# Response of First Order RL and RC Circuits

## First Order Circuits: Overview

In this chapter we will study circuits that have dc sources, resistors, and either inductors or capacitors (but not both). Such circuits are described by **first order differential equations**. They will include one or more switches that open or close at a specific point in time, causing the inductor or capacitor to see a new circuit configuration. This in turn will cause a time-dependent change in voltages and currents.

We will find that the equations describing the voltages and currents in these circuits (i.e., the circuit responses) are exponential in time, and characterized by a single time constant. In other words, we will have responses of the form

$$v(t)\ or\ i(t) \propto e^{-t/\tau}$$

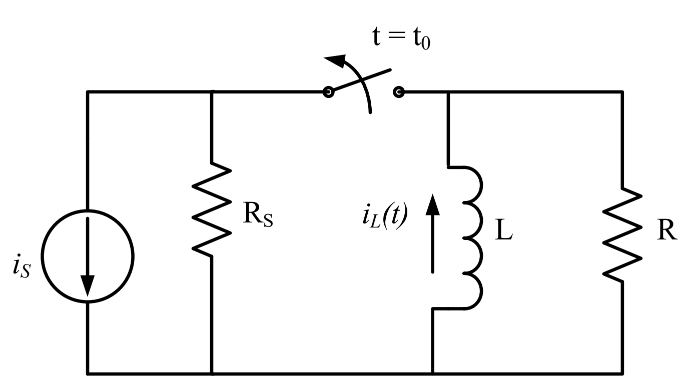
For the natural response, and

$$v(t)\ or\ i(t) \propto (1 - e^{-t/t})$$

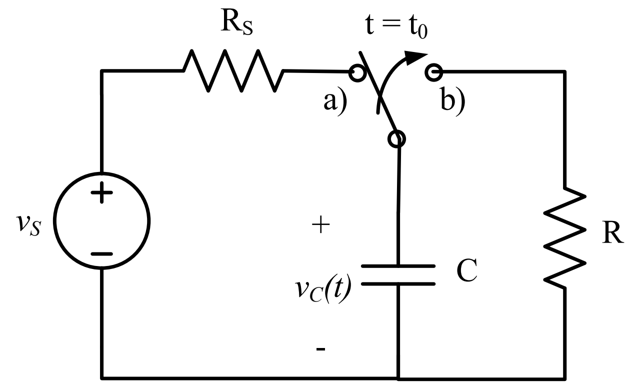
for the step response, where τ is the ***time constant***. These are **single time constant circuits**.

***Natural response*** occurs when a capacitor or an inductor is connected, via a switching event, to a circuit that contains only an equivalent resistance (i.e., no independent sources). In that case, if the capacitor is initially charged with a voltage, or the inductor is initially carrying a current, the capacitor or inductor will release its energy to the resistance.

The circuit below shows an inductor that was initially connected to a current source, which establishes a current in the inductor. The switching event at $t = t_0$ results in the inductor being connected to only a resistance. In this case, the inductor current $i_L(t)$ will decrease exponentially in time.



We next consider a capacitor initially connected to a voltage source, which establishes a voltage across the capacitor. The switching event at $t = t_0$ results in the capacitor being connected to only a resistance. In this case, the capacitor voltage $v_C(t)$ will decrease exponentially in time.



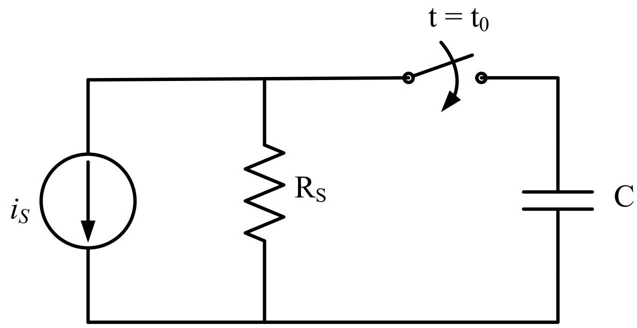

In the circuits above, L and C may be the result of combining inductors or capacitors in series or parallel.

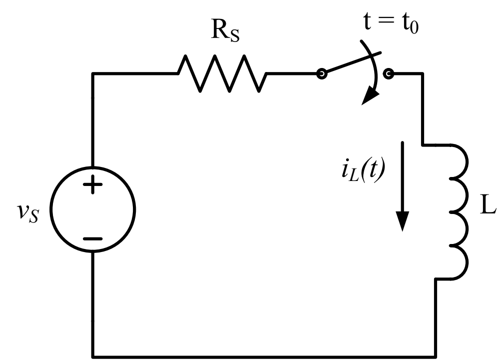
Note that the circuit connected to the inductor before $t = t_0$ is a Norton equivalent, and the circuit connected to the capacitor is a Thevenin equivalent. The resistance Rs may be interpreted as a source resistance. Here is an example of the use of Thevenin and Norton equivalent circuits to help us think in general terms about circuit analysis. We could have had considerably more complex circuits connected to the inductor or to the capacitor – and later we will have that - but we can always reduce those circuits to a Norton or Thevenin equivalent.

After the switching event, the inductor and the capacitor are connected only to a resistance. That resistance should be interpreted as the equivalent Thevenin equivalent resistance seen by the capacitor. In more complex problems, there may be several resistors in series or parallel as well as dependent sources, all of which combine to a single Thevenin equivalent resistance.

Note also that we could have chosen the Norton equivalent for the capacitor example, and the Thevenin equivalent for the inductor example. These are equivalent by the source transformation theorem, but making these choices now will make our later analysis easier.

***Step response*** occurs when an inductor or capacitor is connected, via a switching event, to a circuit containing one or more independent sources. Examples for the inductor and for the capacitor are shown below.



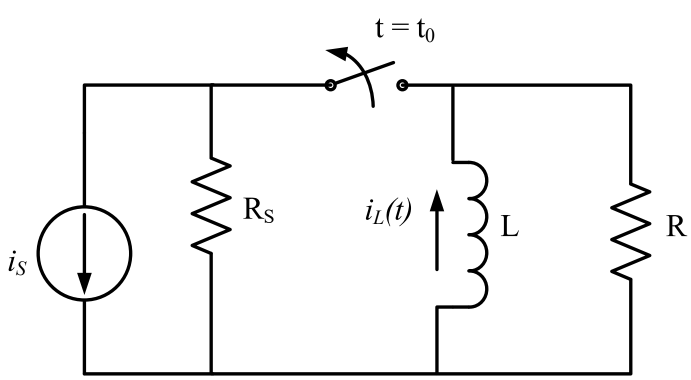
Again, we are using Thevenin and Norton equivalent circuits to represent what gets connected to the inductor and capacitor. Later we will consider more complex circuits.

## The Natural Response of an RL Circuit

The circuit below shows the natural response configuration we introduced earlier. We specify that the switch had been closed for a ***long time***, and then opened at $t = t_0$. After the switch opened, the inductor was connected to the resistance R. We want to know what happens to the inductor current after the switch is thrown.

***Terminology*** When we write "*t*", we mean the variable representing time. When we write $t_0$, we mean a particular moment time, e.g., $t_0 = 0$, or $t_0 = 5$ [ms].

The "natural response" is one in which the inductor, with current flowing through it, undergoes a switching event that connects it to a resistance only. As we pointed out above, that resistance can be a single resistor, or it can be an equivalent resistance that arises from a circuit containing multiple resistors and/or dependent sources.

