Sinusoidal Steady-State Analysis

In this chapter we consider circuits that have sinusoidal voltage and/or current sources, and that are in ***steady state***.

Sinusoidal refers to circuits in which sources are of the form

$$v_S(t) = V_m \cos(\omega t + \varphi_v) \quad i_S(t) = I_m \cos(\omega t + \varphi_i).$$

Steady State was discussed in the chapter on single time constant circuits. There, we said that steady state means there are no changes in voltages or currents. In this chapter, currents and voltages will be changing with time, because they are sinusoidal, but there will be no change in amplitude or phase with time. In this chapter, that is what steady state means.

Our development will use the cosine function, but we could reproduce all of it with the sine function. These are of course related:

$$\sin(\omega t + \varphi) = \cos(\omega t + \varphi + 90^o).$$

The Sinusoidal Source

We plot below an example of a sinusoid voltage. The equation describing this function is $v(t) = V_m \cos(\omega t + \varphi)$.

Definitions

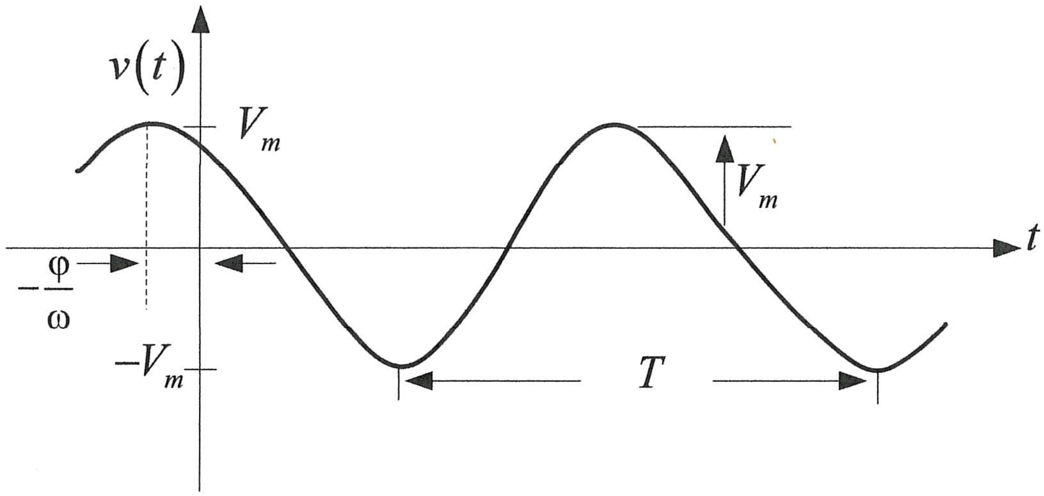
The ***amplitude*** of the sinusoid is V_m. It is the value of the sinusoid measured from zero to peak. We may also refer to the "peak-to-peak" amplitude, which is $V_{p-p} = 2V_m$.

The ***frequency*** f is measured in *Hertz*; the ***angular frequency*** $\omega = 2\pi f$ is measured in *radians per second*. The fundamental unit of both these quantities is sec^{-1}.

The ***period*** of the sinusoid T is related to the frequency as $T = 1/f$.

The ***phase*** of the sinusoid is φ, and it is measured in degrees. Note that ωt is in radians, and φ is in degrees. This use of mixed units is standard. In the figure below, the phase shifts the maximum of the function to the left by an amount φ / ω. We can see that by noting that at the peak value, $\cos(\omega t + \varphi) = 1 \Rightarrow \omega t + \varphi = 0$, so $t = -\varphi / \omega$.

The conversion from radians to degrees is 2π radians $= 360$ degrees.



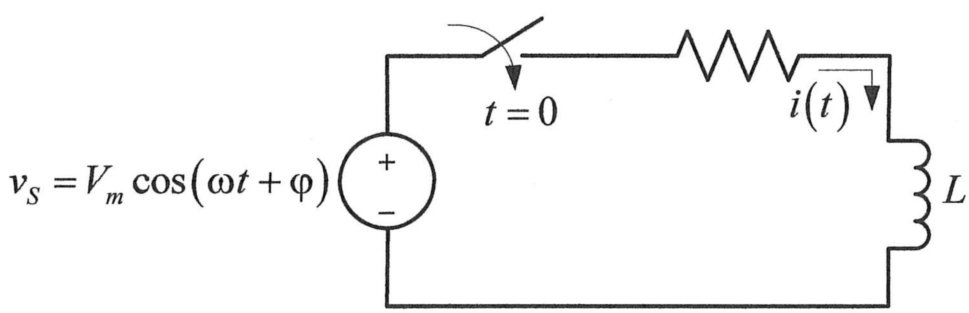
Periodic Functions and Fourier Analysis

The sinusoid is a periodic function. You will learn in later courses that any periodic function can be expressed mathematically as a sum of sinusoids, if we choose the proper frequencies and amplitudes for those sinusoids. Finding the sinusoids that comprise a periodic function is the subject of ***Fourier analysis***. We will not do that in this course.

