Chapter 4: Techniques of Circuit Analysis

This chapter gives us many useful tools for solving and simplifying circuits. We saw a few simple tools in the last chapter (reduction of circuits via series and parallel combinations of resistances, for example) but in this chapter we take circuit simplification much farther…

4.1 Terminology

Planar Circuits can be drawn on a plane with no crossing branches.

Non-Planar Circuits cannot be drawn on a plane without branches crossing one another somewhere.

Figure 4.1 (left) shows a circuit that has crossing branches but that can be re-drawn without any branches crossing. This is a planar circuit.

Figure 4.2 (right) shows a circuit that is not planar: there is no way to draw it without branches crossing.

In Circuit Analysis we will consider planar circuits only. Many of the techniques we learn here cannot be applied to non-planar circuits.
Definitions

A **node** is a point in a circuit where two or more circuit elements meet. The number of nodes in a circuit is \( n \).

A **path** is formed when adjoining (connected) circuit elements are traced, in order, without passing through any node more than once. A **closed path** is a path whose starting and ending points are the same.

A **branch** is a path connecting two nodes.

A **mesh** is a closed path that does not contain any other closed paths. The number of meshes in a circuit is \( m \).

Looking at the circuit in Figure 4.1b (shown again below), we can identify: 4 meshes; 8 closed paths; 6 nodes. Two of the four meshes are shown in green; two nodes are shown as red dots; one possible closed path which is not a mesh is shown in blue. The blue closed path contains two meshes. (The arrows on the paths are just for show: we can trace a path in either direction.)

![Circuit diagram](image)

**Essential Node:** A useful kind of node is the essential node, which is a node at which at least three circuit elements meet. In the circuit above there are four of these; one of them is labeled as a green dot. One of the red dots is also an essential node; the other is not.

An **essential branch** is a branch that connects two essential nodes. There are quite a few in the circuit above.
**Where are we going with all this??** This terminology helps us figure out **how many equations** we will need to solve the circuit:

We need as many equations as there are essential branches with unknown currents; call this number $b_e$. Then we can write $(n_e - 1)$ KCL equations so we need $[b_e - (n_e - 1)]$ KVL equations. Here, $n_e$ is the number of essential nodes.

We will shortly look at two very important and very powerful circuit analysis techniques:

- **Node Voltage Method** (NVM): allows us to solve a circuit with $(n_e - 1)$ KCL equations.

- **Mesh Current Method** (MCM): allows us to solve a circuit with $[b_e - (n_e - 1)]$ KVL equations.

Using either method, we save quite a few equations, since without them we would need $(n_e - 1)$ KCL equations and $[b_e - (n_e - 1)]$ KVL equations.
4.2 Introduction to the Node Voltage Method

To use the Node Voltage Method (NVM) we define new circuit variables called the node voltages. We find the node voltages by solving simultaneous equations called the node voltage equations. These are generally much fewer in number than we need with basic KVL, KCL, Ohm’s Law techniques. If we have the node voltages, we can find any other current or voltage in the circuit with a single equation (i.e., no more simultaneous equations will be necessary).

Node Voltage Method Algorithm

Basic steps in using NVM (we will clarify and expand on this shortly):

- Find and label all essential nodes.
  - There are \( n_e \) essential nodes, and we will need to solve \( n_e - 1 \) node voltage equations.

- Choose one of the essential nodes as a reference and label it with the symbol \( \downarrow \).
  - This is where the “- 1” arises in “\( n_e - 1 \) equations”: we do not need a node voltage equation at the reference.
  - Typically we will choose the reference node to be the one with the largest number of circuit elements connected to it. But we do not have to do this, and we will see there may be reasons for choosing a different node as the reference.

- Define the non-reference node voltages.
  - The non-reference nodes are the ones you did not choose as the reference node. The non-reference node voltages are the voltage drops from each non-reference node to the reference node (an example or two will clarify this). These are labeled with the positive sign at the node, and the negative sign at the reference node.

- Apply Kirchhoff’s Current Law (KCL) to each non-reference node, writing currents at each node in terms of the node voltages and any sources present.
  - Although there is more than one way to do this, in this class we will always set the sum of the currents leaving the node equal to zero.