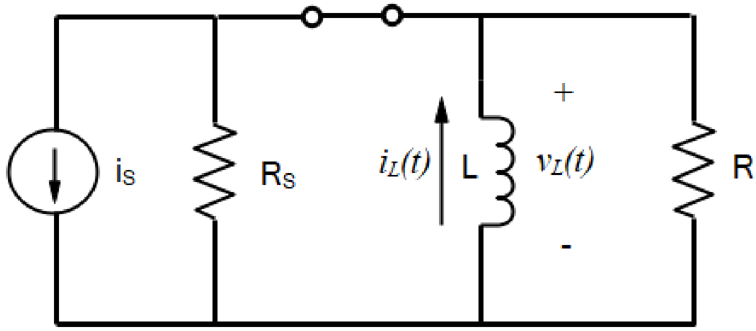



## *A Long Time?*

What do we mean that the switch had been closed "for a long time"? We will be specific later about what that means, but it is important to know that before the switch opens, enough time had passed that voltages and currents are no longer changing. Recall that we said that voltages and currents will be decaying exponentially in time for circuits like this. Therefore, after enough time has passed, voltages and currents will stop changing. Once that happens, we say the circuit is at ***steady state***.

***Analysis***

We begin the analysis at $t < t_0$, i.e., before the switch opened. The circuit for this time domain is shown below: the switch is closed.

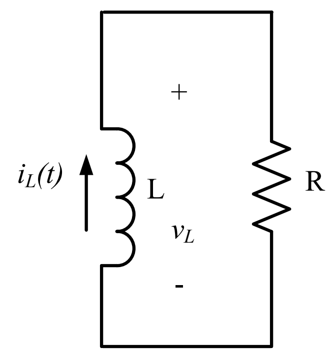



Since we know the circuit was in steady state (because it has been like this *a long time*), we have

$$\frac{di_L}{dt} = 0 \Rightarrow v_L = 0 .$$

From the current-voltage relationship for an inductor, we know that at constant current, ***the inductor acts like a short***. Thus, the resistors R and $R_S$ have no current flowing through them, so the only current flowing is through the inductor, and $i_L = i_S$.

Now we look at the situation for $t > t_0$. The switch has been thrown, and the inductor sees only a resistance. We don't need to worry about the current source or the resistor $R_S$ at this point.




We are no longer in steady state: currents and voltages now begin to change. To find out how, we do a KVL:

$$\boxed{L\frac{di_L(t)}{dt} + i_L(t)R = 0}$$

Note that the inductor current and voltage are in the active sign relationship. You should convince yourself that the signs in this equation are correct.

This is a first-order differential equation for $i_L(t)$.

***Solution***

$$\int_{i(t_0)}^{i(t)} \frac{di_L}{i_L} = -\int_{t_0}^{t} \frac{R}{L} dt$$

$$\ln \frac{i_L(t)}{i_L(t_0)} = -\frac{R}{L}(t - t_0)$$

$$\therefore i_L(t) = i_L(t_0) e^{-R/L(t-t_0)} \quad t \geq t_0$$

We have integrated over time from the switching time $t_0$ to an arbitrary time $t$, and we have integrated over current from its value at $t_0$ to its value at $t$.

***Initial Condition***

For a complete solution, we need the ***initial condition*** $i_L(t_0)$. We found earlier that before the switching event, the current in the inductor was $i_S$. But that was the current flowing in the inductor ***before*** the switch was thrown. We need to know the inductor current ***after*** the switch is thrown, which is want the differential equation refers to.

To do that, we make use of the inductor property that ***we cannot have an instantaneous change in current through an inductor***. That means the inductor current flowing **immediately before** the switch moved, $i_L(t_0^-)$ is equal to the current flowing in the inductor **immediately after** the switch moved, $i_L(t_0^+)$. But $i_L(t_0^-)$ is the current we found above, that is, $i_L(t_0^-) = i_S$, and $i_L(t_0^+)$ is what we have called $i_L(t_0)$ in the equation above. So, $i_L(t_0) = i_L(t_0^+) = i_S$, which is the ***initial condition*** we need in order to solve the equation.

***Valid Time Range***

To be clear about the validity of the solution, we indicate that $t \geq t_0$, meaning that the solution as written is valid **at time $t_0$ and later**. It is not valid for $t < t_0$.

Typically, L and $t_0$ will be known, and we need to find the initial condition and the resistance seen by the inductor, R. As noted above, R will be the Thevenin equivalent resistance of whatever the inductor is connected to. We have a natural response problem if the Thevenin equivalent voltage is 0, that, the Thevenin equivalent is a resistance only.

To simplify notation, we will define the initial value of the current as $i_L(t_0) \equiv I_0$. So in this case we have, as an initial condition, $I_0 = i_S$ (the value of the current source).

### The Time Constant

The form of the response can be simplified by noting that the units of R/L are $s^{-1}$ (inverse seconds):

$$\left[\frac{R}{L}\right] = \frac{\cancel{ohm} \cdot \cancel{Amp}}{\cancel{volt} \cdot \mathrm{s}} = s^{-1}.$$

We can then write

$$i_L(t) = I_0 e^{-(t-t_0)/\tau_L} \qquad t \geq t_0 \;.$$

$$\tau_L = \frac{L}{R}$$

Very often we have $t_0 = 0$, in which case

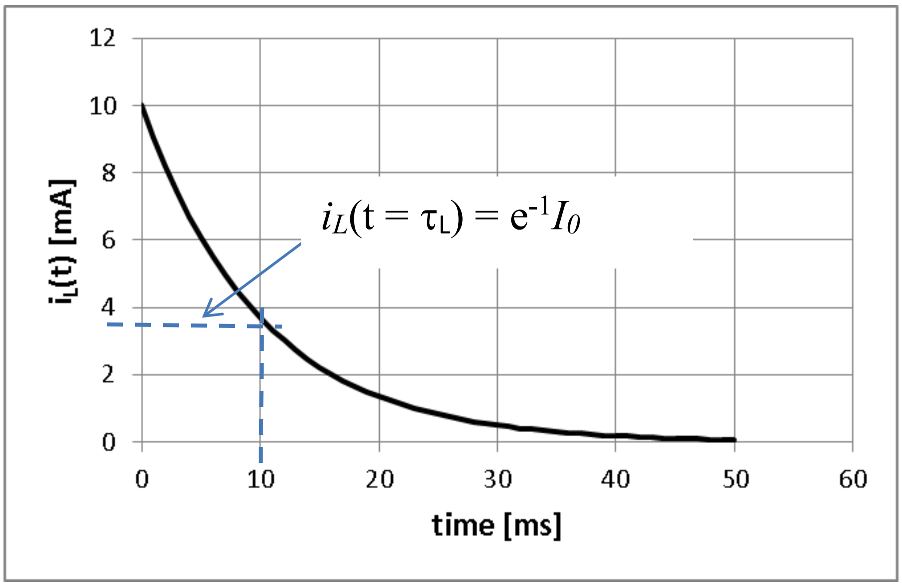
$$i_L(t) = I_0 e^{-t/\tau_L} \quad t \geq 0 \ .$$

We can interpret this equation to mean that the inductor current starts at the initial value $I_0$, and then decays exponentially in time once it is removed from the current source and connected to the resistor.