For science and engineering, this is an extremely powerful concept. For linear systems, it means that the response of a system to a complex periodic signal can be found by adding the responses from each of the single sinusoids that make up the more complex function. Therefore, knowing the response of a system to a sinusoid of arbitrary frequency is extremely important.

The Sinusoidal Response

Consider a sinusoidal source connected to the inductor and resistor at t = 0. We considered problems of this type in the previous chapter, but there the sources were all DC sources.



Let's do the analysis. KVL gives

$$L\frac{di}{dt}+Ri=V_m\cos(\omega t+\varphi)$$

We will not solve this differential equation here, but simply give the solution:

$$i(t)=\frac{-V_m}{\sqrt{R^2+\omega^2L^2}}\cos(\varphi-\vartheta)e^{-\frac{Rt}{L}}+\frac{V_m}{\sqrt{R^2+\omega^2L^2}}\cos(\omega t+\varphi-\vartheta).$$

In this equation, $\vartheta=\tan^{-1}\left(\frac{\omega L}{R}\right)$.

Note there are two terms in this equation for the current. The first one is decaying exponentially, and is similar to the kind of solutions we found for single time constant circuits in the previous chapter. This is a ***transient response***, meaning that after a long time, it decays to zero.

The second term does not decay to zero. It is the ***steady state solution*** for the current. This is the part of the solution that we will learn how to obtain in this chapter.

If we compare the steady state solution for the current to the sinusoid describing the voltage source, we notice that the time dependence is the same: $\cos(\omega t)$. However, the amplitude is different (and of course has different units since one is a voltage and the other a current), and the phase of the sinusoid has changed. We will find that voltages and currents in linear circuits with sinusoidal sources have the same time dependence, but different amplitudes and phases. This is a characteristic of linear circuits. We will only consider linear circuits in this course.

The Phasor Transform

As we saw in the chapter on single time constant circuits, and in the simple example above, circuits that contain inductors and capacitors are described by differential equations. Although we know how to solve differential equations, it is much more convenient to use ***transform methods***; these are mathematical tools used to convert differential equations to algebraic equations. Then, instead of having to solve differential equations to find circuit variables, we can solve algebraic equations.

There are many such transforms defined in mathematics, and you will study some of them in your courses in electrical and computer engineering. For this class, we will define a simple transform method called the ***phasor transform***.

As mentioned above, in linear circuits with sinusoidal sources, the time dependence of the circuit variables will be the same as the time dependence of the sources - either $\cos(\omega t)$ or $\sin(\omega t)$ - , but amplitudes and phases may change. We have a convenient mathematical structure that allows us to keep track of an amplitude and a phase in a single quantity: the ***complex number***. This idea is at the heart of the phasor transform method.

Before we introduce the phasor transform, let's have a brief review of complex numbers.

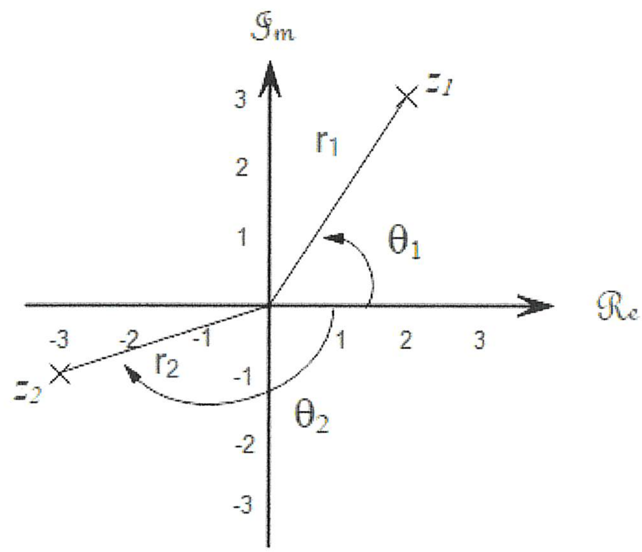
Complex Numbers

A complex number is a number that has a "real" and an "imaginary" part. A generic complex number often is denoted by z so we have

$$z = a + jb$$

where the real part of z is a, and the imaginary part is b. We also have $j = \sqrt{-1}$. Note that $j^2 = -1$.