- Solve the node voltage equations. The resulting node voltages are considered the solution to the circuit. If you have those, you can find any current or voltage in the circuit with one equation (no more simultaneous equations needed).
To see how to express the current leaving a branch in terms of the node voltages, we look at the following example.

The essential nodes are labeled 1, 2, and \( \uparrow \) (reference node). The node voltages are defined as described above: positive at the non-reference node and negative at the reference. The current \( i_{12} \) is leaving node 1 and heading from node 2. It can be written in terms of node voltages as

\[
i_{12} = \frac{v_1 - v_2}{2[\Omega]}.
\]

Where did we get that equation for \( i_{12} \)? From KVL:

\[
v_1 - v_2 - i_{12} (2[\Omega]) = 0
\]

Solving this equation for \( i_{12} \) gives the previous equation. We continue in this way to set up KCL at each non-reference node. So, at node 1, the sum of the currents leaving node 1 is

\[
\frac{v_1}{5} + \frac{v_1 - v_2}{2} + \frac{v_1 - 10}{1} = 0
\]

The sum of the currents leaving node 2 is

\[
\frac{v_2 - v_1}{2} + \frac{v_2}{10} - 2 = 0
\]

If the third term of the first equation is confusing, do a KVL around the loop containing the 10 [V] source, the 1 [\Omega] resistor, and the 5 [\Omega] resistor. Solve this KVL for the current leaving node 1, and you will get the third term in that equation. Eventually, you will get to the point where these terms can be written down quickly without having to examine each KVL individually.
The third term in the second equation is easy, since we know the current in that branch: it’s 2 [A] entering node 2, so the current leaving is – 2 [A]. In this case, we don’t try to express it in terms of the node voltages. Note also that we have summed the currents leaving each node. This is an arbitrary choice, but we must choose something, and be consistent.

We have two equations for two unknowns. The solution is…

\[ v_1 = 9.091 [V] \quad v_2 = 10.91 [V] \]

Now that we have the node voltages, we can find any other voltage or current in the circuit. To do that we probably need to solve one equation, but we will not have to solve simultaneous equations anymore. For example, the current in each branch is easy – we have already shown how to calculate \( i_{12} \) in the equation above. Also, the voltage across any individual component can be found with one KVL. As one more example, the voltage across the current source in the circuit above is nothing but \( v_2 \).

**Notation Rules (you knew we were going to have this…)**

All node voltages must be labeled just as they are in the diagram above. Those node voltages have a ‘+’ sign at the non-reference node, and a ‘-’ sign near the reference node. It is not sufficient to simply label \( v_1 \), for example, near node 1. It is certainly not acceptable to leave out the node voltages from your diagram. If you do, significant credit will be subtracted from quizzes and exams.
4.3 The Node Voltage Method and Dependent Sources

If we have dependent sources, we will need one equation for each controlling variable. We call these auxiliary equations. We will need to write these equations in terms of the node voltages. This is because we are trying to find the node voltages (which are the circuit variables), and we don’t want to introduce any extra variables.

Here is a circuit with a dependent source.

![Circuit Diagram]

We have identified the node voltages and the reference node. Our equations are

\[
\frac{v_1}{400} + \frac{v_1 - 5}{230} + \frac{v_1 - v_2}{25} = 0
\]

\[
\frac{v_2 - v_1}{25} + \frac{v_2}{50} + \frac{v_2 - 10i_q}{150} = 0
\]

Just as in using Kirchhoff’s Laws, we need another equation for the dependent variable…but be sure to express it in terms of the node voltages.

\[
i_q = \frac{v_2 - v_1}{25}
\]

4.4 The Node Voltage Method: Some Special Cases

Special cases arise when we have a voltage source connected either (1) between an essential node and the reference node, or (2) between two non-reference essential nodes.

(1) Voltage source between an essential node and the reference node.

In the circuit below the 100 [V] source is connected from essential node 1 to the reference node.
This is a simple case. Remember that we are trying to find the node voltages, which are the voltage values between each of the essential nodes and the reference node. But we already know that in this case!!

\[ v_1 = v_s = 100 \text{V} \]

This is the node voltage equation for node 1. We have still have the same number of node voltage equations, but one of them is trivial. This works for dependent sources, too, although we of course need the auxiliary equation.