In the figure below we show a plot of $i_L(t)$ for a long enough time that the exponential has decayed to 0 (or at least to a very small value). The initial current $i_S$ is chosen to be 10 [mA] and the time constant is 10 [mS]. The plot also shows the point at which the time $t$ is equal to the time constant, i.e., $t = \tau_L$. At this point, the current has decayed to a value given by

$$i_L(t = \tau_L) = I_0 e^{-\tau_L/\tau_L} = e^{-1} I_0 \ .$$

The numerical value of $e^{-1}$ is approximately 0.368, or roughly 3/8.




***Long Time***

We note that after five time constants, or $t$ = 50 [ms], the current has decayed almost to 0. What is left is less than 1% of the initial value: $t = 5\tau_L \Rightarrow i_L(t) = 0.0067 i_L(t_0)$. For that reason, we can define a "long time" as five time constants.

## The Inductor is a Short in Steady State

Earlier we defined steady state as the condition in which voltages and currents are no longer changing. We said that in switching problems we will need to know that the switch has been in its initial position for a long time so that we know the circuit is in steady state. Looking at the solution for the inductor current, we see that after a long time, the inductor current becomes 0, and again there is no longer any change in voltage or current. In other words, the circuit has returned to steady state.

We also said above that ***in steady state, the inductor acts like a short circuit*** because although there is current flowing in it, there is no voltage across it. The inductor behaves like a short while the switch is in its initial position, and later returns to a short a long time after the switching event. In between those times, the inductor current and voltage are changing exponentially. These are important ideas that we will return to many times in this course and in future courses.

### Power and Energy Delivered to R

We can find expressions for the power delivered to the resistor and the energy dissipated by it as a function of time. Note that these equations are valid only for $t \geq t_0$ .

$$p_{abs\,by\,R}(t) = i_L^2(t)R = I_0^{\ 2} R\,\mathrm{e}^{-2(t-t_0)/\tau_L}$$

$$w_{abs\,by\,R}(t) = \int_{t_0}^{t} p_{abs,R}\,dt = \frac{1}{2} L\, I_0^{\ 2}\left(1 - e^{-2(t-t_0)/\tau_L}\right)$$

As we expect, after a long time, the total energy dissipated in the resistor is $\frac{1}{2} L\, I_0^{\ 2}$, i.e., it is the total energy delivered by the inductor, which was initially carrying a current $i_L(t_0) = I_0$.

### Inductor Voltage

We can also find the voltage across the inductor as follows (recall we have an active sign relationship between $v_L$ and $i_L$.

$$v_L\left(t\right) = -L\frac{di_L}{dt} = -L\frac{d}{dt}\left(I_0 e^{-(t-t_0)/\tau_L}\right) \quad \mathrm{t} > \mathrm{t}_0 .$$

Then,

$$v_L\left(t\right) = -L\, I_0 \cdot \left(-\frac{1}{\tau_L}\right) e^{-(t-t_0)/\tau_L}$$
$$= R\, I_0 e^{-(t-t_0)/\tau_L} .$$

***Valid Time Range for the inductor voltage***
Although the current through the inductor cannot change instantaneously, the voltage across the inductor **can** change instantaneously, and in this case it does. Note that immediately before the switch was moved, the voltage across the inductor was 0. However, at t = $t_0$ in the expression

above, the voltage across the inductor is $v_L(t_0) = I_0 R$. Therefore, the inductor voltage has changed instantaneously.

Because the voltage changes instantaneously, we cannot say what its value is at t = $t_0$. The inductor voltage is indeterminate at t = $t_0$, meaning it is not possible to say what it is. Therefore, we have indicated the valid time range as being t > $t_0$. This implies that the solution for the voltage across the inductor is not valid at t = $t_0$.

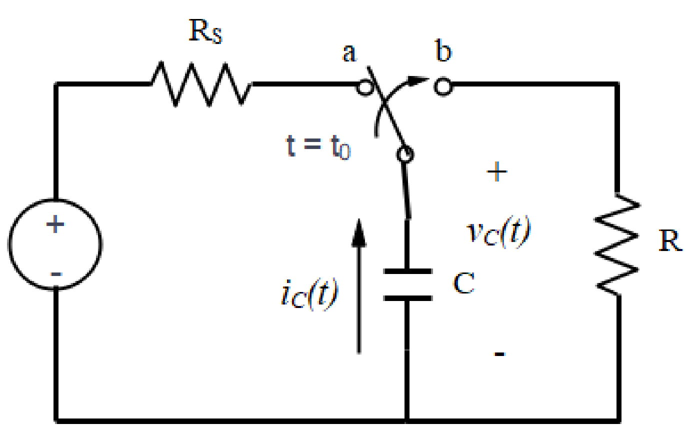
### *Find the Inductor Current First*

We found the inductor current, and from that we found the inductor voltage. There was an important reason for doing it in this order: it is straightforward to find the initial condition for the differential equation that describes the current, which we have done. Although it is possible to find the initial condition for the equation that describes the inductor voltage, this is less straightforward. Therefore, it is a good idea to find the inductor current first, and then if necessary, differentiate to find the inductor voltage.

## The Natural Response of an RC Circuit

We now go through a similar analysis for the RC circuit.

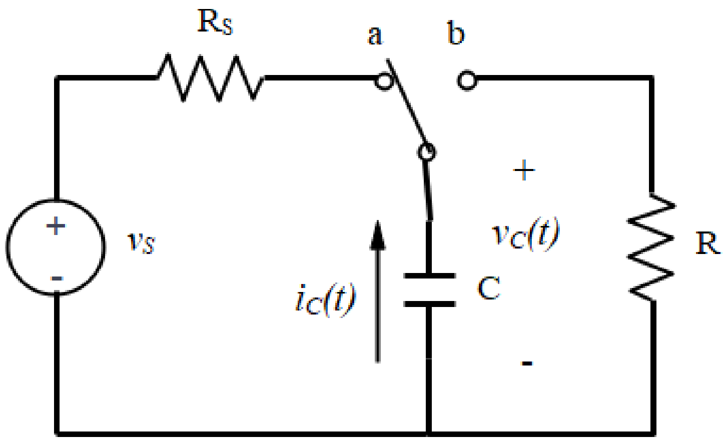
The circuit below shows the natural response configuration for the RC circuit. In this case, a switch is moving from position *a* to position *b*. We specify that the switch had been in position *a* for a long time, and then moved to position *b* at time $t = t_0$. After the switch moved, the capacitor was connected to the resistance R, and we want to know what happens to the capacitor voltage after the switch is moved.




The "natural response" is one in which the capacitor, with a voltage across it, undergoes a switching event that connects it to a resistance only. As for the inductor, that resistance can be a single resistor, or it can be an equivalent resistance that arises from a circuit containing multiple resistors and/or dependent sources.

***Analysis***

We begin the analysis at $t < t_0$, i.e., before the switch opened. The circuit for this time domain is shown next: the switch is in position *a*.



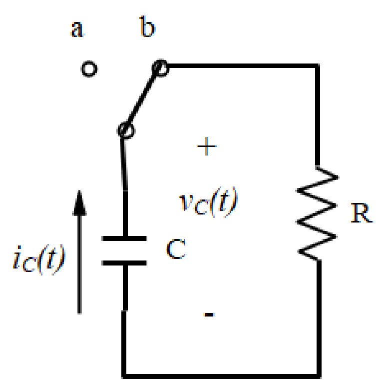

Since we know the circuit was in steady state (because it has been like this *a long time*), we have

$$\frac{dv_C}{dt} = 0 \Rightarrow i_C = 0 .$$

This follows from the current-voltage relationship for the capacitor. Since currents and voltages are no longer changing, the capacitor is in steady state, and ***the capacitor acts like an open circuit***, because there is a voltage across it but no current flowing through it. Thus, the voltage across the capacitor is $v_C = v_S$.