We can visualize complex numbers by plotting them on a graph with the horizontal axis representing the real part of the number and the vertical axis representing the imaginary part. The numbers $z_1 = 2 + 3j$ and $z_2 = -3 - j1$ are shown on the ***complex plane*** below.



When identifying real and imaginary parts, we will use the following notation.

$$\text{Re}\{z\} = a$$
$$\text{Im}\{z\} = b$$

Also shown in the diagram is the idea that instead of thinking of a complex number as having a real and an imaginary part displayed in rectangular coordinates, we can also think of it in polar coordinates, where the magnitude is r and the phase angle is ϑ.

We need to be able to convert from rectangular to polar coordinates and back. Most calculators can do this, so you will want to become familiar with your calculator and how it handles these numbers. The transformations are based on the Pythagorean Theorem.

$$a = r\cos(\vartheta)$$
$$b = r\sin(\vartheta)$$
$$r = \sqrt{a^2 + b^2}$$
$$\vartheta = \tan^{-1}(\frac{b}{a})$$

Looking at our complex plane above, we have that $\vartheta_1 = \tan^{-1}(3/2) = 56.31\deg$, and $\vartheta_2 = \tan^{-1}(-1/-3) = -161.56\deg$. This second angle is equivalent to (360 – 161.56)° = 198.44°. Note that angle is measured from the positive real axis, and is positive in the counterclockwise direction.

Getting the angle right when real and/or imaginary parts are negative is a bit of a trick, because (-1/-3) and (1/3) are of course the same number, but if these are the imaginary and real parts of a complex number, they have angles that are 180° apart. A good calculator will do this correctly, if you know how to use it!

Because we are working with sines and cosines, we will find it convenient to use Euler's relation:

$$e^{\pm j\vartheta} = \cos\vartheta \pm j\sin\vartheta.$$

Based on our notation we have the following.

$$\cos\vartheta = \mathrm{Re}\{e^{j\vartheta}\}$$
$$\sin\vartheta = I_m\{e^{j\vartheta}\}$$

From the diagram above we see that

$$z = a + jb = r\cos(\vartheta) + j\sin(\vartheta),$$

which we can write more compactly as $z = re^{j\vartheta}$. We often use a shorthand notation for this expression:

$$z = re^{j\vartheta} r\angle\vartheta.$$

$z = re^{j\vartheta} \equiv r\angle\vartheta$

Addition, Subtraction, Multiplication, Division

When adding and subtracting complex numbers, it's best to use the rectangular form:

$$z_1 \pm z_2 = (a_1 + jb_1) \pm (a_2 + jb_2)$$
$$= (a_1 + a_2) \pm j(b_1 + b_2)$$

The real and imaginary parts add separately.

When multiplying and dividing, it's best to use the polar form:

$$z_1 \bullet z_2 = r_1\angle\vartheta_1 \bullet r_2\angle\vartheta_2$$
$$= r_1 r_2 \angle(\vartheta_1 + \vartheta_2).$$

Note that the magnitudes multiply, but the angles add because $e^{j\vartheta_1} \bullet e^{j\vartheta_2} = e^{j(\vartheta_1 + \vartheta_2)}$. For division we have

$$\frac{z_1}{z_2} = \frac{r_1\angle\vartheta_1}{r_2\angle\vartheta_2}$$
$$= \frac{r_1}{r_2}\angle(\vartheta_1 - \vartheta_2).$$

Complex Conjugate

We define the ***complex conjugate*** of a complex number as the number that results from changing the sign of the imaginary part. So we have, with z* representing the complex conjugate,

$$z = a + jb \rightarrow z^* = a - jb.$$

In polar form this becomes

$$z = r\angle\vartheta \rightarrow z^* = r\angle-\vartheta.$$

It is easy to show that $zz^* = r^2 = a^2 + b^2$. This number will always be purely real.

The Phasor

We are now ready to use complex numbers to represent the basic circuit elements. Having done that, we will be able to solve circuits with sinusoidal sources using algebraic equations instead of differential equations.

