ECE 3317
Applied Electromagnetic Waves

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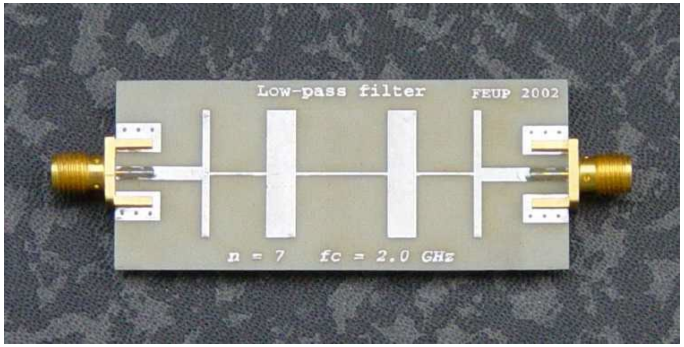
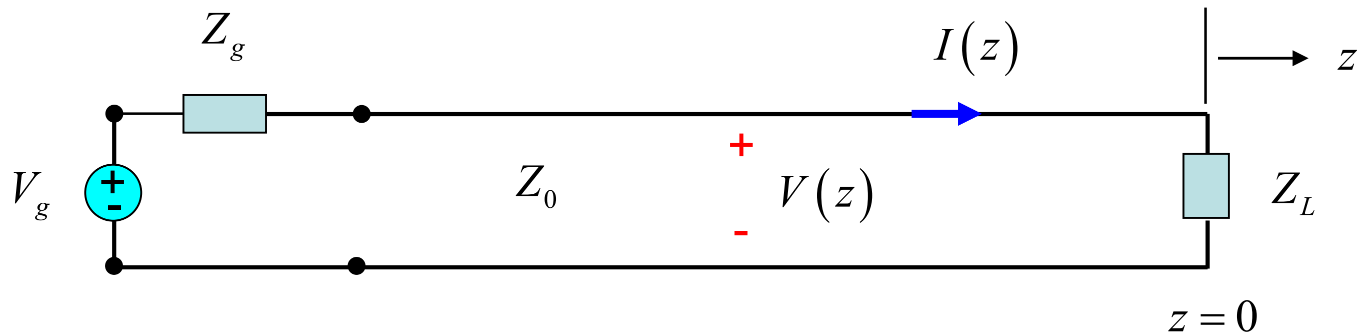
Fall 2023

Notes 10
Transmission Lines
(Reflection and Impedance)



Reflection

Consider a transmission line that is terminated with a load:



$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

Note:
In the frequency domain notes,
the load is always at $z = 0$.

Voltage and current on the line:

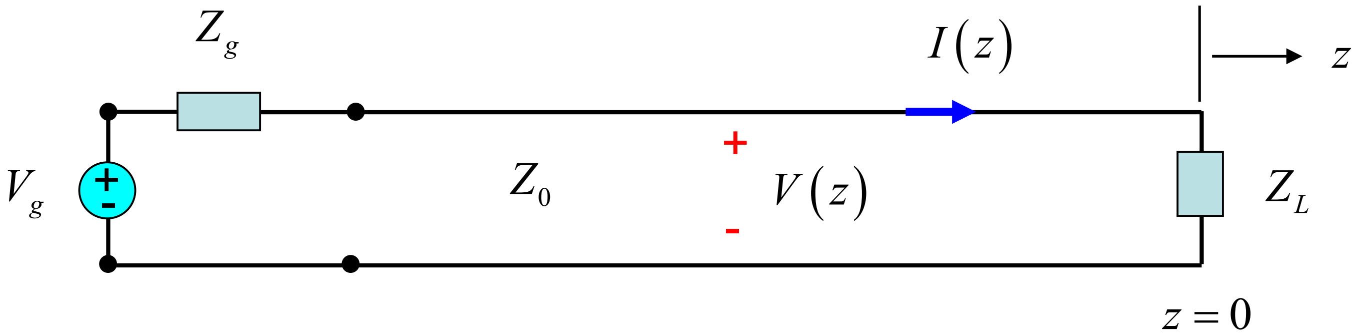
$$V(z) = \underbrace{Ae^{-\gamma z}}_{V^{+}(z)} + \underbrace{B\,e^{+\gamma z}}_{V^{-}(z)}$$

$$I(z) = \underbrace{\left(\frac{A}{Z_0}\right)e^{-\gamma z}}_{I^{+}(z)} - \underbrace{\left(\frac{B}{Z_0}\right)e^{+\gamma z}}_{I^{-}(z)}$$

Reflection (cont.)

Important point:

The forward-traveling and backward-traveling wave amplitudes are the amplitudes that describe the two waves in sinusoidal steady-state (after all bounces have died down and we are in steady state).

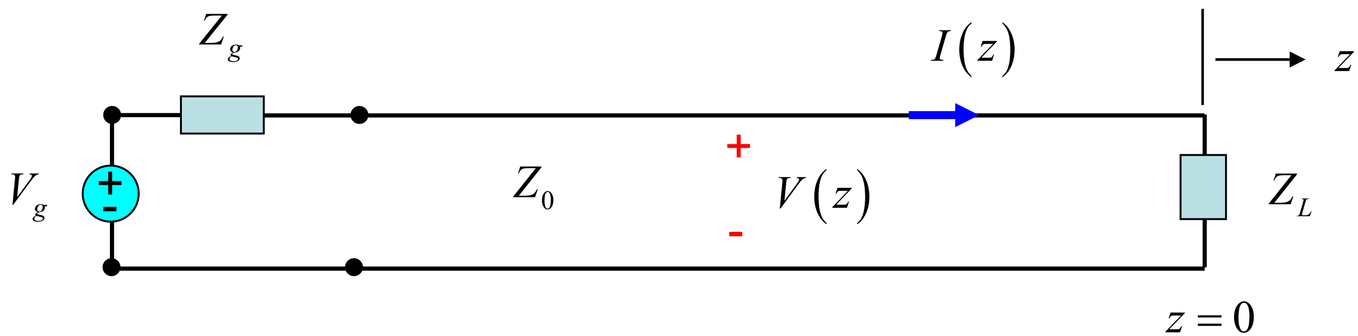


$$V(z) = Ae^{-\gamma z} + B\,e^{+\gamma z} \qquad I(z) = \left(\frac{A}{Z_0}\right)e^{-\gamma z} - \left(\frac{B}{Z_0}\right)e^{+\gamma z}$$

$A =$ amplitude of net forward-traveling wave

$B =$ amplitude of net backward-traveling wave

Reflection (cont.)