Now we look at the situation for $t > t_0$. The switch has been moved to position *b*, and now the capacitor sees only a resistance. We don't need to worry about the current source or the resistor $R_S$ at this point.

We are no longer in steady state: currents and voltages now begin to change. To find out how, we do a KCL:




$$\boxed{C\frac{dv_C(t)}{dt} + \frac{v_C(t)}{R} = 0}$$

Note that the capacitor current and voltage are in the active sign relationship. You should convince yourself that the signs in this equation are correct. This is a first-order differential equation for $v_C(t)$.

***Solution***

This equation has the same form as that for the inductor current, so we can write down the solution directly. The only difference is that the constant in the exponent changes to the 1/RC.

$$v_C(t) = v_C(t_0)e^{-\frac{(t-t_0)}{RC}} \quad t \geq t_0.$$

***Valid Time Range***
As for the inductor current, the capacitor voltage is valid at the time t = $t_0$, and at times greater than this value.

***Initial Condition***
For a complete solution, we need the initial condition $v_C(t_0)$. We found earlier that before the switching event, the voltage across the capacitor was $v_S$. We then make use of the capacitor property that ***we cannot have an instantaneous change in voltage across a capacitor***.

That that means the capacitor voltage **immediately before** the switch moved, $v_C(t_0^-)$ is equal to the voltage across the capacitor **immediately after** the switch moved, $v_C(t_0^+)$. But $v_C(t_0^-)$ is the voltage we found above, that is, $v_C(t_0^-) = v_S$, and $v_C(t_0^+)$ is what we have called $v_C(t_0)$. So, $v_C(t_0) = v_C(t_0^+) = v_S$. This is the ***initial condition*** we need in order to solve the equation.

Typically, C and $t_0$ will be known, and we need to find the initial condition and the resistance seen by the capacitor, R. The resistance R will be the Thevenin equivalent resistance seen by the capacitor. We have a natural response problem if the Thevenin equivalent voltage is 0, that, the Thevenin equivalent is a resistance only.

To simplify notation, we will define the initial value of the voltage as $v_C(t_0) \equiv V_0$. So in this case we have, as an initial condition, $V_0 = v_S$ (the value of the voltage source

### The Time Constant

The form of the response can be simplified by noting that the units of RC are in fact $s^{-1}$ (inverse seconds):

$$[RC] = Ohm\,Farad = \frac{Ohm\,Volt}{Amps} = s^{-1}.$$

We can then write

$$v_C(t) = V_0 e^{-(t-t_0)/\tau_C} \qquad t \geq t_0 \;.$$

As for the inductor current, we indicate that $t \geq t_0$, meaning that the solution as written is valid **at time $t_0$ and later**. It is not valid for $t < t_0$.

If we simplify by setting $t_0 = 0$, i.e., we have

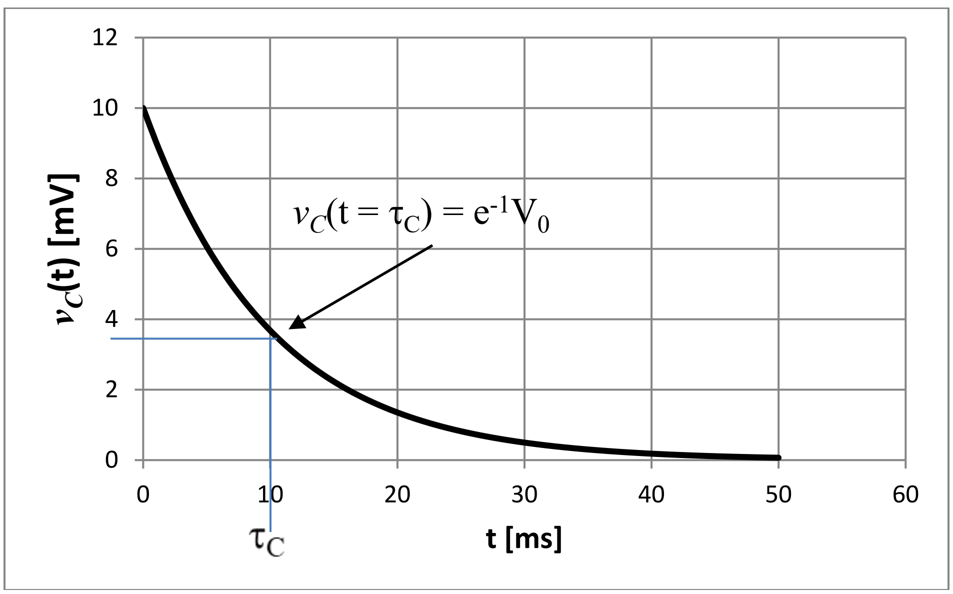
$$v_C(t) = V_0 e^{-t/\tau_L} \quad t \geq 0 \ .$$

We can interpret this equation to mean that the capacitor voltage starts at some initial value $V_0$, and then decays exponentially in time once it is removed from the voltage source and connected to the resistor.

In the figure below we show a plot of $v_C(t)$ for a long enough time that the exponential has decayed to 0 (or at least to a very small value). The initial voltage $v_S$ is chosen to be 10 [mV] and the time constant is 10 [mS]. The plot also shows the point at which the time $t$ is equal to the time constant, i.e., $t = \tau_C$. At this point, the voltage has decayed to a value given by

$$v_C(t = \tau_C) = V_0 e^{-\tau_C/\tau_C} = e^{-1} V_0 .$$

The numerical value of $e^{-1}$ is approximately 0.368, or roughly 3/8.




As for the inductor, after five time constants, or $t$ = 50 [ms], the voltage has decayed almost to 0. What is left is less than 1% of the initial value: $t = 5\tau_C \Rightarrow v_C(t) = 0.0067 v_C(t_0)$. As we pointed out above, we define a "long time" as five time constants.

## *The Capacitor is an Open Circuit in Steady State*

Just as in inductor problems, we need to know that the switch has been in its initial position for a long time so that we know the circuit is in steady state. Looking at the solution for the capacitor voltage, we see that after a long time, the voltage becomes 0, and again there is no longer any change in voltage or current. In other words, the circuit has returned to the steady state.

We also said above that ***in steady state, the capacitor acts like an open circuit*** because although there is a voltage across it, there is no current flowing in it. The capacitor behaves like an open circuit while the switch is in its initial position, and later returns to an open a long time after the switching event. In between these times, the capacitor current and voltage are changing exponentially.

### Power and Energy Delivered to R

We can find expressions for the power delivered to the resistor and the energy dissipated by it as a function of time. Note that these equations are valid only for $t \geq t_0$ .

$$p_{abs\,by\,R}(t) = v_C^2(t)R = V_0{}^2 R\,\mathrm{e}^{-2(t-t_0)/\tau_C}$$

$$w_{abs\,by\,R}(t) = \int_{t_0}^{t} p_{abs,R}\,dt = \frac{1}{2} C\,V_0{}^2 \left(1 - e^{-2(t-t_0)/\tau_C}\right)$$

As we expect, after a long time, the total energy dissipated in the resistor is $\frac{1}{2} C\,V_0{}^2$, i.e., it is the total energy delivered by the capacitor, which was initially had a voltage $v_C(t_0) = V_0$.

