_1 R_1 + i_3 R_3$$

$$v_{S2} = -i_2 R_2 + i_3 R_3$$

If we solve the KCL equation for i_3 and substitute the result into the KVL equations we get:

$$v_{S1} = i_1(R_1 + R_3) - i_2 R_3$$

$$v_{S2} = -i_2(R_1 + R_3) + i_1 R_3$$

We now have two equations, in the form of KVLs, in two unknowns. This substitution, and the resulting equations, can be done "automatically" using the mesh current method …

To use the mesh current method to get the last two equations directly, we do the following.

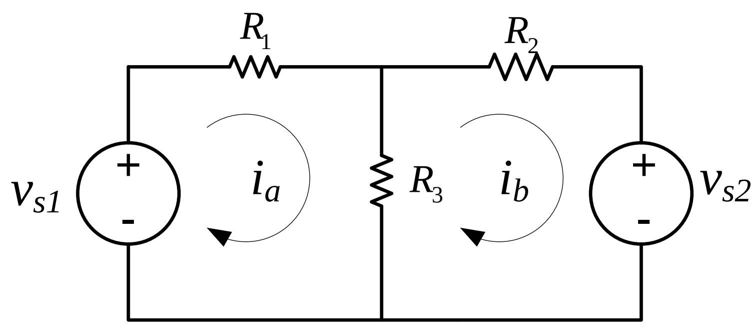
Mesh Current Method Algorithm

1. **Find the meshes and label each with a current.** Remember that we learned how to count meshes m, essential nodes n_e, and essential branches b_e. Then the number of meshes will be

$$m = b_e - (n_e - 1)\ .$$

- The two mesh currents in the circuit below are labeled i_a, i_b. The direction shown (clockwise for both) is arbitrary – we could have chosen either one (or both) to go in the other direction. We will refer to the two meshes as "mesh a" and "mesh b".

- Note that the mesh currents are defined as the currents going "around" the perimeter of the loops. It is important to realize that these are not necessarily the same as branch currents i_1, i_2, i_3.



2. **Apply KVL around each mesh, writing voltages across resistors in terms of the mesh currents.** The result for the circuit above is:

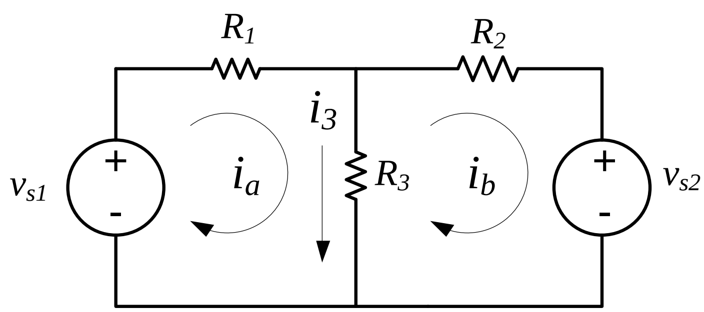
Mesh a $$-v_{S1} + i_a R_1 + (i_a - i_b) R_3 = 0$$

Mesh b $$(i_b - i_a) R_3 + i_b R_2 + v_2 = 0$$

3. **Solve for the mesh currents. Then, find the branch currents in terms of the mesh currents.**

Going back to the original circuit where we defined the branch currents i_1, i_2, i_3, is should be clear that $i_1 = i_a$ and $i_2 = i_b$.

What about i_3? Looking at the circuit below you should be able to convince yourself that $i_3 = i_a - i_b$. So we have found the branch currents in terms of the mesh currents. This is the reason for the statement above that the mesh currents are *not necessarily* equal to the branch currents.



If we substitute our results for the branch currents in terms of the mesh currents into the mesh current equations, we can show that the mesh current equations are equivalent to the two equations we got earlier: Specifically, the equations

$$v_{S1} = i_a\left(R_1 + R_3\right) - i_b R_3 = 0$$
$$-v_{S2} = -i_a R_3 + i_b\left(R_2 + R_3\right)$$

are the same as the previous KVL equations if we make the assignments $i_1 = i_a$, $i_2 = i_b$, and $i_3 = i_a - i_b$.

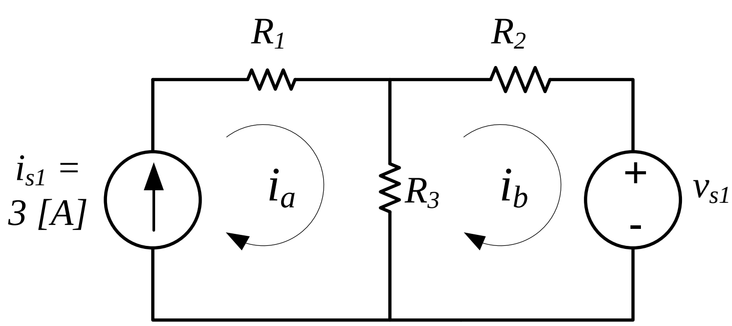
4.6 The Mesh Currents and Dependent Sources

For each dependent source, we need an additional equation defining the controlling variable. This is always the case, but now we want to write the defining equations in terms of the mesh currents. If we use, say, branch currents to define the controlling variables, we will have introduced additional unknowns.

4.7 The Mesh Currents Method: Some Special Cases

Two special cases arise when current sources are present. (Note that these cases are analogous to the case of voltage sources in the node voltage method.)

