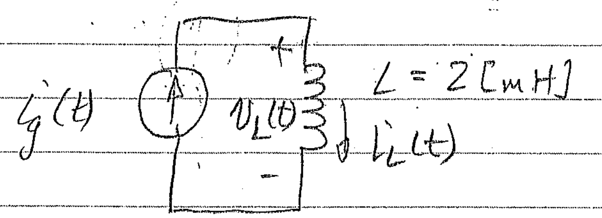
Problem 6.1

For the inductor in the circuit below, find:

i) $v_L(0)$, i.e. the voltage at $t=0$
ii) the time at which $v_L=0$.
iii) the power delivered to L.
iv) the time at which the power delivered to L reaches a maximum
v) the maximum power delivered to L.
vi) the energy in L at the time the power is a max

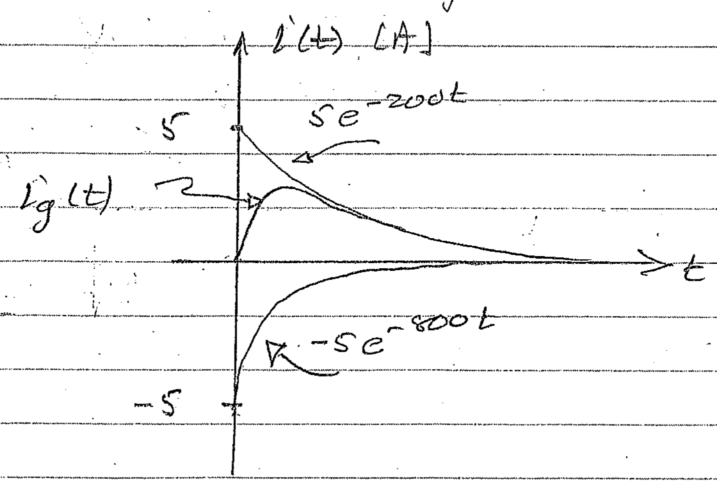
Given:



$i_g(t) = 0 \qquad t<0$

$i_g(t) = 5e^{-200t} - 5e^{-800t}\ [A] \quad t \geq 0$

Let's look at $i_g(t)$ vs. t.



$i_g(t=0) = 0$
$i_g(t=\infty) = 0$

So $i_g(t)$ maximizes between 0 and ∞.

i)

$$v_L(t) = L\frac{di_L}{dt}$$

$$= 0{,}002\,[5(-200)e^{-200t} - 5(-800)e^{-800t}]$$

$$v_L(t=0) = 0{,}002(-1000 + 4000) = \underline{6\,[V]}$$

Let's look at $i_g(t) = i_L(t)$ and $v_L(t)$ at $t=0$.

$i_L(t<0) = 0$ $\quad i_L(t=0) = 5e^{-0} - 5e^{-0} = 0$

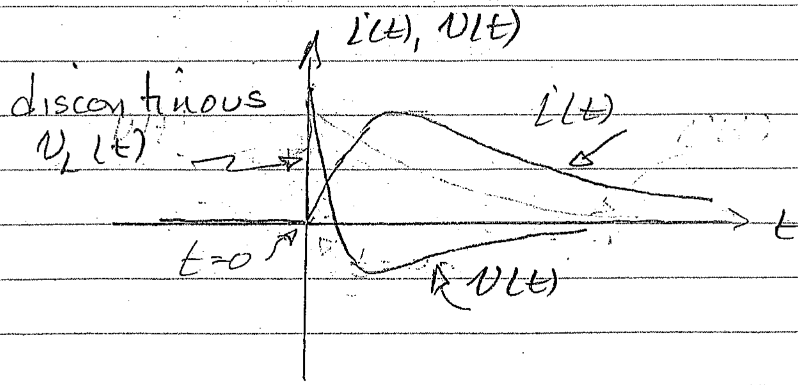
so current is continuous at $t=0$, as it must be for an inductor.

$v_L(t<0) = 0$ (there is no current)

$v_L(t=0) = 6\,[V]$

So there is a discontinuous jump from 0 to 6 [V] at $t=0$ for v_L. This is allowed since it is only inductor current that may not change instantaneously.

Graphically, we have...