We can also find the current through the capacitor as follows.

$$i_C(t) = C\frac{dv_C}{dt} = C\frac{d}{dt}\left(V_0 e^{-(t-t_0)/\tau_C}\right).$$

Then,

$$\begin{aligned} i_C(t) &= C\,V_0 \cdot \left(-\frac{1}{\tau_C}\right) e^{-(t-t_0)/\tau_C} \quad \mathrm{t} > \mathrm{t}_0 \\ &= \frac{V_0}{R} e^{-(t-t_0)/\tau_C} \quad \mathrm{t} > \mathrm{t}_0 . \end{aligned}$$

We note that although the voltage across the capacitor cannot change instantaneously, the current through it **can** change instantaneously, and in this case it does. Immediately before the switch was moved, the capacitor current was 0. However, at $t = t_0$ in the expression above, the current through the capacitor is $i_C(t_0) = \frac{V_0}{R}$. Therefore, the capacitor current has in fact changed instantaneously.

As we discussed in the case of the inductor voltage, the value of the capacitor current at $t = t_0$ is indeterminate, and we indicate this by saying that the valid time range for the solution $t > t_0$.

### *Find the Capacitor Voltage First*

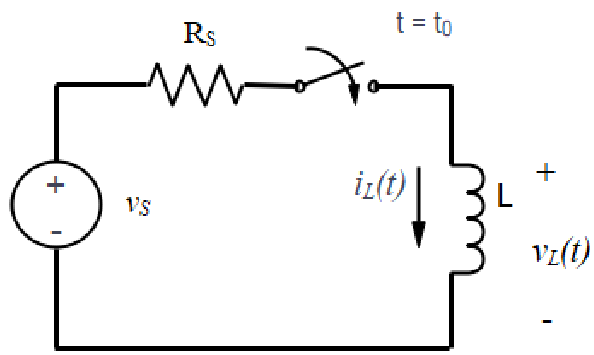
In this analysis we first found the capacitor voltage, and from that we found the capacitor current. We did this because it is straightforward to find the initial condition for the differential equation that describes the voltage. Although it is possible to find the initial condition for the equation that describes the capacitor current, this is less straightforward. Therefore, it is a good idea to find the capacitor voltage first, and then if necessary, differentiate to find the current.

## The Step Response of an RL Circuit

The step response occurs when a switching event results in connecting a capacitor or an inductor to a circuit that contains an ***independent*** voltage source or an ***independent*** current source. Whereas the natural response resulted in exponential decay of an initial current or voltage, the step response will result in an exponential buildup of current or voltage from an initial value.

We follow a similar process here as for the natural response. That is, we examine the circuit before the switching event, establish an initial condition, and then examine the circuit after the switching event.

The circuit below shows an inductor in a switching circuit. Before the switch is thrown, the inductor is not connected to anything, and therefore carries no current. When the switch closes, the inductor is connected to a circuit containing an independent voltage source.




***Analysis***

Analysis of this circuit will again result in a first order differential equation, and therefore we will need an initial condition. To do this, we will need to know that the switch has been open for a long time, so that any changes in inductor current will have decayed to zero.

We will again use the property of an inductor that the current in it cannot change instantaneously. Since the current before the switching event is 0, the current immediately after the switching event will also be zero: $i_L(t_0^-) = i_L(t_0^+) = 0$.

A KVL establishes the differential equation.

$$\boxed{i_L R_S + L\frac{di_L}{dt} = v_S}$$

$$\frac{di_L}{dt} = \frac{v_S}{L} - i_L\frac{R_S}{L}$$

$$\int_{i_L(t_0)}^{i_L(t)} \frac{di_L}{i_L - \frac{v_S}{R_S}} = -\int_{t_0}^{t} \frac{R_S}{L}dt$$

We want to know the inductor current at an arbitrary time t, so we integrate time from the switching time $t_0$ to time t, and we integrate current from its value at $t_0$ to its value at time t.

$$\ln\left(\frac{i_L(t) - \frac{v_S}{R_S}}{i_L(t_0) - \frac{v_S}{R_S}}\right) = -\frac{R_S}{L}(t - t_0)$$

Thus, the expression for the current as a function of time is as follows.

$$i_L(t) = \frac{v_S}{R_S} + \left(i_L(t_0) - \frac{v_S}{R_S}\right)e^{-(t-t_0)/\tau_L} \quad t \geq t_0$$

We have defined the time constant $\tau_L \equiv \frac{L}{R_S}$.

Because the inductor current cannot change instantaneously, this solution is valid at time $t = t_0$ and at values of time greater than this. We have indicated this in the solution.

***Final Value***

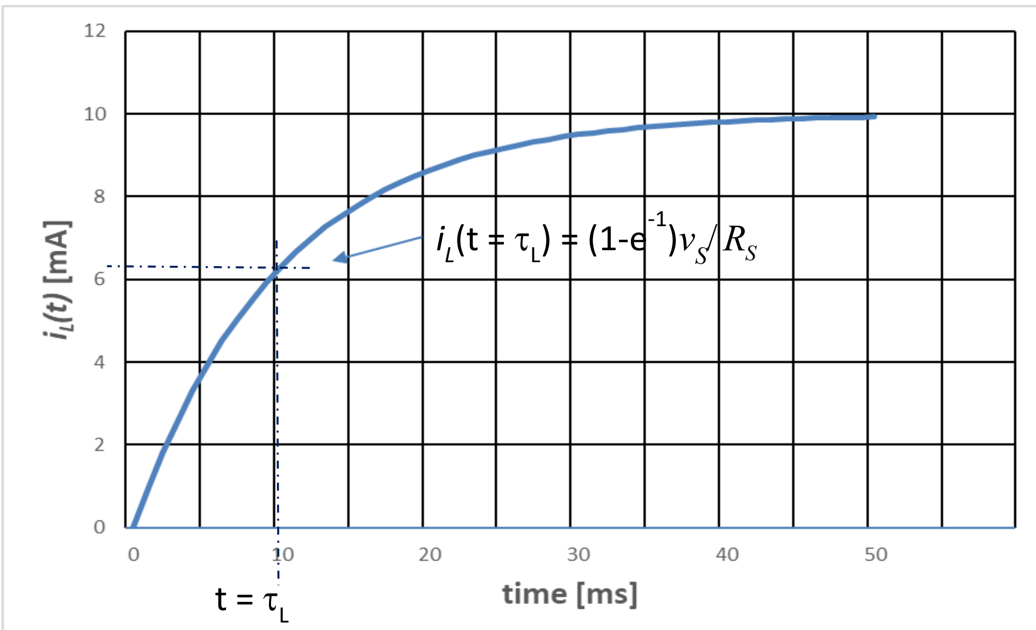
In the figure below we show a plot of $i_L(t)$

$$i_L(t) \text{ for } t \to \infty .$$

The initial current $i_L(t_0) \equiv I_0$ in the plot below is chosen to be 0, in which case the final value is $v_S$/R, which was chosen to be 10 [mA]. The time constant is 10 [mS]. The plot also shows the point at which the time $t$ is equal to the time constant, i.e., $t = \tau_L$. At this point, the current has decayed to a value given by

$$i_L(t = \tau_L) = \frac{v_S}{R_S}(1 - e^{-\tau_L/\tau_L}) = \frac{v_S}{R_S}(1 - e^{-1}) .$$

The numerical value of $(1-e^{-1})$ is approximately (1 - 0.368), or roughly 5/8.