The rest of the circuit is solved as follows:

\[ \frac{v_2 - v_1}{10} + \frac{v_2}{50} - 5 = 0 \]

This gives \( v_1 = 100 \text{ [V]} \); \( v_2 = 125 \text{ [V]} \).

(2) Voltage source between two non-reference essential nodes.

In the circuit below, a voltage source is connected between two non-reference essential nodes (nodes 2 and 3). We are using an independent source for illustration, but our remarks are valid for dependent sources as well.
The node voltage equation for $v_1$ is:

\[
\frac{v_1}{40} + \frac{v_1 - 50}{30} + \frac{v_1 - v_2}{5} = 0. 
\]

But when we try to write a node voltage equation at node 2, we have a problem: there is no way to express the current $i$ as a ‘v/R’ term using the node voltages, as we have been doing. Let’s write a KCL anyway, using node voltages where we can. We will denote the unknown current as a variable.

\[
\frac{v_2 - v_1}{5} + \frac{v_2}{50} + i = 0
\]

Now at node 3 we have the same situation:

\[
\frac{v_3}{100} - 4 - i = 0.
\]

What should we do about this unknown current? Let’s add the last two equations together:

\[
\frac{v_2 - v_1}{5} + \frac{v_2}{50} + \frac{v_3}{100} - 4 = 0
\]

Hmmm…this last equation is what we have got if we had treated node 2 and node 3, with the dependent source in between, as one big node…it’s as if it were a … a…

Supernode!!

In the figure above, the red dashed line encircles the supernode, and the arrows show currents leaving it. Let’s apply our usual node voltage technique to those currents.

The currents leaving node 2 are
\[
\frac{v_2}{50} + \frac{v_3 - v_1}{5} + \frac{v_3}{100} - 4 = 0
\]

which is the equation we got immediately above. This last equation, and the one above that was written for node 1, give us two equations, but we have three unknowns: \(v_1\), \(v_2\), and \(v_3\). We need another equation. Note that we have not included any information about the 10 [V] source. We include this by simply doing a KVL around a loop that includes the dependent source:

\[
v_2 - v_3 = 10[V]
\]

This equation comes from the fact that the voltage between node 2 and node 3 is “constrained” by the voltage source. This is the constraint equation.

Now we have three equations in three unknowns and we can solve for the node voltages.

**Important Note** The supernode equation (the first of our four equations above) contains \(v_2\) AND \(v_3\), so we do not have any fewer node voltages to solve for. But the kind of equations is different from the case where we do not have a supernode: Instead of two node voltage equations, we have a supernode equation and a constraint equation.

**Counting Equations**

It is good practice to count essential nodes and determine how many equations, and of what type, you are going to need. We will do this in the examples in class. You will need…

- …one node voltage equation for each non-reference essential node.
- …one constraint equation for each supernode
- …one auxiliary equation for each dependent source.

**Choosing the Reference Node**

You can choose any essential node as the reference node. Typically this will be the one with the largest number of branches connected to it, since then you will avoid having to write an equation with a lot of terms.

One reason you might not want to choose the node with the largest number of branches is that you might be able to simplify the problem by choosing the reference node so that one or more of them is trivial. If you choose the reference so that a voltage source (dependent or independent) is connected to it, with the other end of the voltage source at another essential node, then that node voltage is trivial.
4.5 Introduction to the Mesh Current Method

To use the Mesh Current Method (MCM) we define new circuit variables called the mesh currents. We find the mesh currents by solving simultaneous equations called the mesh current equations. These are generally much fewer in number than we need with basic KVL, KCL techniques. If we have the mesh currents, we can find any other current or voltage in the circuit with a single equation (i.e., no more simultaneous equations will be necessary).

Consider the circuit below, where KCL is applied to the node at the top.

\[ i_1 = i_2 + i_3 \]

KCL around the left and right loops gives…

\[ v_{s1} = i_1 R_1 + i_3 R_3 \]
\[ v_{s2} = -i_2 R_2 + i_3 R_3 \]

If we solve the KCL equation for \( i_3 \) and substitute the result into the KVL equations we get:

\[ v_{s1} = i_1 (R_1 + R_3) - i_2 R_3 \]
\[ v_{s2} = -i_2 (R_1 + R_3) + i_1 R_3 \]

We now have two equations, in the form of KVLs, in two unknowns. This substitution, and the resulting equations, can be done “automatically” using the mesh current method …

To use the mesh current method to get the last two equations directly, we do the following.
**Mesh Current Method Algorithm**

1. **Find the meshes and label each with a current.** Remember that we learned how to count meshes $m$, essential nodes $n_e$, and essential branches $b_e$. Then the number of meshes will be

\[ m = b_e - (n_e - 1) \]

   - The two mesh currents in the circuit below are labeled $i_a$, $i_b$. The direction shown (clockwise for both) is arbitrary – we could have chosen either one (or both) to go in the other direction. We will refer to the two meshes as “mesh a” and “mesh b”.