$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

$$V(z) = Ae^{-\gamma z} + B\,e^{+\gamma z}$$

$$I(z) = \left(\frac{A}{Z_0}\right)e^{-\gamma z} - \left(\frac{B}{Z_0}\right)e^{+\gamma z}$$

At the load ($z = 0$): $\quad V(0) = Z_L\, I(0)$

Hence, we have $\quad A + B = Z_L\left(\frac{1}{Z_0}(A - B)\right)$

or

$$B\left(1 + \frac{Z_L}{Z_0}\right) = A\left(\frac{Z_L}{Z_0} - 1\right)$$

Reflection Coefficient

We define the load reflection coefficient:

$$\Gamma_L \equiv \frac{V^-(0)}{V^+(0)}$$

Hence $\Gamma_L = \dfrac{B}{A}$

$$V(z) = Ae^{-\gamma z} + B\,e^{+\gamma z}$$

$$V^+(z) = Ae^{-\gamma z}$$

$$V^-(z) = B\,e^{+\gamma z}$$

We then have, from the last slide,

$$\Gamma_L = \frac{B}{A} = \frac{\left(\dfrac{Z_L}{Z_0} - 1\right)}{\left(1 + \dfrac{Z_L}{Z_0}\right)}$$

or

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$$

This is the same formula that we had in the time domain, but here the load impedance and the characteristic impedance may be <u>complex</u>.

Voltage and Current

We can then use $B = \Gamma_L A$ to write:

$$V(z) = A\left(e^{-\gamma z} + \Gamma_L e^{+\gamma z}\right)$$

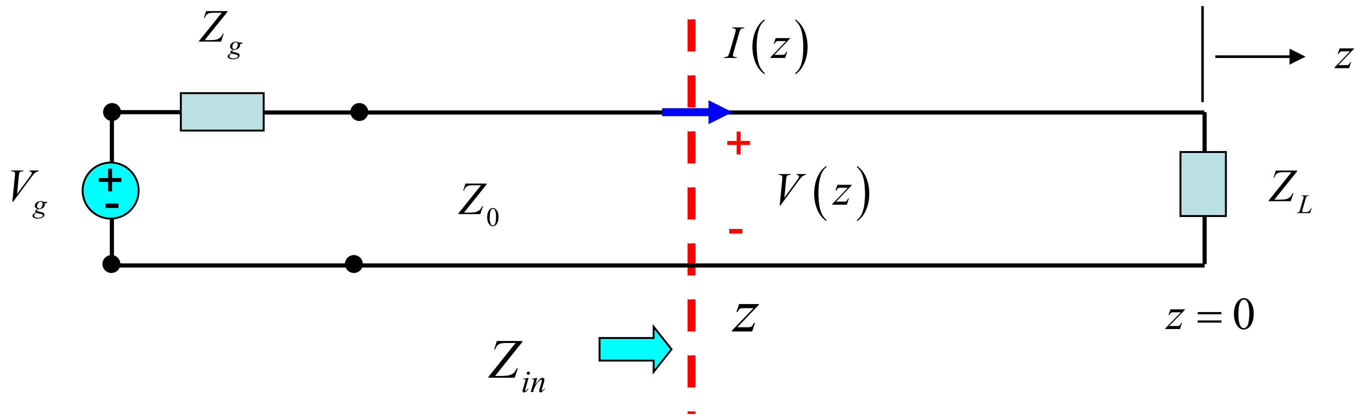
$$I(z) = \frac{1}{Z_0} A\left(e^{-\gamma z} - \Gamma_L e^{+\gamma z}\right)$$

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$$

Note: The generator (source) will determine the unknown (complex) constant A.

Impedance

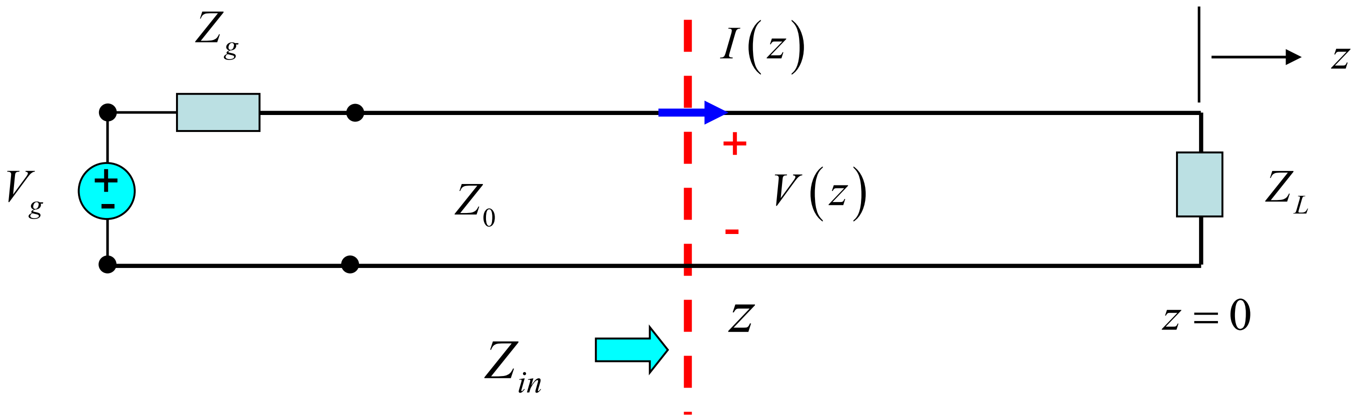
We define the input impedance at any point z on the line:



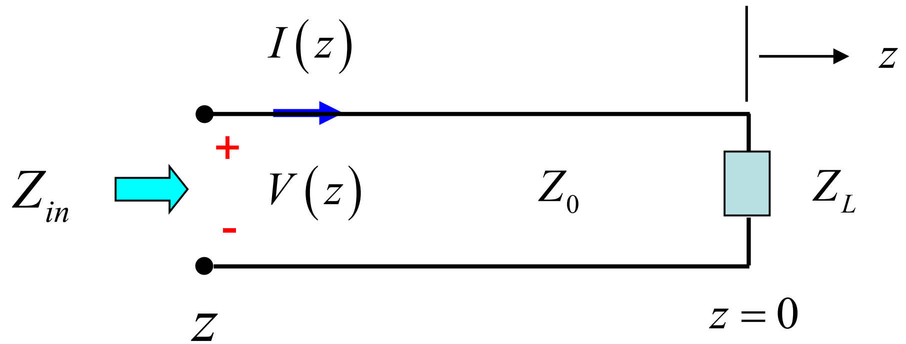
$$Z_{in}(z) \equiv \frac{V(z)}{I(z)}$$

The input impedance is the impedance "seen" looking to the right.