We can now use the current voltage relationship for the inductor to find the voltage across the inductor.

$$v_L(t) = L\frac{di_L(t)}{dt}$$

$$v_L(t) = (v_S - I_0 R_S)e^{-(t-t_0)/\tau_L} \quad t > t_0 .$$

We have indicated that the valid time range for this equation is $t > t_0$, because the voltage across the inductor is indeterminant at $t = t_0$.

As for the natural response, we first found the inductor current, and then differentiated it to find the inductor voltage.

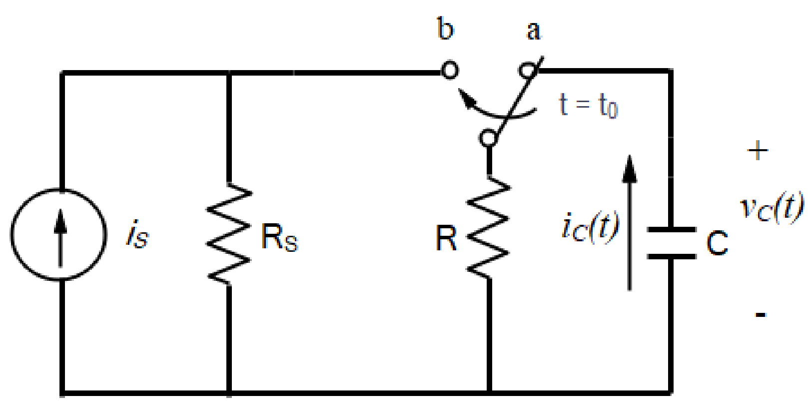
We could find relationships for power and energy, but we will not do that here.

## The Step Response of an RC Circuit

The RC circuit step response analysis is similar to the step response for the RL circuit. We will again make use of the fact that the voltage across a capacitor cannot change instantaneously.

The circuit below shows a capacitor connected to a current source in parallel with a resistor when the switch is thrown. The current source and resistor should be thought of as a Norton equivalent, and therefore can represent a more complex circuit connected to the capacitor.

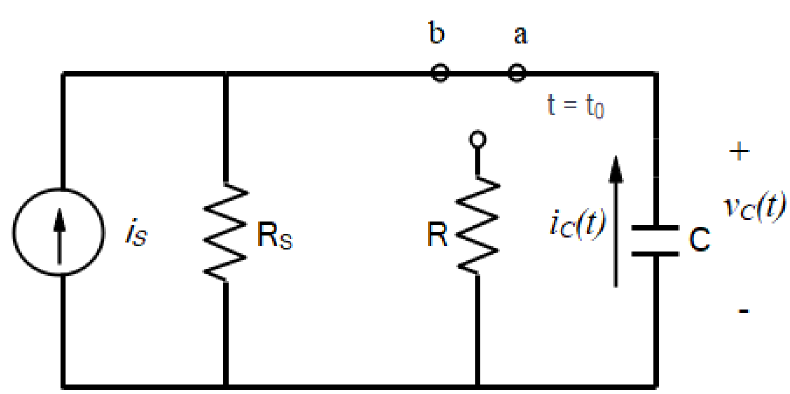
Before the switch is thrown, the capacitor is connected to a resistor. We specify that it has been in that position for a long time, so the capacitor has no voltage across it. When the switch closes at $t = t_0$, the capacitor is connected to the Norton equivalent, and voltages and currents begin to change.




***Analysis***

We again use the property of a capacitor that the voltage across it cannot change instantaneously, so we have $v_C(t_0^-) = v_C(t_0^+) = v_C(t_0)$. Since the voltage before the switching event in this case is 0, the voltage immediately after the switching event will also be zero.

The circuit after the switching event is shown below.



A KCL establishes the differential equation.

$$\boxed{\frac{v_C}{R_S} + C\frac{dv_C}{dt} = i_S}$$

$$\frac{dv_C}{dt} + \frac{v_C}{R_S C} = \frac{i_S}{R_S C}$$

This equation has the same form as for the inductor current step response, so we can write the solution directly.

$$v_C(t) = i_S R_S + \left(V_0 - i_S R_S\right)e^{-(t-t_0)/\tau_C} \quad t \geq t_0$$

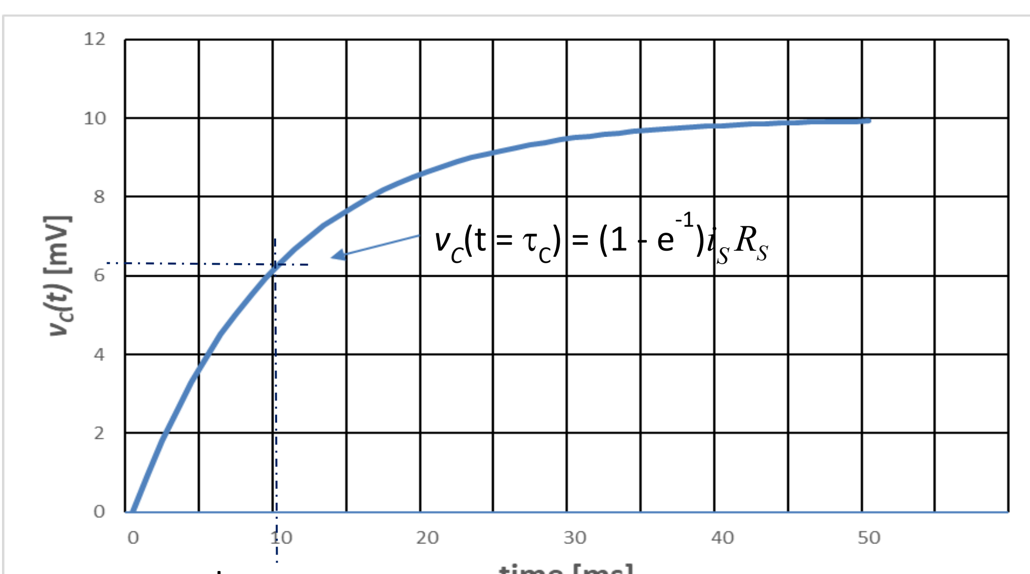
Note that the time constant $\tau_C \equiv R_S C$ is the same as it was for the natural response. Also, we have defined $v_C(t_0) = V_0$, which we found above is 0 for this circuit.

Because the capacitor voltage cannot change instantaneously, this solution is valid at time $t = t_0$ and at values of time greater than this.

In the figure below we show a plot of $v_C\left(t\right)$ for a long enough time that the exponential has very nearly reached its final value. The initial voltage $v_C\left(t_0\right) \equiv V_0$ in the plot below is chosen to be 0, in which case the final value is $i_S$R, which was chosen to be 10 [mV]. The time of the switching event $t_0$ is also chosen to be 0, and the time constant is 10 [mS]. The plot also shows the point at which the time $t$ is equal to the time constant, i.e., $t = \tau_C$. At this point, the current has decayed to a value given by

$$v_C\left(t = \tau_C\right) = i_S R_S(1 - e^{-\tau_C/\tau_C}) = i_S R_S(1 - e^{-1}).$$

The numerical value of $(1-e^{-1})$ is approximately (1 - 0.368), or roughly 5/8.