   - Note that the mesh currents are defined as the currents going “around” the perimeter of the loops. It is important to realize that these are not necessarily the same as branch currents $i_1$, $i_2$, $i_3$.

![Circuit Diagram](image)

2. **Apply KVL around each mesh, writing voltages across resistors in terms of the mesh currents.** The result for the circuit above is:

   - **Mesh a**
     \[ -v_1 + i_a R_1 + (i_a - i_b) R_3 = 0 \]

   - **Mesh b**
     \[ (i_a - i_b) R_3 + i_b R_2 + v_2 = 0 \]

3. **Solve for the mesh currents. Then, find the branch currents in terms of the mesh currents.**

   Going back to the original circuit where we defined the branch currents $i_1$, $i_2$, $i_3$, is should be clear that $i_1 = i_a$ and $i_2 = i_b$.

   What about $i_3$? Looking at the circuit below you should be able to convince yourself that $i_3 = i_a - i_b$. So we have found the branch currents in terms of the mesh currents. This is the reason for the statement above that the mesh currents are *not necessarily* equal to the branch currents.
If we substitute our results for the branch currents in terms of the mesh currents into the mesh current equations, we can show that the mesh current equations are equivalent to the two equations we got earlier: Specifically, the equations

\[
\begin{align*}
    v_1 &= i_a (R_1 + R_3) - i_b R_3 = 0 \\
    -v_2 &= -i_a R_3 - i_b (R_2 + R_3)
\end{align*}
\]

are the same as the previous KVL equations if we make the assignments \(i_1 = i_a, i_2 = i_b,\) and \(i_3 = i_a - i_b.\)
4.6 The Mesh Currents and Dependent Sources

For each dependent source, we need an additional equation defining the controlling variable. This is always the case, but now we want to write the defining equations in terms of the mesh currents. If we use, say, branch currents to define the controlling variables, we will have introduced additional unknowns.

4.7 The Mesh Currents Method: Some Special Cases

Two special cases arise when current sources are present. (Note that these cases are analogous to the case of voltage sources in the node voltage method.)

(1) A branch contains a current (dependent or independent)

In the circuit shown above, it should be clear that \( i_a = 3 \text{ [A]} \), which means that we have one less mesh current equation to solve because one of the unknowns \( (i_a) \) is now known. So there is only one mesh current equation for this circuit, which is

\[
v_{s1} + (i_b - i_a) R_3 + i_b R_2 = 0
\]

So a current source in a branch means that the number of mesh current equations is reduced by 1.
(2) A current source (dependent or independent) is being shared by two meshes

The mesh current equations for the circuit above are as follows.

Mesh b)

\[ i_{b}R_{1} + (i_{b} - i_{c})R_{3} + (i_{b} - i_{a})R_{2} = 0 \]

KVL around either mesh a or c includes the current source. We cannot assume that the voltage across the current source is 0, but we don’t know what it is, so we will simply label it “\( v \)”. Then we have

\[ -v_{S1} + (i_{a} - i_{b})R_{2} + v + i_{a}R_{4} = 0 \]

\[ (i_{c} - i_{b})R_{3} + v_{S2} + i_{c}R_{5} - v = 0 \]

Adding these last two equations gives

\[ -v_{S1} + (i_{a} - i_{b})R_{2} + (i_{c} - i_{b})R_{3} + v_{S2} + i_{c}R_{5} + i_{a}R_{4} = 0 \]

But this equation is just what we would have obtained if we had done a KVL around the path shown in red below. Why, it’s almost as if we had…a…a…
Supermesh !!