So $v_L(t)$ crosses the t-axis. This is explored in part ii)

(PROBLEM 6.1 con't)

ii) $$v_L(t) = -2e^{-200t} + 8e^{-800t} \quad [V] \quad \text{from i)}.$$

$$v_L(t) = 0 \Rightarrow 2e^{-200t} = 8e^{-800t}$$

$$\ln 2 - 200t = \ln 8 - 800t$$

$$t = \frac{\ln 8 - \ln 2}{600} = 2.31 \text{ [ms]}$$

This is where $v_L(t)$ crosses the t-axis.

iii) $$p_{del L}(t) = v_L(t)\, i_L(t)$$

$$= (-2e^{-200t} + 8e^{-800t})(5e^{-200t} - 5e^{-800t})$$

$$= -10e^{-400t} - 40e^{-1600t} + 50e^{-1000t} \quad [W].$$

iv)

$$\frac{dp_{del L}(t)}{dt} = 4000e^{-400t} + (6.4\times10^4)e^{-1600t} - (5\times10^4)e^{-1000t} \quad (1)$$

We need to set this to 0, to find the time at which the power reaches a maximum.

<u>Analytical Solution</u>: This requires some thought because we cannot proceed as in ii) since
$\ln(a+b+c) \neq \ln a + \ln b + \ln c$

So try this: divide through by the first term:

$$1 + 16e^{-1200t} - 12.5e^{-600t} = 0$$

Now note that $(e^{-600t})^2 = e^{-1200t}$. Define $x \equiv e^{-600t} \Rightarrow$

$$1 + 16x^2 - 12.5x = 0 \qquad (2)$$

$$x = 0.09048,\ 0.69077$$

$$\Rightarrow e^{-600t} = 0.09048,\ 0.69077$$

$$\therefore t = 4\ [ms],\ 0.6\ [ms]$$

A plot of $p(t)$ shows that 0.6 [ms] is a maximum, and 4 [ms] is a minimum.

Computer Solution: I waited more than 2 minutes for my TI-89 to solve the original problem (Eqn. 1) and then I lost patience and stopped it. But it took virtually no time to solve the quadratic (Eqn. 2)

v) $$P_{del,L}^{max} = P_{del,L}(t_0 = 0.6\ [ms])$$

$$= -10e^{-400t_0} - 40e^{-1600t_0} + 500e^{-1000t_0}$$

$$= 4.26\ [W]$$

vi) $$w_L = \frac{1}{2} L i_L^2(t = 0.6\ [ms])$$

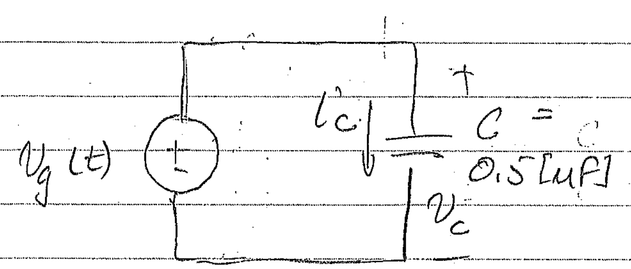
$$= \frac{1}{2}(0.002)\left(5e^{-200\times 0.6\times 10^{-3}} - 5e^{-800\times 0.6\times 10^{-3}}\right)^2$$

$$= 1.797\ [mJ]$$

PROBLEM 6.2

For the capacitor in the circuit below, find:

i) $i_c(0)$, i.e., the current at $t=0$.
ii) $v_c(\pi/80\ [ms])$
iii) the power delivered to C at $\pi/80$ [ms]
iv) the energy in the capacitor at $\pi/80$ [ms]



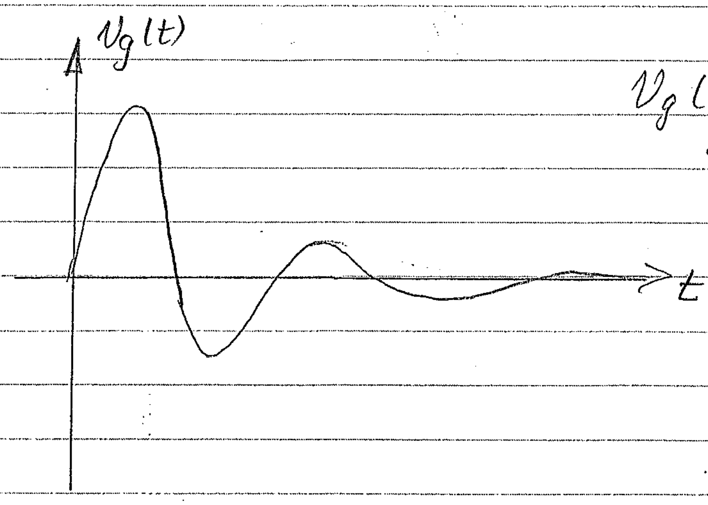
GIVEN:
$v_g(t) = 100e^{-20000t}\sin(40000t)$ [V] $t \geq 0$