Impedance (cont.)



$$Z_{in}(z) \equiv \frac{V(z)}{I(z)}$$



Note:
The input impedance does not care what is to the left of the point z.
We can remove everything to the left if we wish.
(What is to the left of the point z only affects the amplitude level of the voltage and current.)

Impedance (cont.)

We then have

$$Z_{in}(z) = \frac{A\left(e^{-\gamma z} + \Gamma_L e^{+\gamma z}\right)}{\frac{1}{Z_0} A\left(e^{-\gamma z} - \Gamma_L e^{+\gamma z}\right)}$$

Canceling the constant A, we have

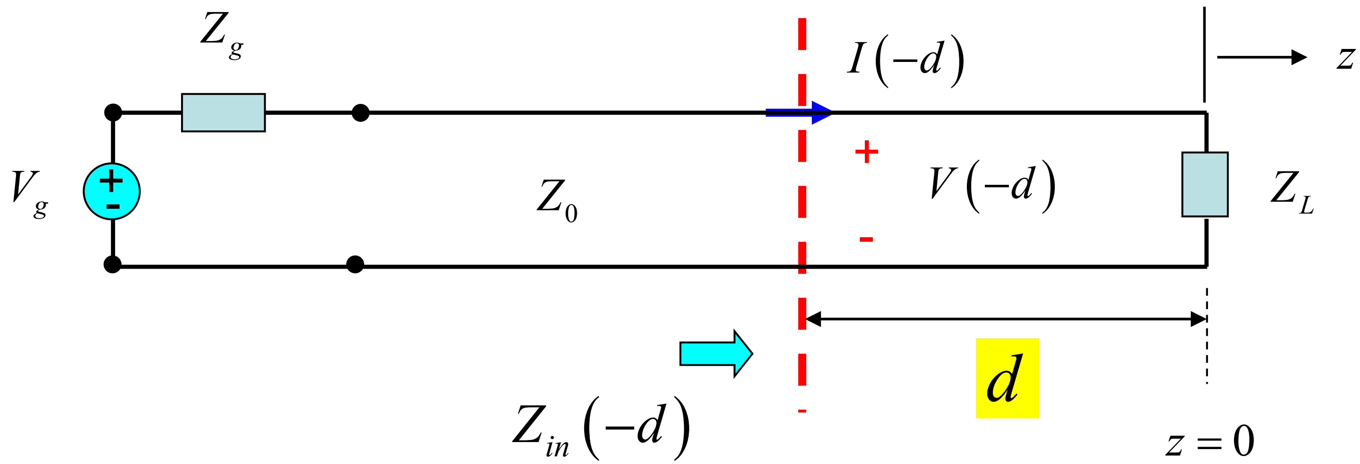
$$Z_{in}(z) = Z_0 \left(\frac{e^{-\gamma z} + \Gamma_L e^{+\gamma z}}{e^{-\gamma z} - \Gamma_L e^{+\gamma z}} \right)$$

or

$$Z_{in}(z) = Z_0 \left(\frac{1 + \Gamma_L e^{+2\gamma z}}{1 - \Gamma_L e^{+2\gamma z}} \right)$$

Impedance (cont.)

Now let $z = -d$: $\quad e^{\gamma z} = e^{-\gamma d}$ (Here d is the distance from the load.)



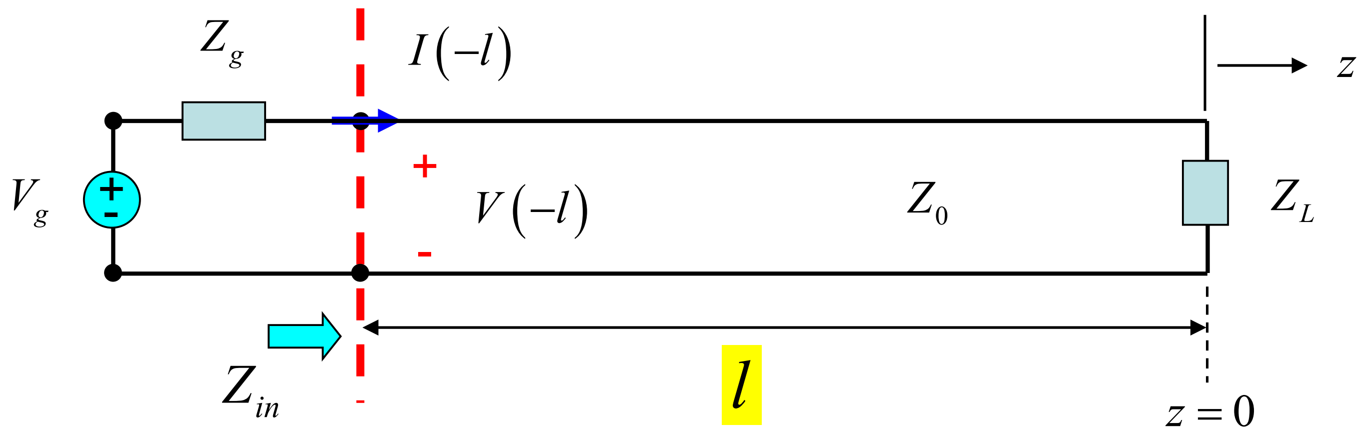
$$Z_{in}(-d) = Z_0 \left(\frac{1 + \Gamma_L e^{-2\gamma d}}{1 - \Gamma_L e^{-2\gamma d}} \right)$$

Impedance (cont.)