We begin with sources. Suppose we have a sinusoidal voltage source:

$$v_s(t) = V_m \cos(\omega t + \varphi)$$

Using Euler's relation, and identifying $\vartheta = \omega t + \varphi$, we have

$$v_s(t) = \text{Re}\left\{V_m e^{j(\omega t+\varphi)}\right\}$$
$$v_s(t) = \text{Re}\left\{V_m e^{j\varphi} e^{j\omega t}\right\}.$$

Note that V_m is real, so it can move into the brackets. We now define the ***phasor*** as

$$\overline{V} = V_m e^{j\varphi}$$

Using the shorter notation defined above, we can write $\overline{V} = V_m \angle \varphi$.

Also, using Euler's relation,

$$\overline{V} = V_m \cos\varphi + jV_m \sin\varphi$$

Note that we have captured the amplitude and phase of the voltage source, but ignored the time dependence. The terminology we will use is "the phasor transform $\mathcal{P}$ of $v_s(t)$ is $\overline{V}$":

$$\mathcal{P}\left\{V_m \cos(\omega t) + \varphi\right\} = \overline{V}.$$

We have transformed the source from the time domain $v_s(t) = V_m \cos(\omega t + \varphi)$ to the phasor domain $\overline{V} = V_m e^{j\varphi}$. This is an important concept then you will see in many applications in engineering.

Notation is Very Important!

The phasor we have defined above transforms a sinusoidal function from the time domain to the phasor domain. We can write this as follows.

$$v_S(t) = V_m \cos(\omega t + \phi) \rightarrow \overline{V} = V_m \angle \phi.$$

In this course, we are going to be very, very fussy about this notation. Here are the rules

The ***time domain*** quantity refers to a function that has the symbol t in it, where t indicates a function of time. A ***phasor domain*** quantity does **not** have time t in it, and it is **not** a function of time; it is a quantity that has been transformed from the time domain. It is critically important that we do not mix time and phasor domain quantities. In class we will give examples of what we will call mixed domains.

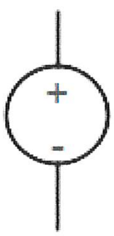
We indicate phasor quantities with a capital letter and a subscript if needed. We also use an "overbar" as shown above. Some textbooks use a bold letter for the phasor: **V**, or an "underbar": $\underline{V}$. In these notes we will use the overbar.

Note that in the expression above we did not put an "equal sign" between the sinusoid and the phasor; we used an arrow indicating that a transform is being made. Using an "equal sign" in place of the arrow would constitute mixing domains. These things are NOT equal! They represent the same physical thing – a voltage source – but they are not "equal".

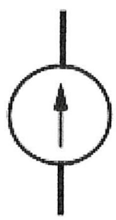
The Circuit Elements in the Phasor Domain

Sources in the Phasor Domain

The transformation we discussed above for the sinusoidal voltage source shows how we can represent a voltage source in the phasor domain. A similar development holds for a current source. We summarize this idea in the following figure.



$\overline{V_S} = V_m \angle \phi_v$



$\overline{I_S} = I_m \angle \phi_i$

We use the same notation for dependent sources. In this case, the dependent source will depend on a phasor voltage or phasor current.

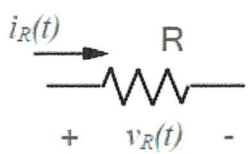
We also need to transform the passive circuit elements into the phasor domain

Passive Circuit Elements in the Phasor Domain

Resistor

If a resistor is connected into a circuit with sinusoidal sources, then the resistor voltage and current will be sinusoidal, and they will be related by Ohm's Law.

Time Domain



$$i_R(t) = I_m \cos(\omega t - \vartheta)$$

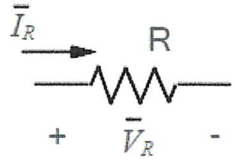
$$v_R(t) = R\, I_m \cos(\omega t - \vartheta)$$

Phasor Domain

If we phasor transform the current $i_R(t)$ and voltage $v_R(t)$, we get

$$i_R(t) \rightarrow \overline{I_R} = I_m \angle \vartheta \;\text{ and }\; v_R(t) \rightarrow \overline{V_R} = R\, I_m \angle \vartheta .$$

But $R\, I_m \angle \vartheta = R\overline{I}_R$, which means that



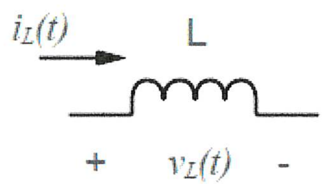
$$\overline{V}_R = \overline{I}_R R$$

So the phasor transform of the resistor is just Ohm's Law, but with phasors substituted for time domain quantities.