Above we have drawn a dashed line around the supermesh. We can write the equation above in one step using this mesh:

\[-v_{S1} + (i_a - i_b)R_2 + (i_c - i_b)R_3 + v_{S2} + i_cR_5 + i_aR_4 = 0\]

But we have to get the value of the current source involved, which we do by recognizing that there is a constraint imposed on \(i_a\) and \(i_c\):

\[-i_a + i_c = 5\]

So we have still have three equations: one regular mesh current equation, and one super mesh equation, and one constraint equation (the constraint equation is the one we just wrote).
Let’s do an example… we will find the power absorbed by the 10 [Ω] resistor and the power delivered by the 3 [A] source in the circuit below.

There are 3 meshes, two of which share a current source. So we will have three mesh current equations:

- one regular mesh current equation
- one supermesh equation
- one constraint equation

The regular mesh current equation is

\[ 5(i_c - i_b) + 10 + 10(i_c - i_a) = 0 \]

The super mesh equation is

\[ -7 + 25i_b + 5(i_b - i_c) + 10(i_a - i_c) + 6i_a = 0 \]

The constraint equation is

\[ i_a - i_b = 3 \]

Solving these equations together gives \( i_a = 2.323 [A] \); \( i_b = -0.667 [A] \); \( i_c = 0.656 [A] \).

Now that we have found the currents, we can find anything else we might need using only one equation. Since we want power for the resistor and the current source, we have labeled the current in the resistor and the voltage across the current source in the next figure.
The current $i_x$ is

$$i_x = i_a - i_c = 1.667 [A]$$

Then

$$p_{abs,10[\Omega]} = i_x^2(10) = 27.8 [W]$$

How did we know that $i_x$ was $i_a - i_c$ and not $i_c - i_a$? Because at the 10 [\Omega] resistor, $i_x$ is defined to be going in the direction of $i_a$ and opposite the direction of $i_c$.

To find $v_s$ we need a KVL:

$$-7 + 30(i_a - i_b) + v_s + 10(i_a - i_c) + 6i_a = 0$$

This gives $v_s = -113.6 [V]$. Then

$$\therefore p_{del,3[A]} = -3(-113.6) = 340.8 [W]$$
4.8 The Node Voltage vs. the Mesh Current Method

We want to think carefully in each case about whether the node voltage or the mesh current method is easier to implement. Some factors to consider:

- One of the methods may require fewer equations to solve simultaneously. For any circuit we will have $n_e - 1$ node voltage equations and $m = b_e - (n_e - 1)$ mesh current equations.

- Voltage sources may reduce the number of node voltage equations by introducing trivial equations. Current sources may reduce the number of mesh current equations by introducing trivial equations. This of course depends on where we put the reference.

- The variable we are looking for may already be a mesh current or a node voltage, in which case solving by the appropriate method will yield the answer directly.

An observation: Most beginning students prefer mesh currents, but in fact the node voltage method almost always involves fewer equations. Also, node voltage method is extremely useful in many electronics applications. I would strongly suggest getting familiar with the node voltage method.
4.9 Source Transformations

We can sometimes simplify a circuit using equivalent circuits…

Two circuits are equivalent if one can be replaced with the other without changing circuit variables (voltages and currents) in the rest of the circuit. We have already seen how to use equivalent circuits to simplify using series and parallel resistor combinations. In that case, we are replacing a group of resistors by a single equivalent resistor. Source transformation is another such simplification technique.

The circuits “1” and “2” below are equivalent with respect to terminals a and b, provided that \( v_s, R_s, i_s, \) and \( R_p \) are related to one another in a particular way. If they are, then a resistor \( R_L \) connected to terminals a and b will have the same voltage across it (and the same current through it) whether it is connected to circuit 1 or to circuit 2. In fact, anything connected to terminals a and b of either circuit will have the same voltage across it and current through it. That is what “equivalent” means.

Note that it is very important to include the qualifier with respect to terminals a and b, because two circuits are not necessarily equivalent at just any two terminals.

**Bottom Line:** If the parameters are related correctly, a voltage source in series with a resistor can be replaced with a current source in parallel with a resistor.
But what relationship must exist among \( v_s, R_s, i_s, \) and \( R_p \) in order for these circuits to be equivalent? We find this relationship by requiring that equivalence must hold for any load resistor \( R_L \), and in particular it must hold for \( R_L = 0 \) and \( R_L = \infty \).