At the beginning of the line ($d = l$) we have:

$$Z_{in}\left(-l\right) = Z_0\left(\frac{1+\Gamma_L\, e^{-2\gamma l}}{1-\Gamma_L\, e^{-2\gamma l}}\right)$$

$l =$ length of line



Impedance (cont.)

We can derive a <u>new</u> form ("tangent form") for the input impedance.

Substituting for the load reflection coefficient, we have:

$$Z_{in}(z) = Z_0 \left(\frac{1+\Gamma_L \, e^{+2\gamma z}}{1-\Gamma_L \, e^{+2\gamma z}} \right)$$

$$= Z_0 \left(\frac{1+\left(\dfrac{Z_L - Z_0}{Z_L + Z_0}\right) e^{+2\gamma z}}{1-\left(\dfrac{Z_L - Z_0}{Z_L + Z_0}\right) e^{+2\gamma z}} \right)$$

$$= Z_0 \left(\frac{(Z_L + Z_0) + (Z_L - Z_0) e^{+2\gamma z}}{(Z_L + Z_0) - (Z_L - Z_0) e^{+2\gamma z}} \right)$$

$$= Z_0 \left(\frac{(Z_L + Z_0) e^{-\gamma z} + (Z_L - Z_0) e^{+\gamma z}}{(Z_L + Z_0) e^{-\gamma z} - (Z_L - Z_0) e^{+\gamma z}} \right)$$

Impedance (cont.)

Rearranging the expression, we have:

$$Z_{in}(z) = Z_0 \left(\frac{(Z_L + Z_0) e^{-\gamma z} + (Z_L - Z_0) e^{+\gamma z}}{(Z_L + Z_0) e^{-\gamma z} - (Z_L - Z_0) e^{+\gamma z}} \right)$$

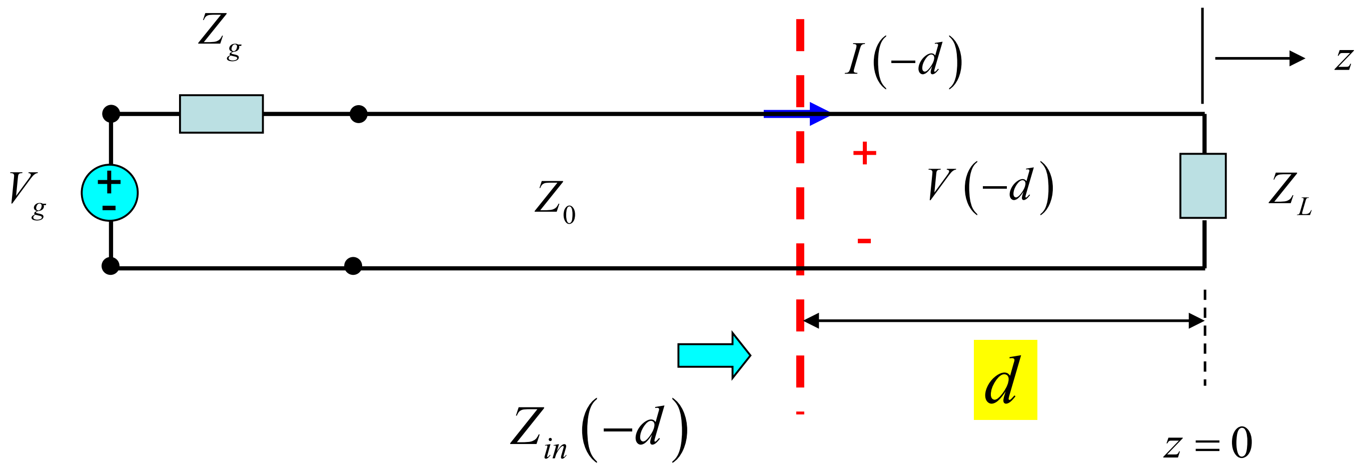
$$= Z_0 \left(\frac{Z_L \left(e^{+\gamma z} + e^{-\gamma z} \right) - Z_0 \left(e^{+\gamma z} - e^{-\gamma z} \right)}{-Z_L \left(e^{+\gamma z} - e^{-\gamma z} \right) + Z_0 \left(e^{+\gamma z} + e^{-\gamma z} \right)} \right)$$

$$= Z_0 \left(\frac{Z_L \left(2\cosh(\gamma z) \right) - Z_0 \left(2\sinh(\gamma z) \right)}{-Z_L \left(2\sinh(\gamma z) \right) + Z_0 \left(2\cosh(\gamma z) \right)} \right)$$

$$= Z_0 \left(\frac{Z_L - Z_0 \tanh(\gamma z)}{Z_0 - Z_L \tanh(\gamma z)} \right)$$

Impedance (cont.)

Now let $z = - d$: $\tanh(\gamma z) = \tanh(-\gamma d) = -\tanh(\gamma d)$



$$Z_{in}(-d) = Z_0 \left(\frac{Z_L + Z_0 \tanh(\gamma d)}{Z_0 + Z_L \tanh(\gamma d)} \right)$$

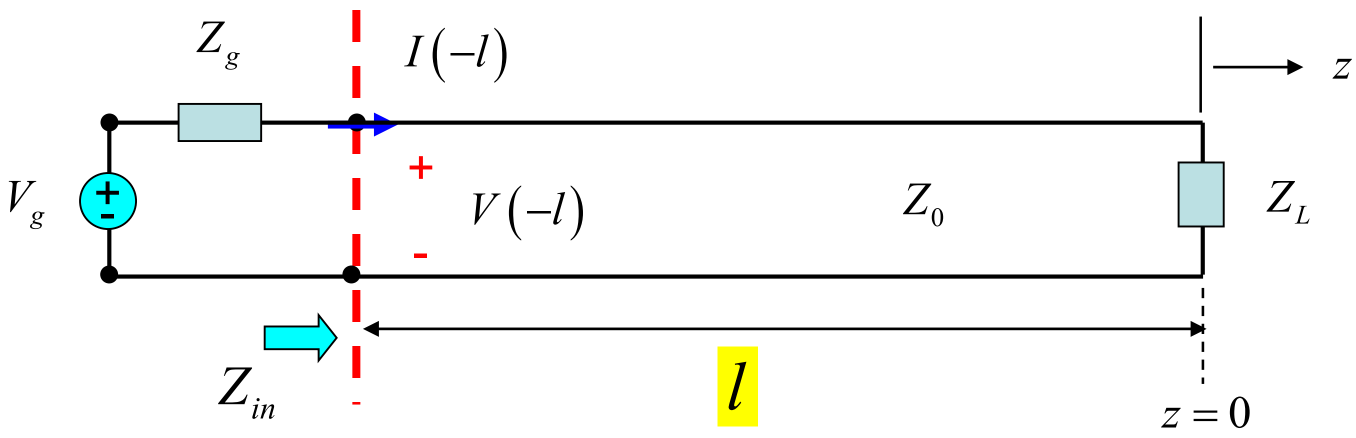
$d =$ distance from load

Impedance (cont.)