(1) *A branch contains a current (dependent or independent)*

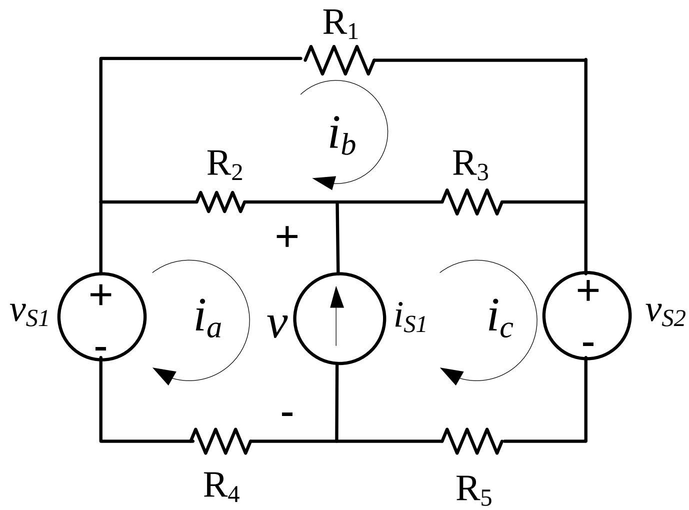


In the circuit shown above, it should be clear that $i_a = 3[\mathrm{A}]$, which means that we have one less mesh current equation to solve because one of the unknowns (i_a) is now known. So there is only one mesh current equation for this circuit, which is

$$v_{s1} + \left(i_b - i_a\right)R_3 + i_b R_2 = 0$$

So a current source in a branch means that the number of mesh current equations is reduced by 1.

(2) *A current source (dependent or independent) is being shared by two meshes*



The mesh current equations for the circuit above are as follows.

Mesh b)

$$i_b R_1 + (i_b - i_c) R_3 + (i_b - i_a) R_2 = 0$$

KVL around either mesh a or c includes the current source. We **cannot** assume that the voltage across the current source is 0, but we don't know what it is, so we will simply label it "*v*". Then we have

$$-v_{S1} + (i_a - i_b) R_2 + v + i_a R_4 = 0$$

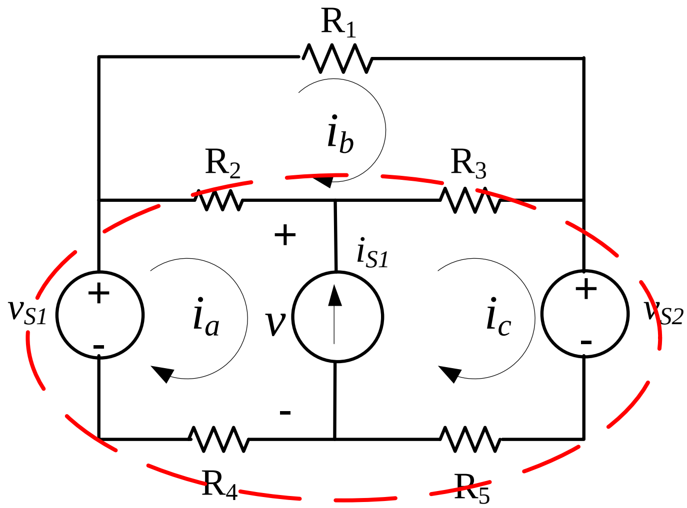
$$(i_c - i_b) R_3 + v_{S2} + i_c R_5 - v = 0$$

Adding these last two equations gives

$$-v_{S1} + (i_a - i_b) R_2 + (i_c - i_b) R_3 + v_{S2} + i_c R_5 + i_a R_4 = 0$$

But this equation is just what we would have obtained if we had done a KVL around the path shown in red below. Why, it's almost as if we had…a…a…

Supermesh !!



Above we have drawn a dashed line around the supermesh. We can write the equation above in one step using this mesh:

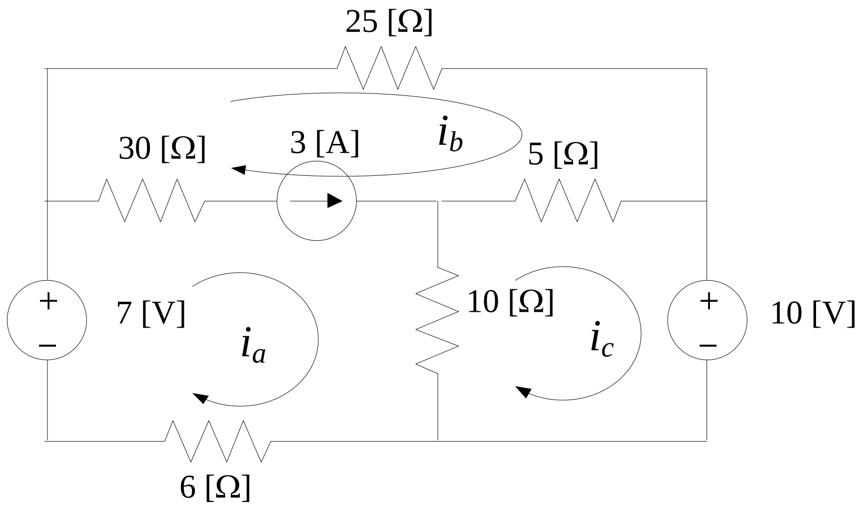
$$-v_{S1} + (i_a - i_b)R_2 + (i_c - i_b)R_3 + v_{S2} + i_c R_5 + i_a R_4 = 0$$

But we have to get the value of the current source involved, which we do by recognizing that there is a *constraint* imposed on i_a and i_c:

$$-i_a + i_c = 5$$

So we have still have three equations: one regular mesh current equation, and one super mesh equation, and one constraint equation (the constraint equation is the one we just wrote).

Let's do an example…we will find the power absorbed by the 10 [Ω] resistor and the power delivered by the 3 [A] source in the circuit below.