The capacitor current can be found from the current voltage relationship. As for the natural response, the current in the capacitor can change instantaneously, and therefore its value at $t = t_0$ is indeterminant. This is indicated as part of the solution.

$$i_C(t) = C\frac{dv_C(t)}{dt}$$

$$i_C(t) = \left(i_S - \frac{V_0}{R_S}\right)e^{-(t-t_0)/\tau_C} \quad \mathrm{t} > \mathrm{t}_0.$$

As for the natural response, we first found the capacitor voltage, and then we differentiated it to find the capacitor current.

## General Solution for the Single Time Constant Response

We have derived solutions for capacitor voltage and inductor current for the natural and step response cases. Here we note that the differential equations describing these cases all have the same mathematical form, and we can find a general solution that is valid for all of them.

The differential equations we are interested in are enclosed in a box in the analyses above. If we denote $x(t)$ as either the ***inductor current*** or the ***capacitor voltage***, we see that the boxed equations all have the following form.

$$\frac{dx(t)}{dt} + \frac{x(t)}{\tau} = \kappa.$$

If $x(t)$ represents the inductor current, then $\tau \equiv L/R$. If $x(t)$ represents the capacitor voltage, then $\tau \equiv RC$, where R is the Thevenin Equivalent resistance seen by the capacitor. The parameter $\kappa$ is (i) 0, for the natural response, (ii) the Norton current in a capacitor problem, or (iii) the Thevenin voltage in an inductor problem.

We noted above that after a long time, any of the circuits we have studied will reach steady state, the condition in which currents and voltages are no longer changing, and therefore time derivatives are zero. If at that time the value of $x$ is defined as the ***final value*** $\boldsymbol{x_f}$, then we see from the equation above that

$$\frac{dx}{dt} \to 0 \Rightarrow x \to x_f = \kappa\tau.$$

Then we can write the differential equation as

$$\frac{dx}{dt} = -\frac{x}{\tau} + \kappa .$$

Solving:

$$\frac{dx}{\left(x - x_f\right)} = -\frac{dt}{\tau}$$

$$x(t) = x_f + \left[ x(t_0) - x_f \right] e^{-(t-t_0)/\tau} .$$

This equation holds for the inductor current or the capacitor voltage in either natural response or step response problems. It requires that we find the initial value $x\left(t_0\right)$ of the variable we're interested in, the final value $x_f$, and the time constant $\tau$.

### Summary of Natural and Step Response Problems

We can summarize the steps needed to solve a natural or step response problem using the general solution just derived as follows.

- For problems involving an inductor, we identify $x\left(t\right)$ as the inductor current. For problems involving a capacitor, we identify $x\left(t\right)$ as the capacitor voltage.

- Examine the circuit before the switching event.
    - It is a good idea to redraw the circuit with the switch in its initial position.

- Find the initial condition $x\left(t_0\right)$ in the circuit ***before the switch was thrown***. It will always be assumed that the switch was in its initial position for a long time.
    - If the circuit contains an inductor, find the current in the inductor before the switching event. If the circuit contains a capacitor, find the voltage across the capacitor before the switching event.

- Examine the circuit after the switching event.
    - It is a good idea to redraw the circuit with the switch in its new position.

- Find the Thevenin equivalent resistance seen by the inductor or capacitor in the circuit created ***after the switch was thrown***.
    - To do this, remove the inductor or capacitor, and find the Thevenin equivalent resistance at the terminals to which it had been connected.

- Find the appropriate time constant: $\tau_L \equiv L/R$ ; $\tau_C \equiv RC$ .

- Using the general solution derived above, write the equation for the inductor current or for the capacitor voltage using the appropriate initial condition and time constant.
    - Be sure to indicate the time region over which the solution is valid.

- If needed, we can find other voltages or currents in the circuit.
    - We can find the voltage across an inductor by differentiating the inductor current, and we can find the current in a capacitor by differentiating the voltage.
    - We can also find any voltage or current in the circuit using Kirchhoff's laws, or whatever tools we find appropriate.

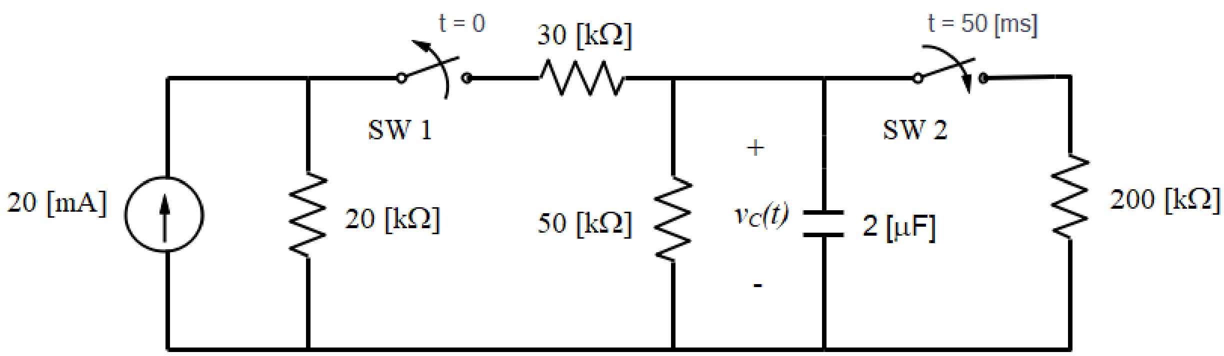
## Sequential Switching

We can have a situation in which there is a switching event at $t = t_0$, and then sometime later there is a second switching event at $t = t_1$. We could have many switching events at $t_2$, $t_3$, …

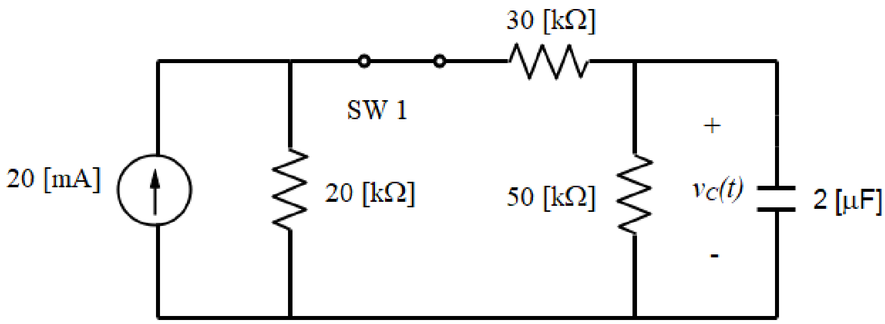
If after the first switching event, the second switching event happens after the circuit has reached steady state, then the sequential switching problem is no different from what we have been studying here. We can simply solve the problem that arises from the first switching event, find a new initial condition, and solve the problem for the second switching event. But if the second switching event occurs before the circuit reaches steady state, we will have to be careful about choosing initial conditions for the second switching event.

We will look at an example here to illustrate the idea of sequential switching in the case that the second switching event happens before the circuit reaches steady state after the first switching event. We will work the details in class, but give only the solutions here in the notes.

***This is Problem 7.6 in Worked Problems***
The circuit below shows an example of a sequential switching event. Switch 1 (SW 1) is assumed to have been closed for a long time, and then it opens at t = 0. Switch 2 (SW 2) is assumed to have been open for a long time, and then at t = 50 [ms], it closes.