At the beginning of the line ($d = l$) we have:

$$Z_{in}(-l) = Z_0 \left(\frac{Z_L + Z_0 \tanh(\gamma l)}{Z_0 + Z_L \tanh(\gamma l)} \right)$$

$l =$ length of line



Impedance (cont.)

Limiting Cases

1) General (lossy) line: $l = 0 \quad \rightarrow \quad Z_{in} = Z_L$

2) Lossless line:

$$\omega = 0 \quad \rightarrow \quad \gamma = 0 \quad \rightarrow \quad Z_{in} = Z_L$$

$$\gamma = \sqrt{(\cancel{R} + j\omega L)(\cancel{G} + j\omega C)} = j\omega\sqrt{LC} \rightarrow 0$$

These limiting cases agree with circuit theory.

Impedance (cont.)

Limiting Cases (cont.)

3) Lossy infinite line: $l \to \infty \quad \to \quad Z_{in} = Z_0$ (complex)

Proof:

$$Z_{in} = Z_0 \left(\frac{Z_L + Z_0 \tanh(\gamma l)}{Z_0 + Z_L \tanh(\gamma l)} \right)$$

$$\gamma = \alpha + j\beta, \quad \alpha > 0$$

$$\tanh(\gamma l) \to 1$$

Note : $$\tanh(\gamma l) = \frac{\sinh(\gamma l)}{\cosh(\gamma l)} = \frac{\frac{1}{2}\left(e^{\gamma l} - e^{-\gamma l}\right)}{\frac{1}{2}\left(e^{\gamma l} + e^{-\gamma l}\right)} = \frac{1 - e^{-2\gamma l}}{1 + e^{-2\gamma l}} = \frac{1 - \overset{\overset{l \to \infty}{\downarrow}}{e^{-2\alpha l}} e^{-j2\beta l}}{1 + e^{-2\alpha l} e^{-j2\beta l}}$$

Impedance (cont.)

Limiting Cases (cont.)

4) Matched load ($Z_L = Z_0$):

$$Z_{in} = Z_0 \left(\frac{Z_L + Z_0 \tanh(\gamma l)}{Z_0 + Z_L \tanh(\gamma l)} \right)$$

$$\Downarrow$$

$$Z_{in} = Z_0 \left(\frac{Z_0 + Z_0 \tanh(\gamma l)}{Z_0 + Z_0 \tanh(\gamma l)} \right)$$

$$\Downarrow$$

$Z_{in} = Z_0$ (independent of line length!)

Impedance (cont.)

Lossless Case

$$\gamma = \alpha + j\beta = \sqrt{(\cancel{R} + j\omega L)(\cancel{G} + j\omega C)} = j\omega\sqrt{LC}$$

Use $\tanh(jx) = j\tan(x)$

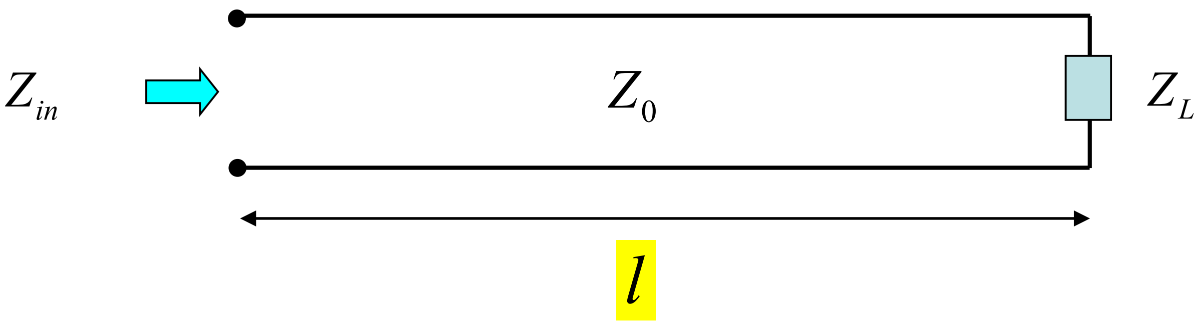
We then have $Z_{in} = Z_0\left(\dfrac{Z_L + jZ_0\tan(\beta l)}{Z_0 + jZ_L\tan(\beta l)}\right)$

where

$$\beta = \omega\sqrt{LC} = \omega\sqrt{\mu\varepsilon} = \frac{2\pi f}{c_d} = \frac{2\pi}{\lambda_d} \qquad \lambda_d = \frac{\lambda_0}{\sqrt{\mu_r\varepsilon_r}}$$

Impedance (cont.)

Summary of final formula for a lossless line:



$$Z_{in} = Z_0 \left(\frac{Z_L + jZ_0 \tan(\beta l)}{Z_0 + jZ_L \tan(\beta l)} \right)$$

$$\beta = \omega\sqrt{LC} = \omega\sqrt{\mu\varepsilon} = \frac{2\pi f}{c_d} = \frac{2\pi}{\lambda_d}$$

$$\lambda_d = \lambda_0 / \sqrt{\mu_r \varepsilon_r}$$

Impedance (cont.)

For a lossless line:

The input impedance repeats every one-half wavelength.

The voltage and current repeat every wavelength.

The voltage and current become their negatives after one-half wavelength.

The magnitude of the voltage and current repeat every one-half wavelength.