Inductor

Now consider an inductor in a circuit with sinusoidal sources. We have

Time Domain



$$i_L(t) = I_m \cos(\omega t + \vartheta)$$

$$v_L(t) = L\frac{di_L(t)}{dt} = -\omega L I_m \sin(\omega t + \vartheta)$$

We want to compare this with $i_L(t)$, so we convert back to cosine:

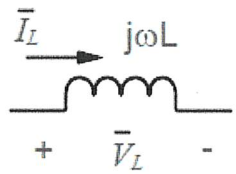
$$v_L(t) = -\omega L I_m \cos(\omega t + \vartheta - 90^\circ)$$

Phasor Domain

Transforming and $v_L(t)$ to the phasor domain, we have

$$\overline{V}_L = -\omega L I_m e^{j\vartheta} e^{-j90} = j\omega L \mathrm{I}_m e^{j\vartheta}$$

We have used $e^{-j90} = \cos(-90) + j\sin(-90) = -j$ to get the last result. So we have



$$\overline{V}_L = j\omega L \overline{I}_L$$

In the phasor domain, we have something that looks like Ohm's Law for an inductor, except that the resistance R is replaced by jωL. We call jωL the ***inductive impedance***. We'll return to the topic of impedance later. Note that because jωL multiplies a current to get a voltage, it must have units of Ohms.

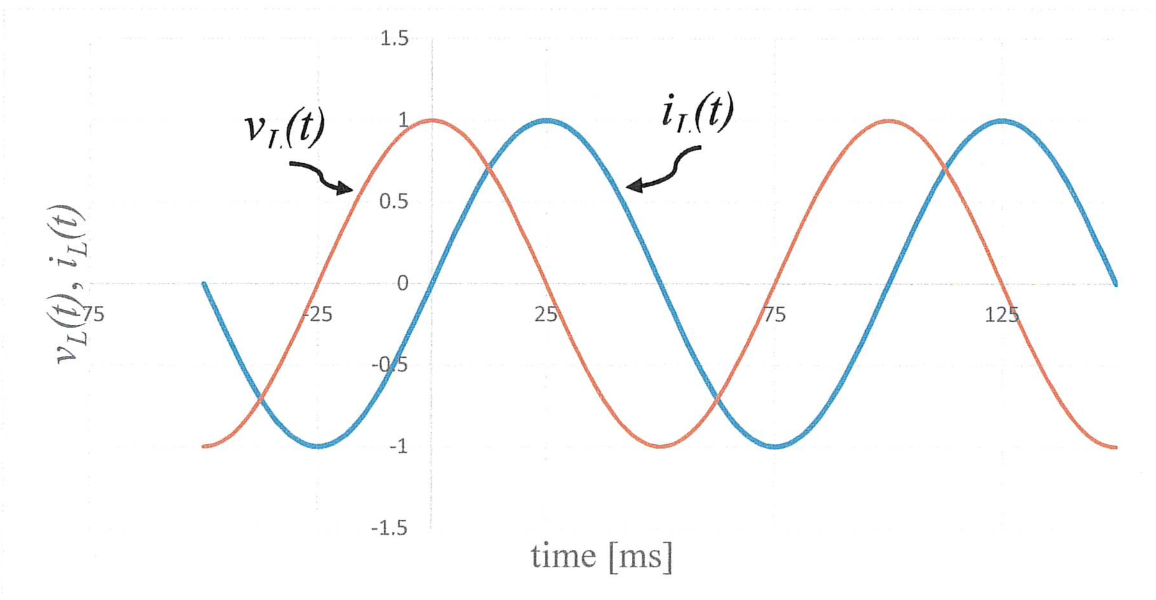
Here we see how the phasor transform makes algebraic equations out of differential equations. The current voltage relationship in the time domain for the inductor involves a time derivative, whereas in the phasor domain it is an algebraic relationship.