There are 3 meshes, two of which share a current source. So we will have three mesh current equations:

- one regular mesh current equation
- one supermesh equation
- one constraint equation

The regular mesh current equation is

$$5(i_c - i_b) + 10 + 10(i_c - i_a) = 0$$

The super mesh equation is

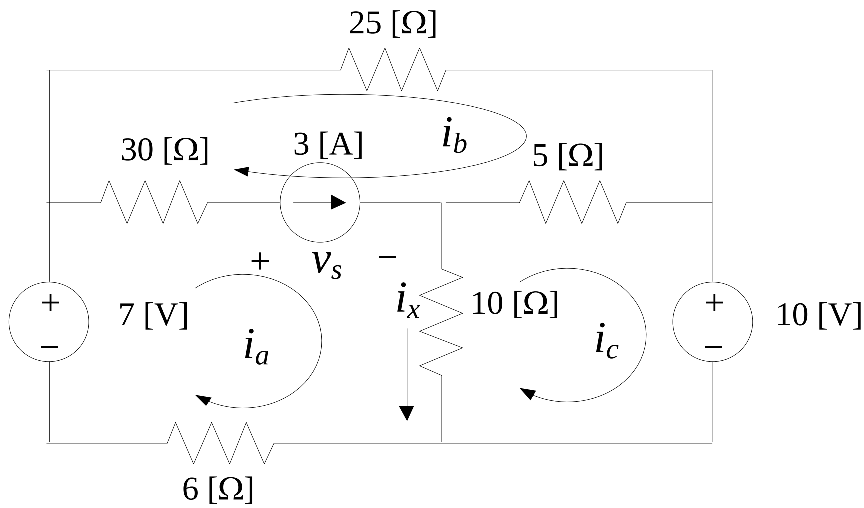
$$-7 + 25i_b + 5(i_b - i_c) + 10(i_a - i_c) + 6i_a = 0$$

The constraint equation is

$$i_a - i_b = 3$$

Solving these equations together gives i_a = 2.323 [A]; i_b = -0.667 [A]; i_c = 0.656 [A].

Now that we have found the currents, we can find anything else we might need using only one equation. Since we want power for the resistor and the current source, we have labeled the current in the resistor and the voltage across the current source in the next figure.



The current i_x is

$$i_x = i_a - i_c = 1.667\ [A]$$

Then

$$p_{abs,10[\Omega]} = i_x^2(10) = 27.8\ [\text{W}]$$

How did we know that i_x was $i_a - i_c$ and not $i_c - i_a$? Because at the 10 [Ω] resistor, i_x is defined to be going in the direction of i_a and opposite the direction of i_c.

To find v_s we need a KVL:

$$-7 + 30(i_a - i_b) + v_S + 10(i_a - i_c) + 6i_a = 0$$

This gives v_S = -113.6 [V]. Then

$$\therefore p_{del,3[A]} = -3(-113.6) = 340.8[W]$$

4.8 The Node Voltage vs. the Mesh Current Method

We want to think carefully in each case about whether the node voltage or the mesh current method is easier to implement. Some factors to consider:

- One of the methods may require fewer equations to solve simultaneously. For any circuit we will have $n_e - 1$ node voltage equations and $m = b_e - (n_e - 1)$ mesh current equations.
- Voltage sources may reduce the number of node voltage equations by introducing trivial equations. Current sources may reduce the number of mesh current equations by introducing trivial equations. This of course depends on where we put the reference.
- The variable we are looking for may already be a mesh current or a node voltage, in which case solving by the appropriate method will yield the answer directly.

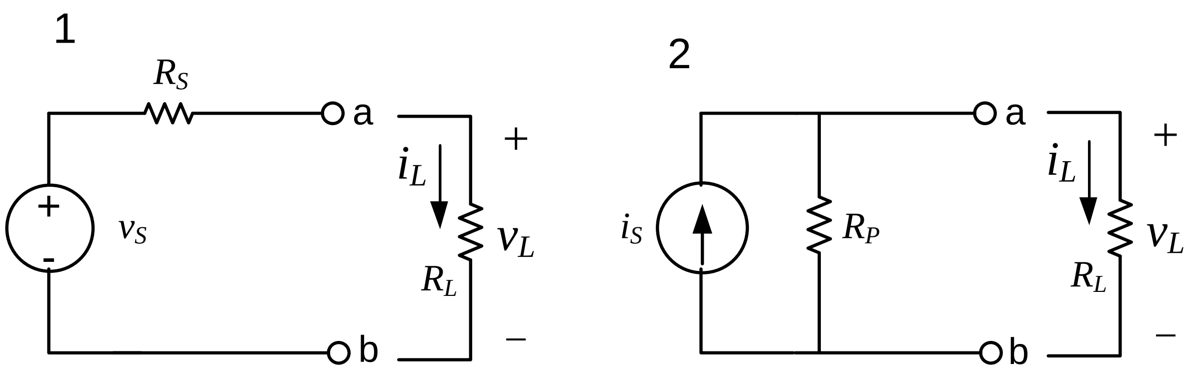
An observation: Most beginning students prefer mesh currents, but in fact the node voltage method almost always involves fewer equations. Also, node voltage method is extremely useful in many electronics applications. I would strongly suggest getting familiar with the node voltage method.

4.9 Source Transformations

We can sometimes simplify a circuit using ***equivalent circuits***…

Two circuits are *equivalent* if one can be replaced with the other without changing circuit variables (voltages and currents) in the rest of the circuit. We have already seen how to use equivalent circuits to simplify using series and parallel resistor combinations. In that case, we are replacing a group of resistors by a single equivalent resistor. ***Source transformation*** is another such simplification technique.