For \( R_L = 0 \), we ask what current \( i_L \) flows through \( R_L \). (By the way, \( R_L = 0 \) means that a short circuit exists at terminals a and b, and the resulting current is the *short-circuit current*). If it is connected to circuit 1, then

\[
  i_L = \frac{v_s}{R_s}
\]

If it is connected to circuit 2, then

\[
  i_L = i_s
\]

For these to be equivalent, we must have

\[
  i_s = \frac{v_s}{R_s}
\]

Now consider \( R_L = \infty \). In that case, terminals a and b are *open circuit*, and the voltage \( v_L \) is the *open-circuit voltage*. Then when \( R_L \) is connected to circuit 1 we have

\[
  v_L = v_s
\]

and when it is connected to circuit 2,

\[
  v_L = i_s R_p
\]

So that means

\[
  v_s = i_s R_p
\]

Comparing this equation to the one for \( i_s \) above shows that we also need \( R_s = R_p \).

**Summary:** A voltage source \( v_s \) in series with a resistor \( R_s \) will be equivalent to a current source \( i_s \) in parallel with a resistor \( R_p \) if

\[
  v_s = i_s R_p \text{ and } R_s = R_p
\]

This is the *source transformation theorem*. 
4.10 Thevenin and Norton Equivalents

We are often interested in the voltage and/or current in a load that is connected to a particular pair of terminals in a circuit. For example, we may want to connect a load resistor, or maybe several different load resistors, to the terminals labeled a), b) in the circuit shown below.

![Circuit Diagram]

The following idea is very powerful, and may help in analyzing a case like this:

**Thevenin Equivalent Circuit:** The behavior of any linear circuit at a specific pair of terminals in a circuit may be modeled by a voltage source \( v_{TH} \) in series with a resistor \( R_{TH} \).

We will look only at linear circuits in this course; *linear circuit* is defined later in the section on superposition. What we are saying is that the circuit below on the right can be modeled by the circuit on the left.
Important Notes:

- We are “modeling” the circuit at two particular terminals with $v_{TH}$ and $R_{TH}$ – we are not suggesting that the only things inside the box are a resistor and a voltage source.
- The model holds for any load but only at terminals a), b). If we different terminals in the original circuit, the values of $v_{TH}$ and $R_{TH}$ will change.
- The circuit must be linear, but it can contain any of the basic circuit elements: voltage sources, current sources (dependent and independent), resistors, capacitors, and inductors.

Finding $v_{TH}$ and $R_{TH}$:

The box in the figure below contains an arbitrary linear circuit. We have labeled terminals a) and b). On the right, we have an open circuit at a), b), resulting in an open-circuit voltage $v_{OC}$. (We can think of this as an infinite load resistance.) On the left, we have connected a short to the terminals, resulting in a short-circuit current $i_{SC}$.

By comparing the drawing on the left with the Thevenin Equivalent drawing above, it should be clear that

$$v_{OC} = v_{TH}.$$  

By comparing the drawing on the left with the Thevenin Equivalent, we can see also that

$$i_{SC} = \frac{v_{TH}}{R_{TH}}.$$  

So we already have an algorithm for finding a Thevenin Equivalent: If we know the open-circuit voltage and the short-circuit current at the terminals a), b), we can find the Thevenin Equivalent:

$$v_{TH} = v_{OC} \quad \text{and} \quad R_{TH} = \frac{v_{TH}}{i_{SC}}.$$
If this were an experiment, we could measure $v_{OC}$ and $i_{SC}$. If it is an analytical problem, we can calculate them using our knowledge of circuit theory.

Is this useful?? Wow, yeah! This idea is used a lot. What it means is that we can talk about a lot of complicated circuits without having to know anything about those circuits except their Thevenin Equivalents. This is extremely useful.

It also means that if we need to analyze how several different load resistors behave when connected to a circuit at two particular terminals, we only need the Thevenin Equivalent, and we can make the calculations much simpler. This idea is shown below.

![Diagram showing Thevenin Equivalent](image)

Generally it is a lot easier to handle the Thevenin Equivalent circuit on the right than it is to analyze the complicated circuit on the left.
4.11 More on Deriving a Thevenin Equivalent

There is another method for finding \( R_{TH} \):

**The Test Source**

Suppose we had a circuit that could be modeled using only a resistor. Then if we were to apply a voltage source \( v_T \) and measure or calculate the current \( i_T \) through it, we could find \( R_{TH} \) as:

\[
R_{TH} = \frac{v_T}{i_T}
\]

Here, \( v_T \) is known as the “test source”. If our circuit is simply a resistance, we can use the test source to find that resistance. This will be the Thevenin Equivalent resistance \( \hat{R} \).