$v_g(t) = 0 \quad t < 0$

i) $i_c(t) = C\dfrac{dv_c}{dt} \qquad v_c = v_g$

$$= 5\times10^{-7}\left[100e^{-20000t}(40000)\cos(40000t) - 20000(100)e^{-20000t}\sin(40000t)\right]$$

$t=0: \; i_c(0) = 5\times10^{-7}[100(40000) - 0] = 2\,[A]$



$v_g(t)$ is a rapidly decaying sinusoid.

Note that $v_g(t)$ does not change instantaneously at $t=0$: it has the value 0 [V] immediately before and immediately after $t=0$.

But the current does change instantaneously. Just before $t=0$, it is 0 [A] (because nothing is happening: $v_g=0$) and just after $t=0$ it is 2 [A].

ii) $$v_c\left(\tfrac{\pi}{80}\times 10^{-3}\,[s]\right) = 100\, e^{-20000\left(\frac{\pi}{80}\times 10^{-3}\right)} \sin\left(40000\times \tfrac{\pi}{80}\times 10^{-3}\right)$$

$$= 45.59\ [V]$$

iii) $$P_{del\,by\,c}\left(\tfrac{\pi}{80}\times 10^{-3}\,[s]\right) = v_c\left(\tfrac{\pi}{80}\times 10^{-3}\,[s]\right)\cdot i_c\left(\tfrac{\pi}{80}\times 10^{-3}\,[s]\right)$$

$$i_c\left(\tfrac{\pi}{80}\times 10^{-3}\,[s]\right) = 5\times 10^{-7}\Big[100\, e^{-20000\cdot\frac{\pi}{80}\times 10^{-3}} \times (40{,}000)\times \cos\left(40000\times\tfrac{\pi}{80}\times 10^{-3}\right) - 20000\, e^{-20000\times\frac{\pi}{80}\times 10^{-3}} \times \sin\left(40000\times \tfrac{\pi}{80}\times 10^{-3}\right)\Big]$$

$$= -0.456\ [A]$$

$$P_{del\,by\,c} = -20.79\ [W]$$

iv) $$w_c\left(\tfrac{\pi}{80}\times 10^{-3}\,[s]\right) = \tfrac{1}{2} C v_c^2$$

$$= \tfrac{1}{2}(5\times 10^{-7})(45.59)^2 = 520\ [\mu J]$$

PROBLEM 6.3

Given $v(t) = -40e^{-5t}$ [V] $t \geq 0$

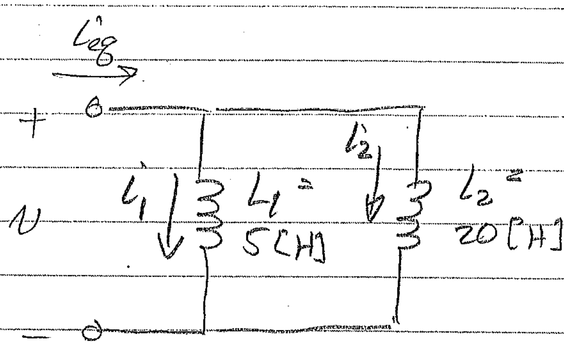
$i_1(0) = -2$ [A] ; $i_2(0) = 4$ [A]

FIND

i) L_{eq}

ii) $i_{eq}(t)$

iii) $i_1(t)$ and $i_2(t)$



i) $L_{eq} = \left(\frac{1}{5} + \frac{1}{20}\right)^{-1} = 4$ [H]

ii)

$$i_{eq}(t) = \frac{1}{L_{eq}} \int_0^t v(t)\,dt + i_{eq}(0)$$

KCL holds at $t=0$, so

$$i_{eq}(0) = i_1(0) + i_2(0) = 2 \text{ [A]}$$

So...