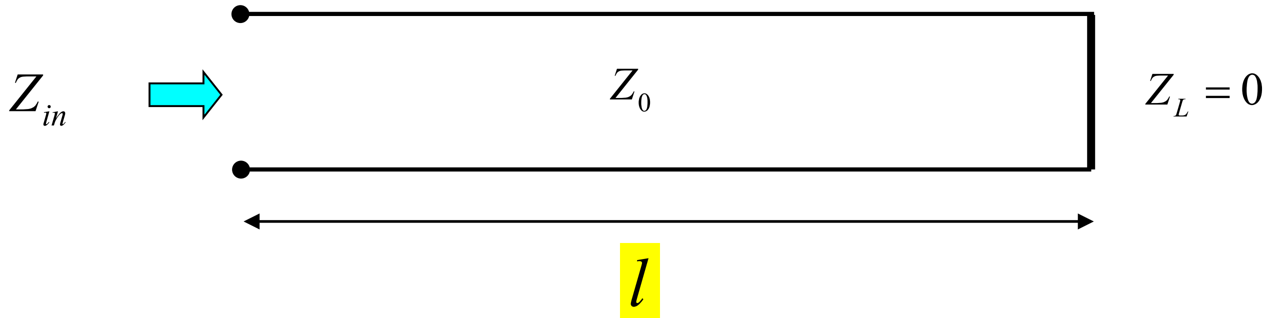
$$Z_{in} = Z_0 \left(\frac{Z_L + jZ_0 \tan(\beta d)}{Z_0 + jZ_L \tan(\beta d)} \right) \qquad \beta = \frac{2\pi}{\lambda_d} \qquad \beta d = 2\pi \left(\frac{d}{\lambda_d} \right)$$

$$V(z) = A\left(e^{-j\beta z} + \Gamma_L e^{+j\beta z}\right) \qquad I(z) = \frac{1}{Z_0} A\left(e^{-j\beta z} - \Gamma_L e^{+j\beta z}\right) \qquad \beta z = 2\pi \left(\frac{z}{\lambda_d} \right)$$

Impedance (cont.)

Special Cases of Lossless Line

Short-circuit line



$$Z_{in} = Z_0 \left(\frac{\cancel{Z_L} + jZ_0 \tan(\beta l)}{Z_0 + j\cancel{Z_L} \tan(\beta l)} \right)$$

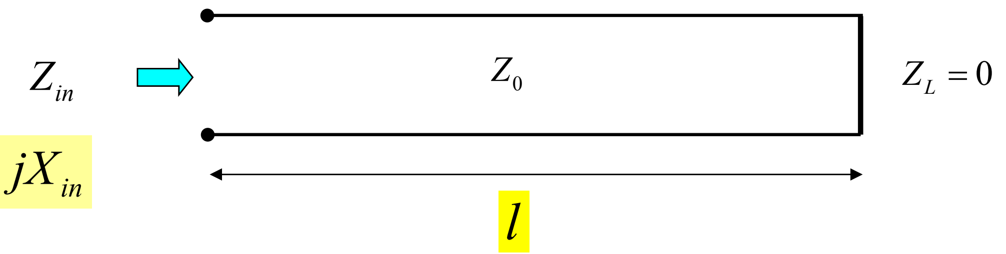
$$\boxed{Z_{in} = jZ_0 \tan(\beta l)}$$ or $$Z_{in} = jZ_0 \tan(2\pi l / \lambda_d)$$

Impedance (cont.)

Short-circuit line

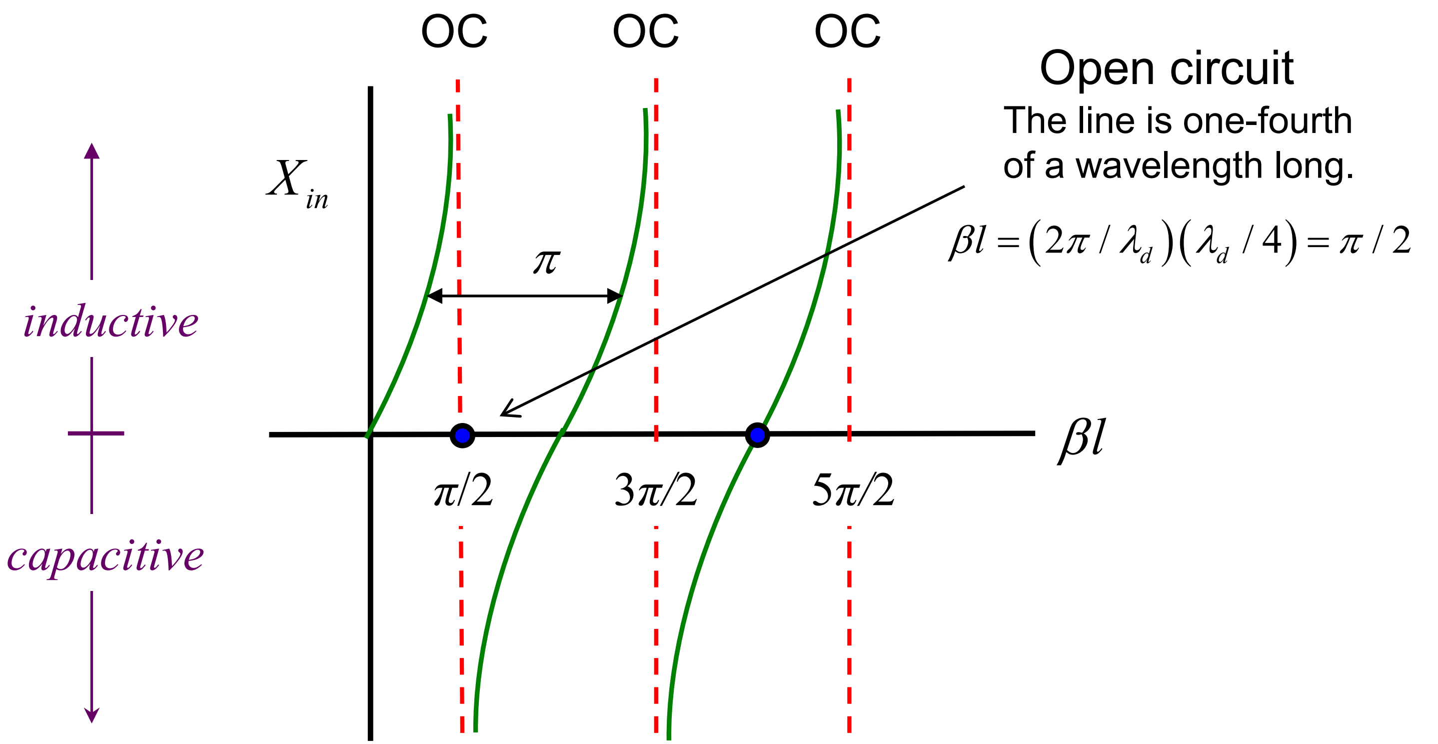
$$X_{in} = Z_0 \tan\left(\beta l\right)$$