The circuits "1" and "2" below are ***equivalent with respect to terminals a and b***, provided that v_s, R_s, i_s, and R_p are related to one another in a particular way. If they are, then a resistor R_L connected to terminals a and b will have the same voltage across it (and the same current through it) whether it is connected to circuit 1 or to circuit 2. In fact, anything connected to terminals a and b of either circuit will have the same voltage across it and current through it. That is what "equivalent" means.



Note that it is very important to include the qualifier *with respect to terminals a and b*, because two circuits are not necessarily equivalent at just any two terminals.

Bottom Line: If the parameters are related correctly, a voltage source in series with a resistor can be replaced with a current source in parallel with a resistor.

But what relationship must exist among v_s, R_s, i_s, and R_p in order for these circuits to be equivalent? We find this relationship by requiring that equivalence must hold for <u>any</u> load resistor R_L, and in particular it must hold for $R_L = 0$ and $R_L = \infty$.

For $R_L = 0$, we ask what current i_L flows through R_L. (By the way, $R_L = 0$ means that a short circuit exists at terminals a and b, and the resulting current is the *short-circuit current*). If it is connected to circuit 1, then

$$i_L = \frac{v_S}{R_S}$$

If it is connected to circuit 2, then

$$i_L = i_S$$

For these to be equivalent, we must have

$$i_S = \frac{v_S}{R_S}.$$

Now consider $R_L = \infty$. In that case, terminals a and b are *open circuit*, and the voltage v_L is the *open-circuit voltage*. Then when R_L is connected to circuit 1 we have

$$v_L = v_S$$

and when it is connected to circuit 2,

$$v_L = i_S R_P.$$

So that means

$$v_S = i_S R_P$$

Comparing this equation to the one for i_S above shows that we also need $R_S = R_P$.

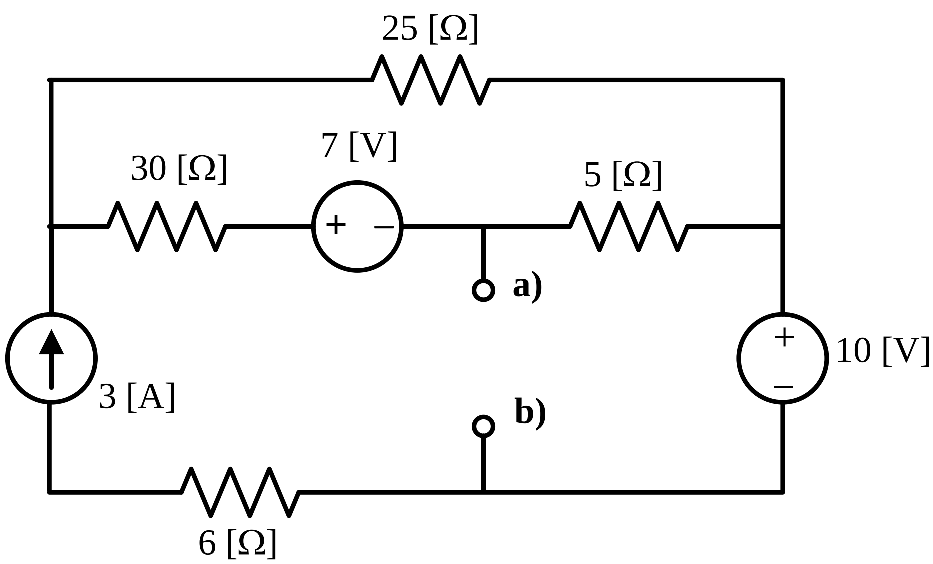
<u>**Summary:**</u> A voltage source v_S in series with a resistor R_S will be equivalent to a current source i_S in parallel with a resistor R_P if

$$v_S = i_S R_P \text{ and } R_S = R_P.$$

This is the ***source transformation theorem***.

4.10 Thevenin and Norton Equivalents

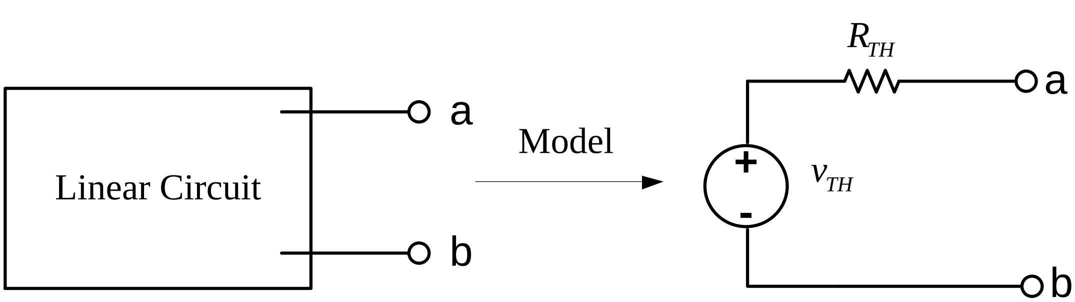
We are often interested in the voltage and/or current in a load that is connected to a particular pair of terminals in a circuit. For example, we may want to connect a load resistor, or maybe several different load resistors, to the terminals labeled a), b) in the circuit shown below.



The following idea is very powerful, and may help in analyzing a case like this:

> ***Thevenin Equivalent Circuit:*** The behavior of any linear circuit at a specific pair of terminals in a circuit may be modeled by a voltage source v_{TH} in series with a resistor R_{TH}.

We will look only at linear circuits in this course; *linear circuit* is defined later in the section on superposition. What we are saying is that the circuit below on the right can be modeled by the circuit on the left.