Phase Relationship for the Inductor

The phase of the inductor voltage is not the same as the phase of the inductor current.

$$\bar{V}_L = j\omega L\bar{I} = \omega L e^{j90} I_m e^{j\vartheta}$$
$$\bar{V}_L = \omega L I_m \angle(\vartheta + 90)$$

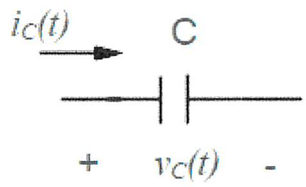
The phase of the voltage is 90° ahead of the phase for the current. When we say "ahead", we mean that the peak of the voltage occurs before the peak of the current. In other words, the voltage is ahead of the current in time. The terminology is that the ***voltage leads the current*** by 90°. Here is a plot (vertical units are arbitrary):



Capacitor

If a capacitor is connected into a circuit with sinusoidal sources, we have…

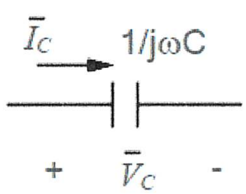
Time Domain



$$v_C(t) = V_m \cos(\omega t + \vartheta)$$
$$i_C(t) = C\frac{dv_C(t)}{dt} = -CV_m\omega \sin(\omega t + \vartheta)$$
$$= -CV_m\omega \cos(\omega t + \vartheta - 90^o)$$

Phasor Domain

Transforming to the phasor domain gives



$$\bar{I}_C = -\omega C\bar{V}e^{-j90} = j\omega C\bar{V}$$
$$\bar{V}_C = \left(\frac{1}{j\omega C}\right)\bar{I}_C$$

Again, we have something that looks like Ohm's Law, with R replaced by $\dfrac{1}{j\omega C}$. This is the ***capacitive impedance***, and also has units Ohms.

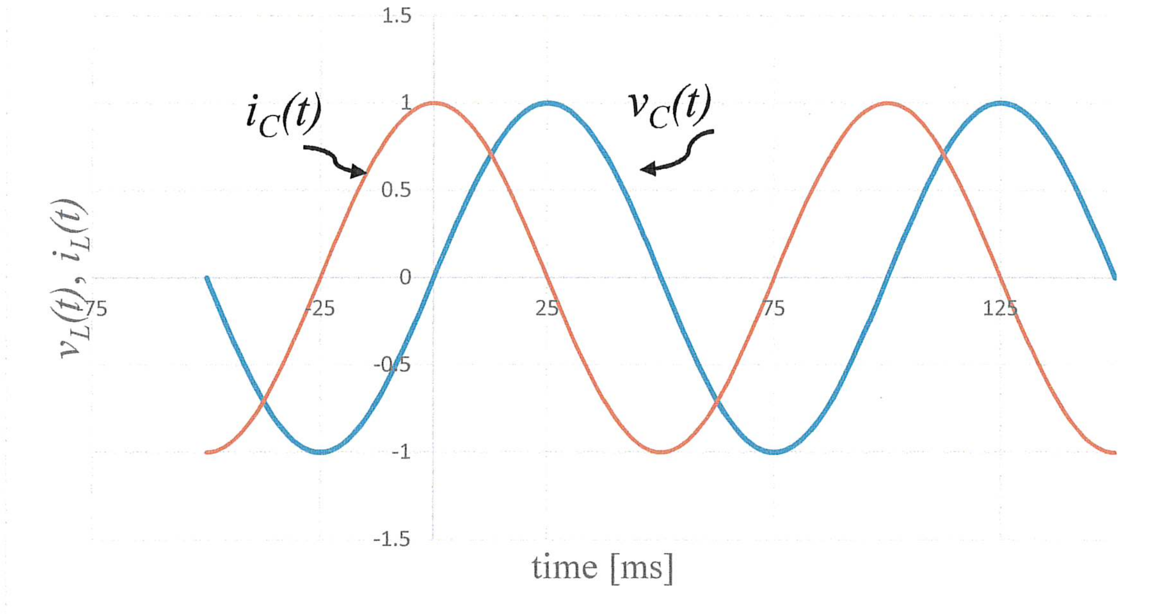
Phase Relationship for the Capacitor

Let's look at the phase for the capacitor voltage compared with the capacitor current.

$$\bar{V}_C = \frac{1}{\omega C}e^{-j90}\bar{I}_C$$
$$= \frac{1}{\omega C}I_m\angle(\vartheta - 90^o)$$

The phase of the current is 90° ahead of the phase for the current, that is, the current is ahead of the voltage in time. The terminology is that the ***current leads the voltage*** by 90°. Here is a plot (vertical units are arbitrary)

:



Example

Here we'll work Problem 9.2 parts a and b in *Worked Problems* as an example of converting sinusoids representing voltage and current sources to the phasor domain.