Be careful: the polarity of \( v_T \) and the current \( i_T \) that we calculate have to be in the active sign convention. Otherwise, we will get the wrong sign for \( R_{TH} \). This can be seen just by noting that if in the circuit above, \( R_{TH} \) is positive and \( v_T \) is positive, \( i_T \) will be positive. If we were to reverse the direction of the current and then take the ratio, we would get the wrong sign for \( R_{TH} \).

But is this useful? If we are finding the Thevenin Equivalent of a circuit that is just resistances, like the one above, we can simply combine these into one resistance (series, parallel, delta-to-wye), and we have \( R_{TH} \). We don’t need a “test source”. But what if our circuit is not just a resistance, and contains sources as well? We can use the test source idea as follows.

1. De-activate all independent sources.

To de-activate an independent voltage source, we replace it with a short. Note that a short is a voltage source of value 0 [V].
To de-activate an independent current source, we replace it with an open circuit. Note that an open circuit is a current source of value 0 [A].

![Diagram showing de-activation of a current source]

2. Apply a test source of known value; it doesn’t matter what the value is. You can even leave the value out and just call it \( v_T \).

3. Calculate the current \( i_T \).

Then \( R_{TH} = \frac{v_T}{i_T} \). If you have not given your test source a value, just calculate the ratio \( v_T/i_T \).

**Notes**

- **You cannot de-activate dependent sources.** You need to leave them in; they affect the equivalent resistance of the circuit.

- If you have nothing but resistances and independent sources, you don’t need the test source: you can simply de-activate the independent sources and find \( R_{TH} \) by resistor combinations. See the example on the next page.

- If there are dependent sources but no independent sources, you have to use a test source, because the open circuit voltage and short circuit current will both be 0, so you can’t take the ratio. See the example two pages forward.
Example: Resistances and independent sources.

Find the Thevenin Equivalent resistance of the circuit below at terminals a), b).

Since we have only independent sources, we can simply de-activate them:

Now simple resistor combinations give us

\[ R_{TH} = 4 + \left( \frac{20}{5} \right) = 8 [\Omega] \]
Example: Resistances and dependent sources.

Find the Thevenin Equivalent resistance of the circuit below at terminals a), b).

We will not do this problem here in the notes. Here we just note that at terminals a), b), both the open circuit voltage and the short circuit current are 0 because there are no independent sources. So this circuit is modeled by a resistance only. But we cannot find $R_{TH}$ from $v_{OC}$ and $i_{SC}$ – we must apply a test source.
On Finding the Thevenin Equivalent

- If the circuit contains independent sources, you can find an open circuit voltage and a short circuit current, or you can use a test source to find $R_{TH}$. We only need to choose two of these things to find the Thevenin Equivalent. It is a smart idea to check to see which of these methods is easier to use: the short circuit current may remove components in parallel to the terminals of interest, for example. The test source method is useful if we want to de-activate independent sources.

- If the circuit consists only of resistances, these can be combined into one to find $R_{TH}$. In that case, the open circuit voltage and short circuit current will be both be 0, which means the Thevenin voltage is 0.

- If the circuit contains only resistances and dependent sources (or only dependent sources), the open circuit voltage and short circuit current will again be 0. In that case, there is no choice but to use a test source.

On a Negative Thevenin Equivalent

We assume that there are no negative-valued resistors (of the type you find in your lab kit, for example). However, when modeling a circuit that contains dependent sources, it is possible that the Thevenin Equivalent resistance is negative. This does not mean that we can have negative valued resistors. It means that the circuit model includes a negative resistance. That resistance is simply part of the model; it is not an actual circuit component.

Only circuits with dependent sources can have negative $R_{TH}$. But just because a circuit has a dependent source does not mean it will have a negative $R_{TH}$. 
Two interesting cases

Consider the circuit below, where we are interested in terminals a), b).