Important Notes:

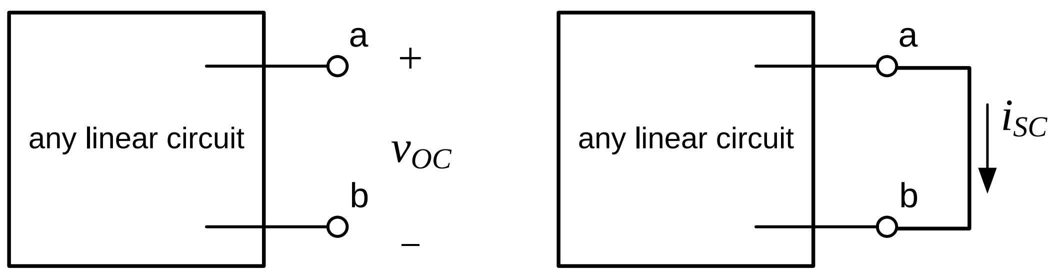
- We are "modeling" the circuit at two particular terminals with v_{TH} and R_{TH} – we are not suggesting that the only things inside the box are a resistor and a voltage source.

- The model holds for any load but only at terminals a), b). If we different terminals in the original circuit, the values of v_{TH} and R_{TH} will change.

- The circuit must be linear, but it can contain any of the basic circuit elements: voltage sources, current sources (dependent and independent), resistors, capacitors, and inductors.

Finding v_{TH} and R_{TH}:

The box in the figure below contains an arbitrary linear circuit. We have labeled terminals a) and b). On the right, we have an open circuit at a), b), resulting in an open-circuit voltage v_{OC}. (We can think of this as an infinite load resistance.) On the left, we have connected a short to the terminals, resulting in a short-circuit current i_{sc}.



By comparing the drawing on the left with the Thevenin Equivalent drawing above, it should be clear that

$$v_{OC} = v_{TH} \ .$$

By comparing the drawing on the left with the Thevenin Equivalent, we can see also that

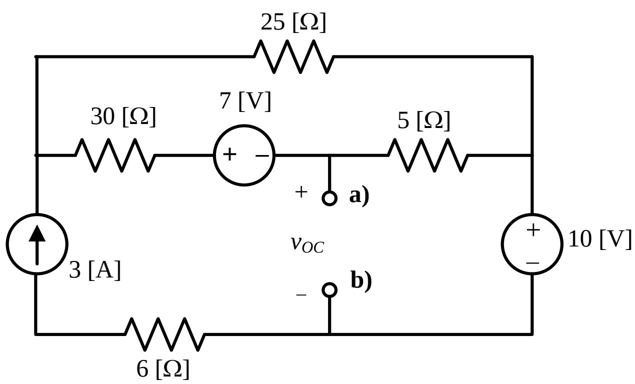
$$i_{SC} = \frac{v_{TH}}{R_{TH}} \ .$$

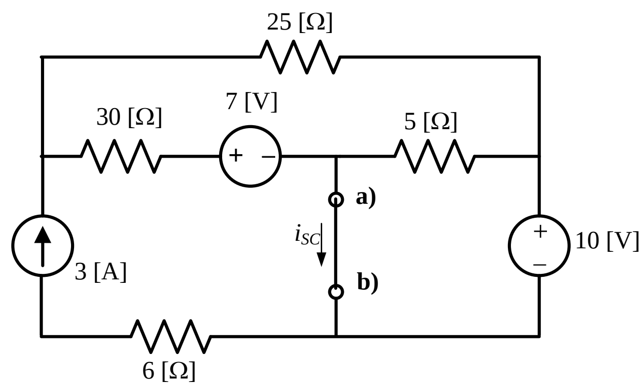
So we already have an algorithm for finding a Thevenin Equivalent: If we know the open-circuit voltage and the short-circuit current at the terminals a), b), we can find the Thevenin Equivalent:

$$v_{TH} = v_{OC} \quad \text{and} \quad R_{TH} = \frac{v_{TH}}{i_{SC}} \ .$$

If this were an experiment, we could measure v_{OC} and i_{SC}. If it is an analytical problem, we can calculate them using our knowledge of circuit theory.

What about that circuit earlier? We can use the node voltage method, or the mesh current method, or KVL, KCL, Ohm's Law, to find the open circuit voltage and short circuit current at terminals a, b. Below we show these variables – NOTE CAREFULLY that if the polarity of v_{oc} is positive at a, then is_c has to be indicated as going from a to b, as we have it below. If we mix those up, we will get the wrong sign for R_{Th}.



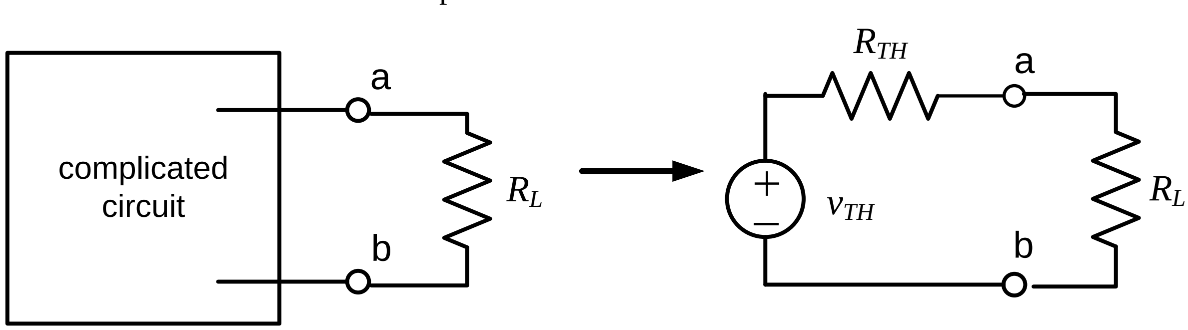


Solving these circuits gives voc = 15.67 [V], isc = 3.418 [A]. That means

$$R_{TH} = \frac{v_{OC}}{i_{SC}} = \frac{15.67\ [V]}{3.418\ [A]} = 4.585\ [\Omega]\ .$$