Problem 9.2 a and b: Find the phasor transform of each time domain function.

a) $v(t) = 170\cos(377t - 40^o)$ [V]

$$v(t) = \text{Re}\{170e^{j377t}e^{-j40}\}$$
$$\Rightarrow \overline{V} = 170e^{-j40} = 170\angle -40^o \text{ [V]}$$

b) $i(t) = 10\sin(1000t + 20^o)$ [A]

Our definition of the phasor was based on a source represented by a cosine function. So we first have to convert the sinusoid to a cosine.

$$i(t) = 10\cos(1000t + 20^o - 90^o)$$
$$\Rightarrow \overline{I} = 10\angle -70^o \text{ [A]}$$

Impedance

We can summarize these results as follows. The resistor, capacitor, and inductor in the phasor domain all have current-voltage relationships expressed by

$$\overline{V} = \overline{I}Z$$

where Z is the impedance, and

$$Z = R \text{ for a resistor,}$$

$$Z = \frac{1}{j\omega C} \text{ for a capacitor, and}$$

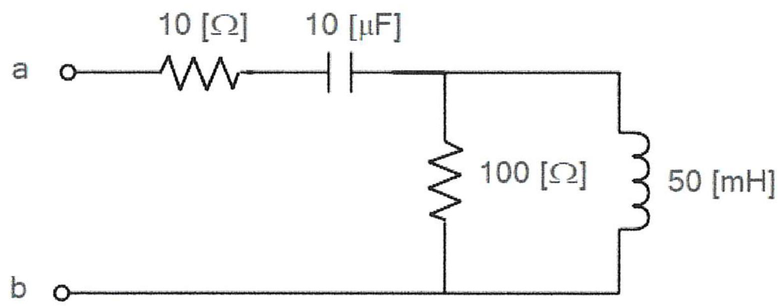
$$Z = j\omega L \text{ for an inductor.}$$

Note that $\overline{V}$ and $\overline{I}$ are phasors, and they get an "overbear" to indicate that. The impedance Z is a complex number, but it is not a phasor, so it does not get an overbar.

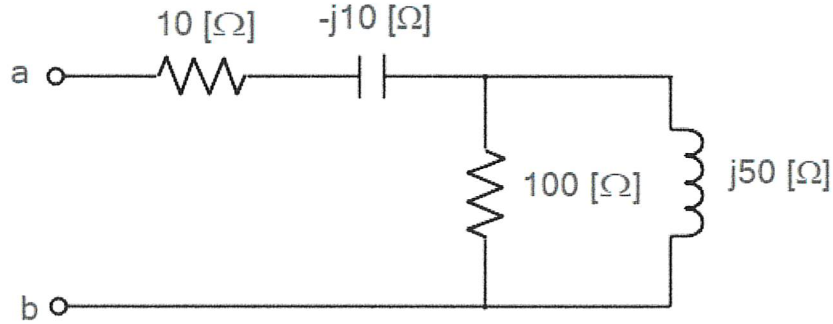
The impedance for a resistor is purely real, and the impedance for a capacitor and for an inductor are purely imaginary. But an impedance can have both a real and imaginary part, as we see in the next problem.

We can combine impedances as we would combine resistances. This is illustrated by Problem 9.3 in the *Worked Problems*. We work part of that problem here:

Problem 9.3 part a: Find the equivalent impedance at terminals a, b for ω = 1000 [rad/s].



There is no way to combine these circuit elements in the time domain. But if we convert them to the phasor domain, we can add the impedances just as we would resistances. The resistors remain as they are, but the capacitor and the inductor transform to the phasor domain as we have indicated above. To do that, we need the angular frequency ω, which we are given.



This is the phasor-domain version of the original circuit. We found the capacitive impedance as follows.

$$Z_C = \frac{1}{j\omega C} = \frac{1}{j1000[\text{rad/s}]10\text{x}10^{-6}[\text{F}]}.$$

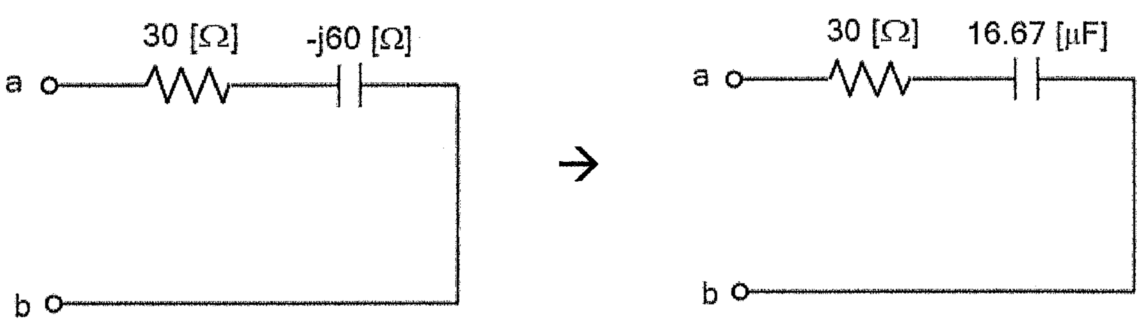
Doing the multiplication and noting that $\frac{1}{j} = -j$ gives $Z_C = \frac{-j}{0.1} = -j10[\Omega]$. Note that the units of impedance are Ohms, as we noted above. The inductor becomes $Z_L = j\omega L = j50[\Omega]$.