$$i_{eq}(t) = \frac{1}{4} \int_0^t (-40e^{-5t})\,dt + 2 \text{ [A]}$$

$$= \frac{1}{4}(-40)\left(\frac{1}{-5}\right) e^{-5t} \Big|_0^t + 2$$

$$= 2(e^{-5t} - 1) + 2 = 2e^{-5t} \text{ [A]}$$

iii) $$i_1(t) = \frac{1}{L_1}\int_0^t v(t)\,dt + i_1(0)$$

$$= \frac{1}{5}\int_0^t (-40e^{-5t})\,dt + (-2)$$

$$= \frac{8}{5}e^{-5t}\Big|_0^t - 2$$

$$= 1.6\,e^{-5t} - 3.6 \quad [A]$$

$$i_2(t) = \frac{1}{L_2}\int_0^t v(t)\,dt + i_2(0)$$

$$= \frac{1}{20}\int_0^t (-40e^{-5t})\,dt + 4$$

$$= \frac{2}{5}e^{-5t}\Big|_0^t + 4$$

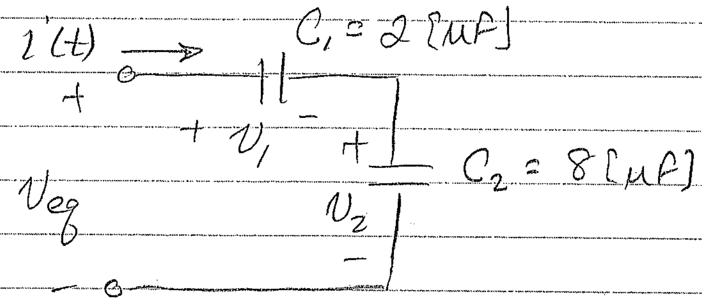
$$= 0.4\,e^{-5t} + 3.6 \; [A]$$

Not surprisingly, $i_1(t) + i_2(t) = i_{eq}(t)$.

PROBLEM 6.4

GIVEN $i(t) = 240\, e^{-10t}$ [μA] $t \geq 0$

$v_1(0) = -10$ [V] $\qquad v_2(0) = -5$ [V]



FIND

i) C_{eq}

ii) $v_{eq}(t)$

iii) $v_1(t)$ and $v_2(t)$

i) $$C_{eq} = \left(\frac{1}{C_1} + \frac{1}{C_2}\right)^{-1} = 1.6 \text{ [}\mu\text{F]}$$

ii) $$v_{eq}(t) = \frac{1}{C_{eq}} \int_0^t i(t)\, dt + v_{eq}(0)$$

$$v_{eq}(0) = v_1(0) + v_2(0) = -15 \text{ [V]}$$

$$v_{eq}(t) = \frac{1}{1.6\times10^{-6}} \int_0^t 240\, e^{-10t}\, dt - 15$$

$$= -15(e^{-10t} - 1) - 15$$

$$= -15 e^{-10t} \text{ [V]}$$

iii) $$v_1(t) = \frac{1}{C_1} \int_0^t i(t)\, dt + v_1(0)$$

$$= \frac{1}{2\times10^{-6}} \int_0^t 240 e^{-10t}\, dt - 10$$

$$= -12 e^{-10t} + 2 \text{ [V]}$$

$$v_2(t) = \frac{1}{C_2}\int_0^t i(t)\,dt + v_2(0)$$

$$= \frac{1}{8\times10^{-6}}\int_0^t 240e^{-10t} - 5$$

$$= -3e^{-10t} - 2 \quad [V]$$

So we see that as expected

$$v_1(t) + v_2(t) = v_{eq}(t)$$

Another thing we can look at is energy as $t \to \infty$. Note that

$$t \to \infty \Rightarrow v_1 \to 2\ [V], \quad v_2 \to -2\ [V]$$

So $v_{eq} \to 0$ [V].