Is this useful?? Wow, yeah! This idea is used a lot. What it means is that we can talk about a lot of complicated circuits without having to know anything about those circuits except their Thevenin Equivalents. In fact, we often don't even need to know what the Thevenin voltage and resistance are. For example, we can analyze an audio amplifier without knowing what will be connected at the input, if we just know that whatever will be connected has a Thevenin equivalent This is extremely useful.

It also means that if we need to analyze how several different load resistors behave when connected to a circuit at two particular terminals, we only need the Thevenin Equivalent, and we can make the calculations much simpler. This idea is shown below.



Generally it is a lot easier to handle the Thevenin Equivalent circuit on the right than it is to analyze the complicated circuit on the left.

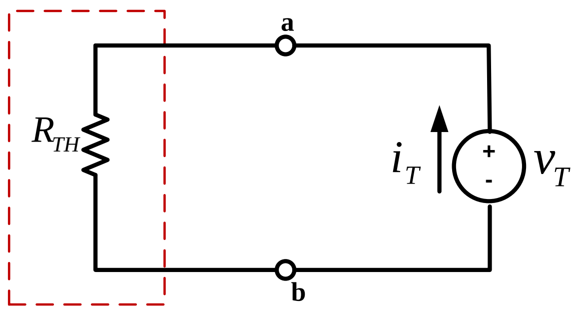
4.11 More on Deriving a Thevenin Equivalent

There is another method for finding R_{TH}:

The Test Source

Suppose we had a circuit that could be modeled using only a resistor. Then if we were to apply a voltage source v_T and measure or calculate the current i_T through it, we could find R_{TH} as:

$$R_{TH} = \frac{v_T}{i_T}$$



Circuit modeled by R_{TH} only

Test source

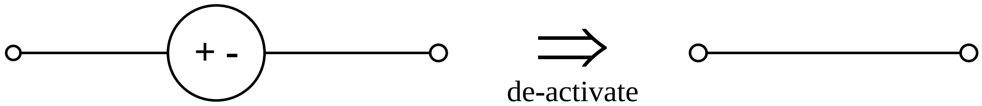
Here, v_T is known as the “test source”. If our circuit is simply a resistance, we can use the test source to find that resistance. This will be the Thevenin Equivalent resistance R_{TH}.

Be careful: the polarity of v_T and the current i_T that we calculate have to be in the active sign convention. Otherwise, we will get the wrong sign for R_{TH}. This can be seen just by noting that if in the circuit above, R_{TH} is positive and v_T is positive, i_T will be positive. If we were to reverse the direction of the current and then take the ratio, we would get the wrong sign for R_{TH}.

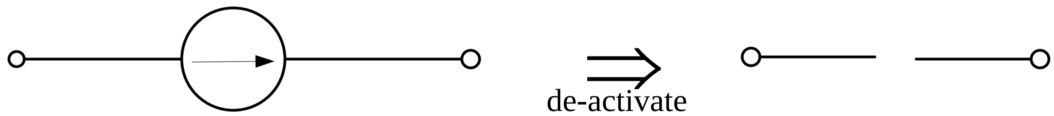
But is this useful? If we are finding the Thevenin Equivalent of a circuit that is just resistances, like the one above, we can simply combine these into one resistance (series, parallel, delta-to-wye), and we have R_{TH}. We don’t need a “test source”. But what if our circuit is not just a resistance, and contains sources as well? We can use the test source idea as follows.

1. De-activate all *__independent__* sources.

To de-activate an independent voltage source, we replace it with a short. Note that a short is a voltage source of value 0 [V].



To de-activate an independent current source, we replace it with an open circuit. Note that an open circuit is a current source of value 0 [A].



2. Apply a test source of known value; it doesn't matter what the value is. You can even leave the value out and just call it v_T.

3. Calculate the current i_T.

Then $R_{TH} = \dfrac{v_T}{i_T}$. If you have not given your test source a value, just calculate the ratio v_T/i_T.

Notes

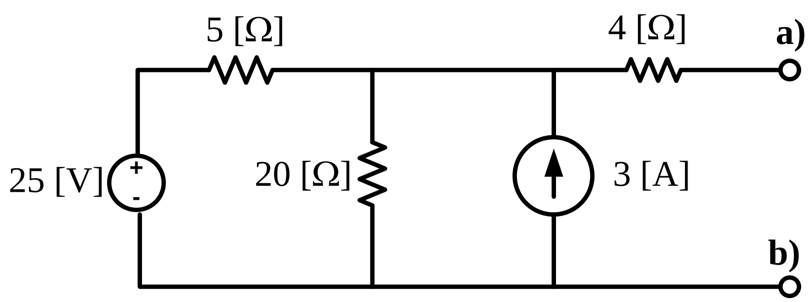
- ***You cannot de-activate dependent sources***. You need to leave them in; they affect the equivalent resistance of the circuit.

- If you have nothing but resistances and independent sources, you don't need the test source: you can simply de-activate the independent sources and find R_{TH} by resistor combinations. See the example on the next page.