![Circuit Diagram]

Any circuit components to the left of the source $v_{S2}$ cannot have any effect on what happens at terminals a), b), because $v_{S2}$ fixes the voltage across those components. So no matter what values $R_1$ and $R_2$ or $v_{S1}$ have, the voltage across them is $v_{S2}$, and something connected to a), b) will see $v_{S2}$ but not those components.

What that means is that as long as we are interested only in what happens at a), b), which is to the right of terminals 1) and 2), we can re-draw the circuit as follows.

![Re-drawn Circuit Diagram]

Now think about the current source. Nothing outside of $R_4$ and the current source can “see” $R_4$, because the current through it is fixed by $i_s$. In other words, if $R_4$ doubled in value, nothing different would happen at terminals a), b) – or terminals 1), 2) for that matter. So as long as we are interested in something outside the branch with $i_s$ and $R_4$, we can remove $R_4$ as well.
**Bottom line**

Circuit components in parallel with a voltage source can be replaced by just the voltage source, provided we are interested only in what is happening outside of those components.

Circuit components in series with a current source can be replaced by just the current source, provided we are interested only in what is happening outside of those components.
4.12 Maximum Power Transfer

Sometimes we are interested in maximizing the power transferred to a load: we want to get as much power to our stereo speakers as possible, for example. The power transferred to a load from a circuit depends on the circuit as well as on the load.

The circuit below shows a Thevenin Equivalent of anything – maybe a stereo system. The resistance connected a), b) represents the load – a speaker, for example. We want to analyze how much power is delivered to the load.

Thevenin Equivalent of any circuit

The power delivered to \( R_L \) is

\[
\rho_{del,R_L} = i_L^2 R_L = \left( \frac{V_{TH}}{R_L + R_{TH}} \right)^2 R_L
\]

Analysis: If \( R_L \) is very small (approaching 0), no power is delivered to \( R_L \). If \( R_L \) is very large (approaching infinity), again no power is delivered. So there is a maximum at some finite value of \( R_L \), which we can find by differentiating with respect to \( R_L \) and equating to 0:

\[
\frac{d\rho_{del,R_L}}{dR_L} = 0 = V_{TH}^2 \left[ \frac{(R_{TH} + R_L)^2 - 2R_L(R_{TH} + R_L)}{(R_{TH} + R_L)^4} \right]
\]

\[
\therefore (R_{TH} + R_L)^2 - 2R_L(R_{TH} + R_L) = 0
\]

\[
\Rightarrow R_L = R_{TH}
\]

So the power delivered to the load is a maximum (i.e., we are getting as much power to the load as possible) if the load resistance is equal to the Thevenin resistance.

The amount of power delivered in that case is
\[ P_{del,R_{l_{\text{max}}}} = \frac{v_{TH}^2}{4R_L} \]

We may not always be able to choose the load resistance, but if we can, it should be as close as possible to the Thevenin resistance. Audio speakers are made to have resistances equal to the output resistance (Thevenin resistance at the output) of typical stereo amplifiers.
4.13 Superposition

Idea: For some kinds of systems, the response of the system to several sources is equal to the sum of the responses to each source individually.

Definitions:
- **System**: For our purposes, a system is any circuit we may be interested in. In mechanical engineering, it could be a collection of parts connected by springs, for example.
- **Source**: For our purposes, a source is any independent voltage or current source.
- **Response**: For our purposes, the response of the system will be any voltage or current generated by the sources.
- **Superposition**: The idea that the system response to several sources is the same as the sum of the responses to the individual sources is the *Superposition Principle*.
- **Linear systems**: A system for which superposition holds is a *linear system*.

How can we use this idea to solve circuits? Here is the algorithm:

**Application of the Superposition Principle**:

1. De-activate all but one of the independent sources. We do not consider dependent sources here; those are always left in the circuit. (*De-active* means to replace voltage sources with a short (0 voltage) and current sources with an open circuit (0 current), as we did for finding Thevenin Equivalent Resistances.)
2. Find the response of the system (a voltage or current we are looking for) to the remaining source.
3. Repeat steps 1 and 2 for each source.
4. Add the responses to each source to get the total response to all sources.

Superposition may make some circuits easier to solve, but it will usually be more trouble than it’s worth for the kind of circuits we have been dealing with so far. But when we go on to ac sources and the phasor domain later in the course, we will have to use it if the sources have different ac frequencies. We return to this idea later…