$$Z_{in} = jX_{in}$$



or

$$X_{in} = Z_0 \tan\left(2\pi l / \lambda_d\right)$$



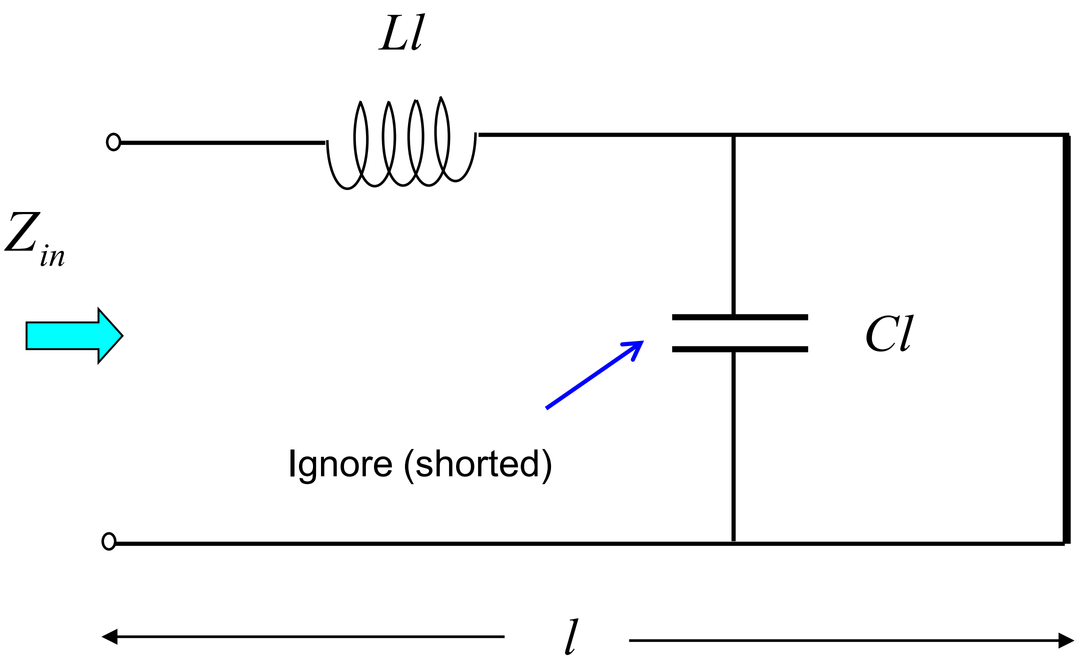
Impedance (cont.)

Short-circuit line

$$X_{in} = Z_0 \tan(\beta l)$$

Low frequency:

$$\begin{aligned} X_{in} &\approx Z_0(\beta l) \\ &= \sqrt{\frac{L}{C}}\left(\omega\sqrt{LC}\, l\right) \\ &= \omega(Ll) \end{aligned}$$

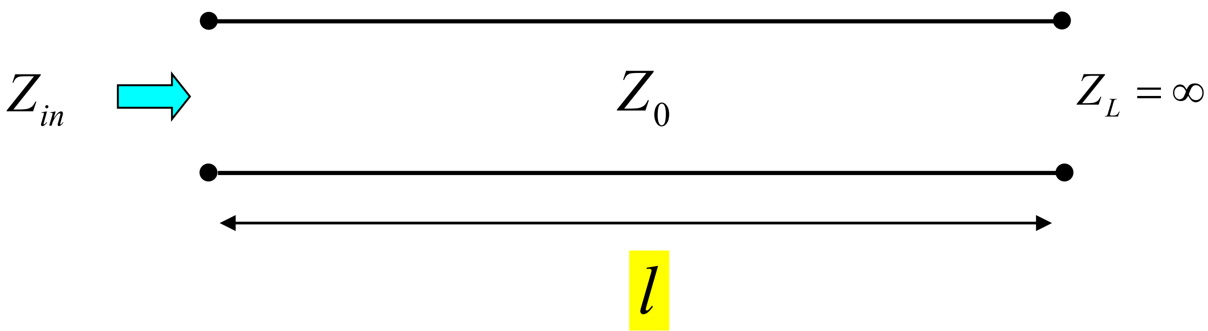


$$X_{in} \approx \omega(Ll)$$

Impedance (cont.)

Special Cases of Lossless Line (cont.)

Open-circuit line



$$Z_{in} = Z_0 \left(\frac{Z_L + jZ_0 \tan(\beta l)}{Z_0 + jZ_L \tan(\beta l)} \right) \quad \Rightarrow \quad Z_{in} = Z_0 \left(\frac{1 + j\cancel{(Z_0 / Z_L)} \tan(\beta l)}{\cancel{(Z_0 / Z_L)} + j \tan(\beta l)} \right)$$

$$\boxed{Z_{in} = -jZ_0 \cot(\beta l)}$$

or

$$Z_{in} = -jZ_0 \cot(2\pi l / \lambda_d)$$

Impedance (cont.)