But that means there is energy stored in the capacitors at $t = \infty$ that is not observable from the terminals of the equivalent circuit.

$$w_\infty = \frac{1}{2}C_1 v_1(\infty)^2 + \frac{1}{2}C_2 v_2(\infty)^2$$

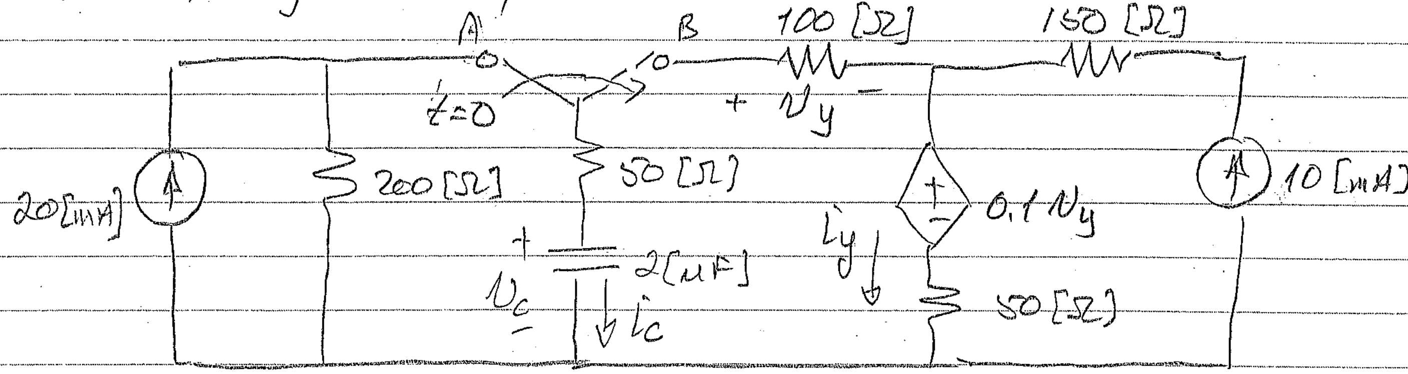
$$= \frac{1}{2}(2\times10^{-6})(2)^2 + \frac{1}{2}(8\times10^{-6})(-2)^2$$

$$= 20\ [\mu J]$$

But w_∞ for $C_{eq} = 0$!

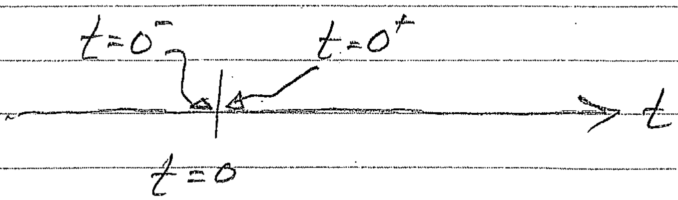
PROBLEM 6.5

The switch in the circuit below has been in position "A" for a long time, so that all voltages and currents have stopped changing. It moved to position "B" at $t=0$. Find $v_C(0^-)$, $v_C(0^+)$, $i_C(0^-)$, $i_C(0^+)$, $i_y(0^-)$, $i_y(0^+)$.

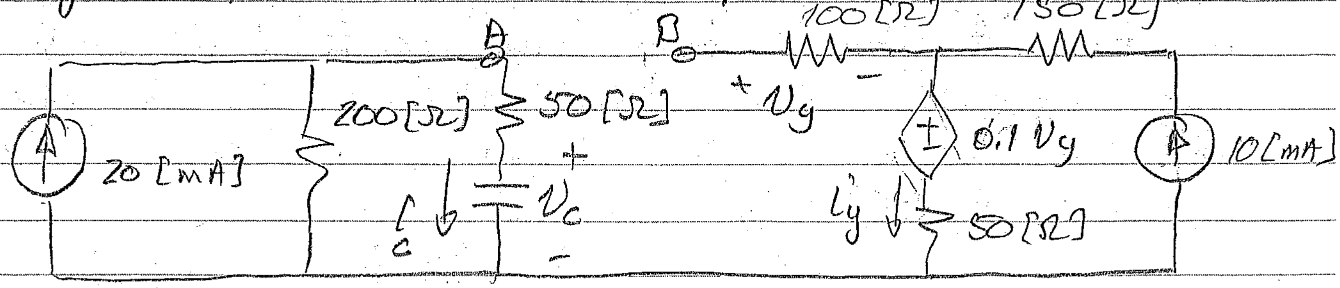


This problem introduces several ideas that we will be exploring in the next chapter on "transient response". There, we will look at switching events in circuits containing capacitors or inductors, in which voltages and currents change with time. The time dependence will be $\sim e^{-at}$, so that after a "long time", voltages and currents will stop changing. For now, we only need to know about basic properties of C and L.

Also: the switch moves at $t=0$. The designation $t=0^-$ is the time just before the switch moves; $t=0^+$ is the time just after the switch moves.