We can now combine these impedances in series and parallel to find that $Z_{ab} = 30 - j60\ [\Omega]$.

In general, an impedance will have a real and an imaginary part. We give names to these parts as follows.

$$Z = R + jX$$

The real part of Z is the ***resistance***, and imaginary part X is called the ***reactance***. Thus, we talk about ***capacitive reactance*** $X_C = \frac{-1}{\omega C}$, and ***inductive reactance*** $X_L = \omega L$. The capacitive reactance is negative (j is in the denominator), and the inductive reactance is positive. So we can interpret Z_{ab} found above as a resistance of 30 [Ω] and a capacitance of $\frac{1}{\omega C} = 60 \Rightarrow C = 16.667\ [\mu\text{F}]$. We can then draw an equivalent impedance in the phasor domain, and we can inverse transform back to the time domain.



Sometimes it is convenient to work with the reciprocal of the impedance, which is the ***admittance*** Y. We then have

$$Y = \frac{1}{Z} = G + jB$$

where G is the ***conductance*** and B is the ***susceptance***.

Solving Circuits in the Phasor Domain

Now that we have a way to represent sources and passive circuit elements in the phasor domain, we can solve circuits with sinusoidal sources by converting all the circuit elements to the phasor domain, finding the phasor corresponding to whatever voltage or current we are looking for, and then converting that phasor back to the time domain. We will illustrate this in class by solving a few problems.

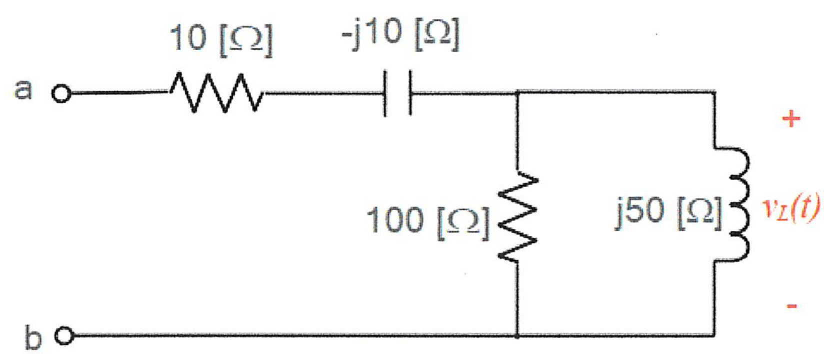
Time domain versus Phasor Domain

We must be very careful to treat the time domain and the phasor domain separately, because they represent very different things. In particular, we need to make sure that the notation for these two domains is distinct.

Time domain functions contain an explicit dependence on the variable representing time, t. We also treat DC voltages as time domain quantities, since a voltage source of 10 Volts, for example, is one that remains constant in time. Phasors, on the other hand, do not contain any dependence on time, and we must not write equations that suggests that they do.

Combining notation for phasor and time domains is called "mixed domains". We must be very careful not to mix domains. We will show examples of mixed domains in class so you will know what to avoid. We give a couple of simple examples here.

Looking at the phasor domain circuit we worked with above, imagine that we had labeled things as in the following diagram.



The problem here is that the circuit diagram is clearly a phasor domain diagram, but we have labeled a circuit variable in the time domain. This is an example of mixed domains, and must be avoided at all costs!

An example of an equation that mixes domains is the following.

$$v(t) = 6.2\angle 40^o .$$

The left side of this equation is a time domain quantity, but the right side is clearly a phasor.

In this class, we use lowercase letters for voltage and current in the time domain. A lowercase v or i indicates a time domain voltage or current, even if the explicit dependence on time is not indicated (that is, even if we do not write $v(t)$). Therefore, a lowercase v is a time domain quantity and must not be mixed with phasor domain quantities. The following equation would also be considered mixed domains.

$$v = 6.2\angle 40^o .$$

We will give other examples of mixed domains in class.