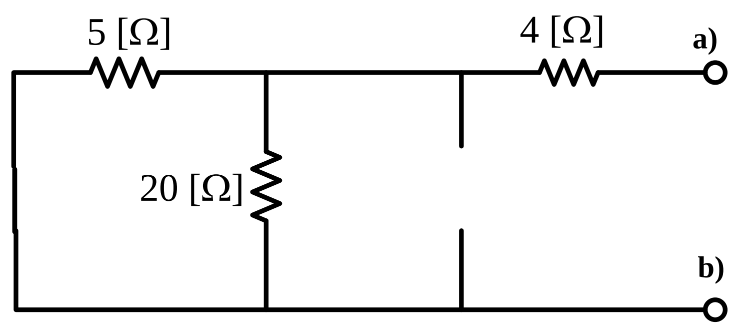
- If there are dependent sources but no independent sources, you have to use a test source, because the open circuit voltage and short circuit current will both be 0, so you can't take the ratio. See the example two pages forward.

Example: Resistances and independent sources.

Find the Thevenin Equivalent resistance of the circuit below at terminals a), b).



Since we have only independent sources, we can simply de-activate them:



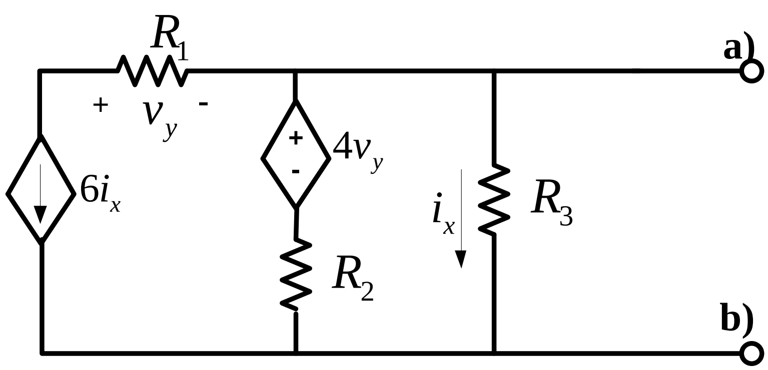
Now simple resistor combinations give us

$$R_{TH} = 4 + (20 \mathbin{//} 5) = 8\,[\Omega]$$

Example: Resistances and dependent sources.

Find the Thevenin Equivalent resistance of the circuit below at terminals a), b).

We will not do this problem here in the notes. Here we just note that at terminals a), b), both the open circuit voltage and the short circuit current are 0 because there are no independent sources. So this circuit is modeled by a resistance only. But we cannot find R_{TH} from v_{OC} and i_{SC} – we must apply a test source.



On Finding the Thevenin Equivalent

- If the circuit contains independent sources, you can find an open circuit voltage and a short circuit current, or you can use a test source to find R_{TH}. We only need to choose two of these things to find the Thevenin Equivalent. It is a smart idea to check to see which of these methods is easier to use: the short circuit current may remove components in parallel to the terminals of interest, for example. The test source method is useful if we want to de-activate independent sources.

- If the circuit consists only of resistances, these can be combined into one to find R_{TH}. In that case, the open circuit voltage and short circuit current will be both be 0, which means the Thevenin voltage is 0.

- If the circuit contains only resistances and dependent sources (or only dependent sources), the open circuit voltage and short circuit current will again be 0. In that case, there is no choice but to use a test source.

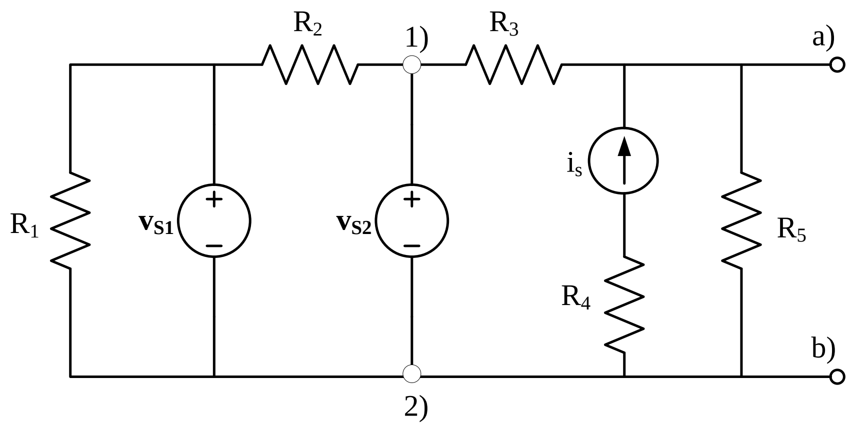
On a Negative Thevenin Equivalent

We assume that there are no negative-valued resistors (of the type you find in your lab kit, for example). However, *when modeling a circuit that contains dependent sources*, it is possible that the Thevenin Equivalent resistance is negative. This does not mean that we can have negative valued resistors. It means that *the circuit model includes a negative resistance*. That resistance is simply part of the model; it is not an actual circuit component.

Only circuits with dependent sources can have negative R_{TH}. But just because a circuit has a dependent source does not mean it will have a negative R_{TH}.