We begin by re-drawing the circuit for $t<0$, that is, before the switch moves.



Because the switch was in this position for a long time, there is no change in voltage or current. That means:

$$i_c = C\frac{dv_c}{dt} = 0 \quad \text{because } \frac{d}{dt} \to 0.$$

So there is no current in 50 [Ω] and

$$v_c = (0.02)(200) = 4 \text{ [V]}$$

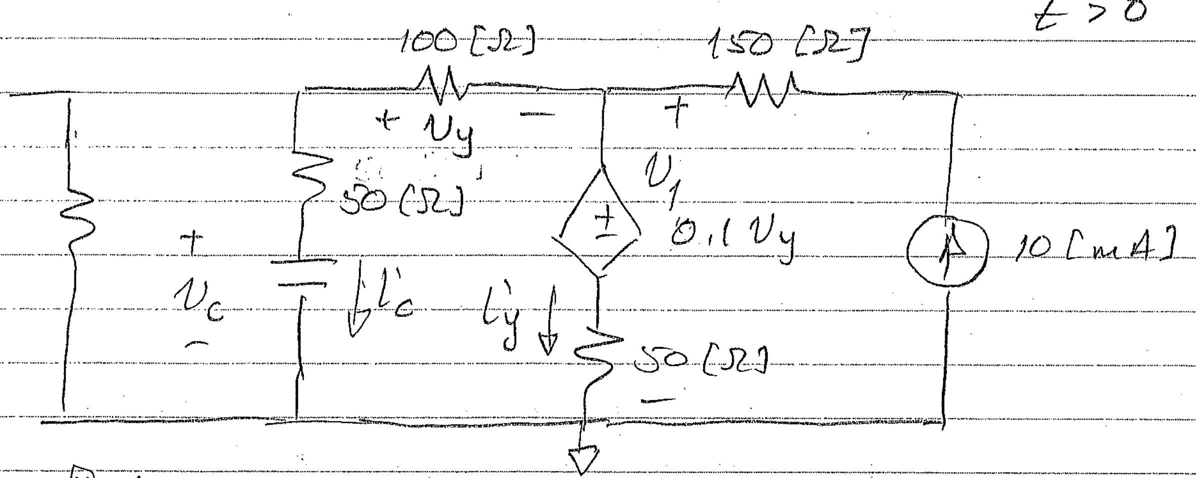
Because these are the values right up to the point that the switch moves, we have

$$i_c(0^-) = 0 \qquad v_c(0^-) = 4 \text{ [V]}$$

Also, $i_y(0^-) = 10$ [mA].

We now re-draw the circuit for $t>0$, that is, after the switch moves to B.

(PROBLEM 6.5)



(don't need this part any more)

Right after the switch closes, we have

$$v_c(0^+) = v_c(0^-) = 4\,(V)$$

because the voltage across a capacitor cannot change instantaneously. We consider the movement of the switch to be instantaneous.

What we will do now is solve the circuit for $t = 0^+$, knowing that $v_c(0^+) = 4$ [V].

$$\frac{v_1 - 0.1\, v_y}{50} - 0.01 + \frac{v_1 - 4}{150} = 0$$

$$v_y = -\frac{v_1 - 4}{150} \cdot 100$$

$$\Rightarrow \qquad v_1 = 1.5\,[V] \qquad v_y = 1.667\,[V]$$

These results hold for $t = 0^+$. After that, they will change, but we are not concerned about that just yet.

We have $i_y(0^+) = \dfrac{v_1 - 0.1 v_y}{50} = 26.67$ [mA]

$i_C(0^+) = \dfrac{v_1 - 4}{150} = -16.67$ [mA]

We review our results:

$$\left.\begin{array}{l} v_c(0^-) = 4 \text{ [V]} \\ v_C(0^+) = 4 \text{ [V]} \end{array}\right\}$$ voltage across a capacitor cannot change instantaneously.

$$\left.\begin{array}{l} i_C(0^-) = 0 \\ i_C(0^+) = -16.67 \text{ [mA]} \end{array}\right\}$$ current through a capacitor can change instantaneously.

$$\left.\begin{array}{l} i_y(0^-) = -10 \text{ [mA]} \\ i_y(0^+) = 26.67 \text{ [mA]} \end{array}\right\}$$ other currents can change instantaneously.