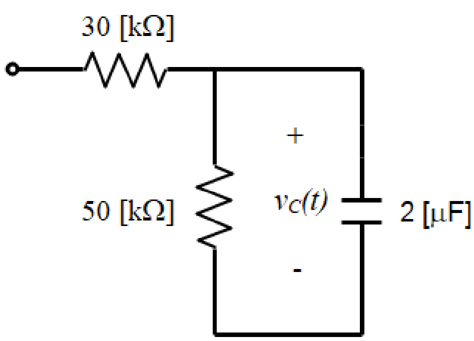



Let's think about what happens here. The circuit below shows what is happening before t = 0. Switch 1 is closed, and switch 2 is open. The current source and resistors create a voltage $v_C(t_0)$ across the capacitor.



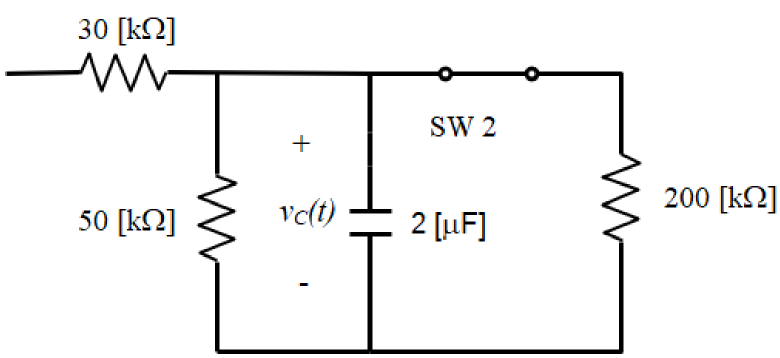

t < 0

After switch 1 opens, but before switch 2 closes, the capacitor is connected only to the 50 [kΩ] resistor. This is a natural response problem.




0 < t < 50 [ms]

Finally, after switch 2 closes, the circuit looks as follows. This is another natural response, but now the Thevenin equivalent resistance seen by the capacitor has changed because of the introduction of the 200 [kΩ] resistor, so the time constant will be different.




t > 50 [ms]

In the last two circuits we could have left the 30 [kΩ] resistor out, but we left it in to make it clear what the circuit configuration is.

Here is our plan.

1. Find the initial condition for the first switching event from the circuit at t < 0.
2. Find the natural response for the circuit in the time range 0 < t < 50 [ms].
3. Find the value of the capacitor voltage at t = 50 [ms]. This will serve as the initial condition for the second switching event.
4. Find the capacitor voltage for t > 50 [ms].

We start with the circuit shown for t < 0. Since the switches have been in their initial positions for a long time, we know that the capacitor is at steady state, and therefore acts as an open circuit. We can use the current divider rule to find that the voltage across the capacitor is 200 [V]. We have now established the initial condition for the first switching event, which is that $v_C(t_0) = 200\ [\mathrm{V}]$. We can be explicit about this:

$$\boxed{v_C(t) = 200\ [\mathrm{V}]\ \ \mathrm{t} \le 0}.$$

We now look at the circuit for 0 < t < 50 [ms]. This is the circuit configuration after the first switch has opened, but before the second switch has closed. This is a natural response problem, and the Thevenin equivalent resistance connected to the capacitor is 50 [kΩ]. Therefore, the time constant is $\tau_C$ = 50 [kΩ] * 2 [μF] = 0.1 [s]. To use our general formula, we assign $\tau_C$ = 0.1 [s]. But we also need to know the final value of the capacitor voltage.

The final value of the capacitor voltage is determined by noting that after a long time, the capacitor will be in steady state, and thus will behave like an open circuit. Therefore, the current through it will be 0, and the voltage across it will be also be zero. In fact, the final value for capacitor voltage or inductor current in any natural response problem will be zero.

Now, we know that there will be a another switching event at t = 50 [ms]. This is less than one time constant, and therefore the capacitor voltage will not reach zero before the second switching event. But the capacitor does not "know" this, and therefore behaves as if there will not be another switching event. Therefore, the solution to the natural response problem is constructed with a final capacitor voltage of 0. Applying our general formula with these parameters, we get (with $t_0$ = 0),

$$v_C(t) = v_{C,f} + \left[v_C(t_0) - v_{C,f}\right]e^{-(t-t_0)/\tau_C}$$

$$v_C(t) = 0 + \left[200\ [\mathrm{V}] - 0\right]e^{-(t-0)/0.01\ [\mathrm{s}]}.$$

Simplifying, and including the time range over which the solution is valid, we get

$$\boxed{v_C(t) = 200e^{-t/0.01\text{ [s]}} \quad 0 \ \le \ \mathrm{t} \ \le \ 50 \text{ [ms]}}.$$

This solution is valid at t = 0, at t = 50 [ms], and at times in between these values.

When the second switching event occurs, the value of the capacitor voltage is

$$v_C(t = 50\ [ms]) = 200e^{-0.05/0.01\text{ [s]}} = 121.31 \text{ [V]}.$$

This is the initial condition for the second switching event.

We now look at the circuit valid for t > 50 [ms]. This is also a natural response problem, but now the Thevenin equivalent resistance is 200 [kΩ] in parallel with 50 [kΩ]. This gives a new time constant $\tau_C$ = 80 [ms]. Referring again to our general formula with the new time constant, the initial condition calculated above, and a final value of 0, we have

$$v_C(t) = v_{C,f} + \left[v_C(t_0) - v_{C,f}\right]e^{-(t-t_1)/\tau_C}$$

Note that the exponent is now $(t - t_1)/\tau_C$, and that the second switching event at occurs at $t_1$ = 50 [ms]. This information must be included in the solution.

$$v_C(t) = 0 + \left[121.31 \text{ [V]} - 0\right]e^{-(t-50\text{ [ms]})/80\text{ [ms]}}$$

$$\boxed{v_C(t) = 121.31e^{-(t-50\text{ [ms]})/80\text{[ms]}} \text{ [V]} \ \ \mathrm{t} \ge 50 \text{ [ms]}}$$

The equations given in the boxes above constitute the solution to the sequential switching problem. When writing these solutions, we need to be sure to include the range of time for which each solution is valid.

As we have been pointing out, the voltage across the capacitor cannot change instantaneously, and in fact these solutions reflect that. We can plug t = 0 into the equation that holds for $0 \ \le \ \mathrm{t} \ \le \ 50 \text{ [ms]}$, and see from that equation that the voltage is 0, as it should be at t = 0. Similarly, we can plug t = 50 [ms] into the equation $\mathrm{t} \ge 50 \text{ [ms]}$,and find that the capacitor voltage is 121.31 volts, as it should be at that time. Thus, the three equations “connect” at the boundaries of the time domains, so the capacitor voltage is always continuous.

## Unbounded Response

We noted in the chapter on Thevenin equivalents that it is possible for a Thevenin equivalent resistance to be negative. If the Thevenin resistance seen by a capacitor or an inductor in a single time constant circuit is negative, the voltage across the capacitor or the current in the inductor will increase exponentially:

$$x(t) = x_f + \left[x(t_0) - x_f\right]e^{+t/|\tau|}$$

This is a valid result, but we have to wonder how to determine the final value. After a long time, $x(t)$ will go to infinity. It turns out that the solution is simple: the final value of $x(t)$ is determined just as it is in the case where the time constant is positive. One way to think about this is that the final value occurs at a time $t \to -\infty$. Thus, the only difference in problems where the time constant is negative is that the solution is growing exponentially. The final value is found in the same as it is for positive time constants. In real circuits, exponential growth will have to stop at some point, since voltages and currents cannot be infinite. In real components, either another switching event must occur to stop the exponential increase, or else some components in the circuit will burn up!