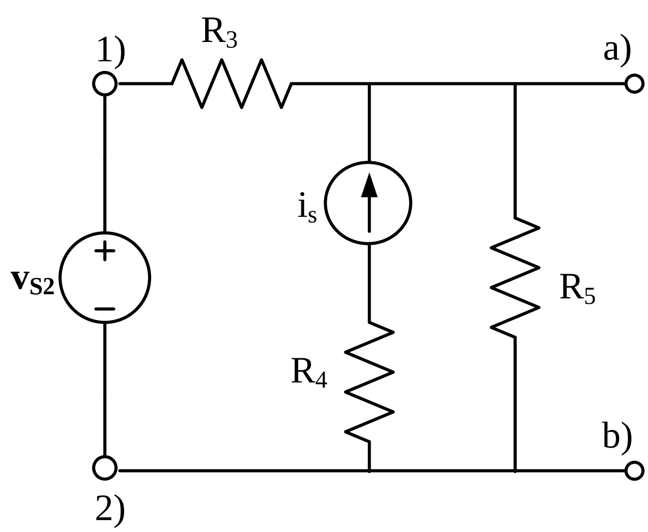
Two interesting cases

Consider the circuit below, where we are interested in terminals a), b).



Any circuit components to the left of the source v_{S2} cannot have any effect on what happens at terminals a), b), because v_{S2} fixes the voltage across those components. So no matter what values R_1 and R_2 or v_{S1} have, the voltage across them is v_{S2}, and something connected to a), b) will see v_{S2} but not those components.

What that means is that *as long as we are interested only in what happens at a), b), which is to the right of terminals 1) and 2),* we can re-draw the circuit as follows.



Now think about the current source. Nothing outside of R_4 and the current source can "see" R_4, because the current through it is fixed by i_S. In other words, if R_4 doubled in value, nothing different would happen at terminals a), b) – or terminals 1), 2) for that matter. So as long as we are interested in something outside the branch with i_S and R_4, we can remove R_4 as well.

Bottom line

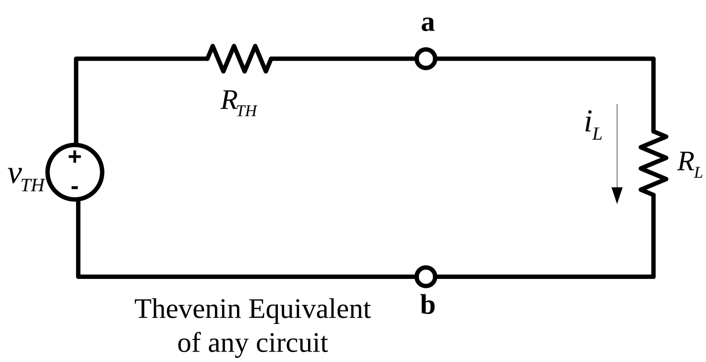
Circuit components in parallel with a voltage source can be replaced by just the voltage source, provided we are interested only in what is happening outside of those components.

Circuit components in series with a current source can be replaced by just the current source, provided we are interested only in what is happening outside of those components.

4.12 Maximum Power Transfer

Sometimes we are interested in maximizing the power transferred to a load: we want to get as much power to our stereo speakers as possible, for example. The power transferred to a load from a circuit depends on the circuit as well as on the load.

The circuit below shows a Thevenin Equivalent of anything – maybe a stereo system. The resistance connected a), b) represents the load – a speaker, for example. We want to analyze how much power is delivered to the load.



The power delivered to R_L is

$$\rho_{del,R_L} = i_L^2 R_L = \left(\frac{v_{TH}}{R_L + R_{TH}}\right)^2 R_L$$

Analysis: If R_L is very small (approaching 0), no power is delivered to R_L. If R_L is very large (approaching infinity), again no power is delivered. So there is a maximum at some finite value of R_L, which we can find by differentiating with respect to R_L and equating to 0:

$$\frac{dp_{del,R_L}}{dR_L} = 0 = v_{TH}^2 \left[\frac{\left(R_{TH} + R_L\right)^2 - 2R_L\left(R_{TH} + R_L\right)}{\left(R_{TH} + R_L\right)^4}\right]$$

$$\therefore \left(R_{TH} + R_L\right)^2 - 2R_L\left(R_{TH} + R_L\right) = 0$$

$$\Rightarrow R_L = R_{TH}$$

So the power delivered to the load is a maximum (i.e., we are getting as much power to the load as possible) if the load resistance is equal to the Thevenin resistance.

The amount of power delivered in that case is

$$p_{del,R_l\,\max} = \frac{v_{TH}^2}{4R_L}$$

We may not always be able to choose the load resistance, but if we can, it should be as close as possible to the Thevenin resistance. Audio speakers are made to have resistances equal to the output resistance (Thevenin resistance at the output) of typical stereo amplifiers.

4.13 Superposition

Idea: For some kinds of systems, the response of the system to several sources is equal to the sum of the responses to each source individually.

Definitions:

- **System**: For our purposes, a system is any circuit we may be interested in. In mechanical engineering, it could be a collection of parts connected by springs, for example.
- **Source**: For our purposes, a source is any independent voltage or current source.
- **Response**: For our purposes, the response of the system will be any voltage or current generated by the sources.
- **Superposition**: The idea that the system response to several sources is the same as the sum of the responses to the individual sources is the ***Superposition Principle***.
- **Linear systems**: A system for which superposition holds is a ***linear system***.

How can we use this idea to solve circuits? Here is the algorithm:

Application of the Superposition Principle:

1. De-activate all but one of the independent sources. We do not consider dependent sources here; those are always left in the circuit. (*De-active* means to replace voltage sources with a short (0 voltage) and current sources with an open circuit (0 current), as we did for finding Thevenin Equivalent Resistances.)

2. Find the response of the system (a voltage or current we are looking for) to the remaining source.

3. Repeat steps 1 and 2 for each source.

4. Add the responses to each source to get the total response to all sources.

Superposition may make some circuits easier to solve, but it will usually be more trouble than it's worth for the kind of circuits we have been dealing with so far. But when we go on to ac sources and the phasor domain later in the course, we will have to use it if the sources have different ac frequencies. We return to this idea later…