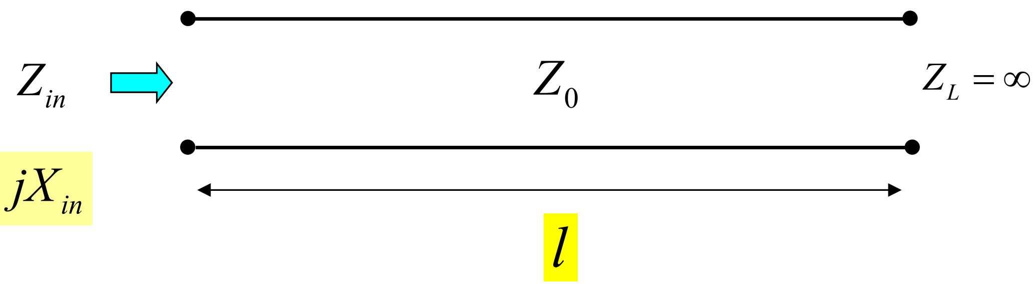
Open-circuit line

$$X_{in} = -Z_0 \cot(\beta l)$$

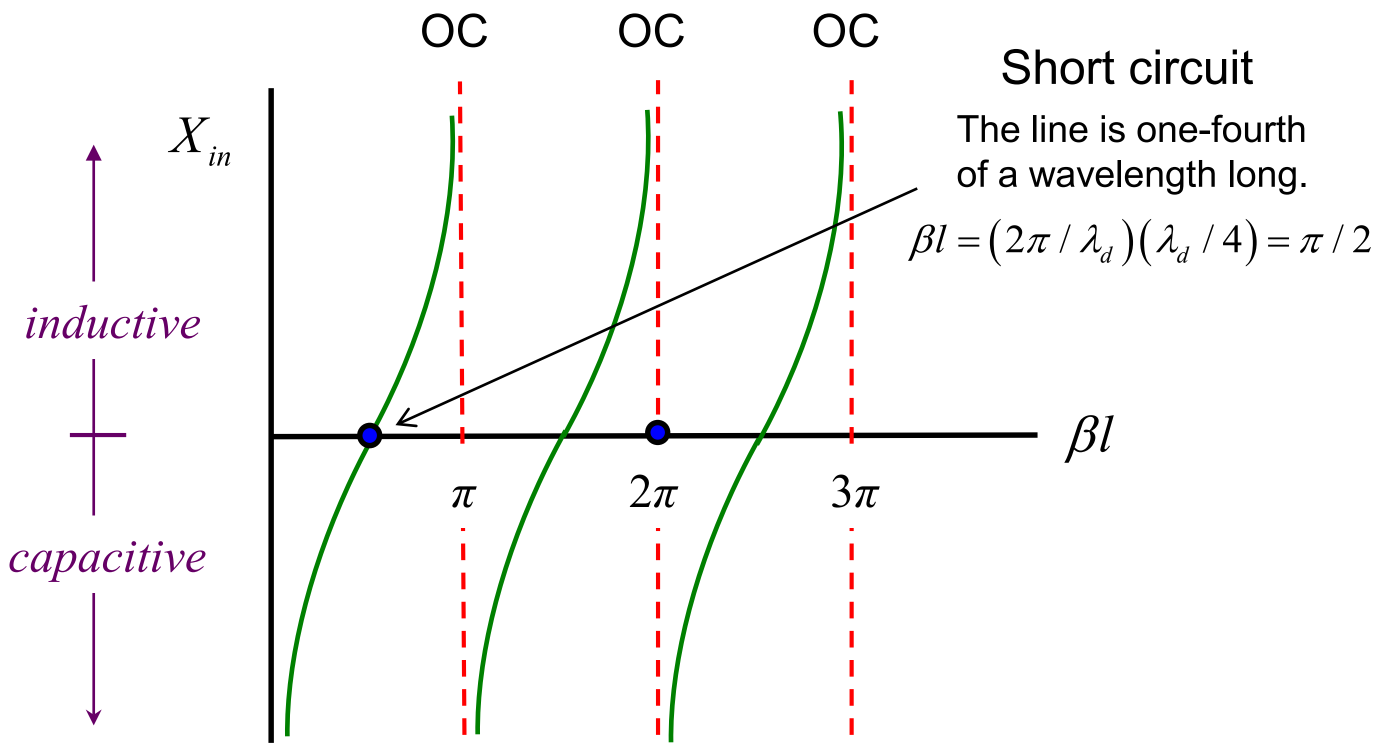
or

$$X_{in} = -Z_0 \cot(2\pi l / \lambda_d)$$

Open-circuit line



$$Z_{in} = jX_{in}$$



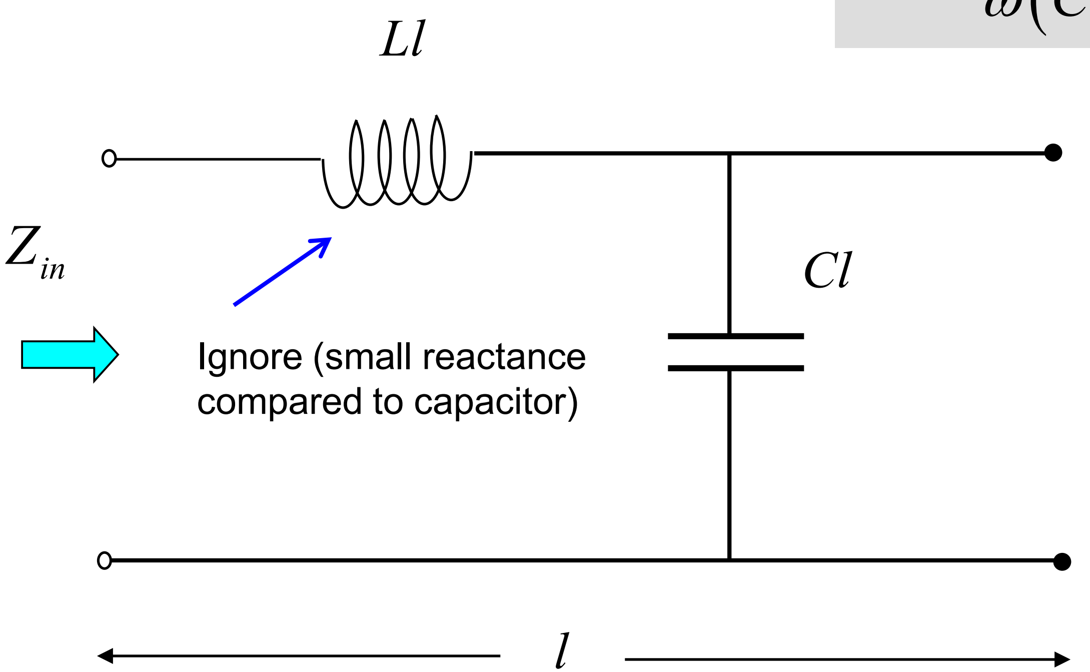
Impedance (cont.)

Open-circuit line

$$X_{in} = -Z_0 \cot(\beta l)$$

Low frequency:

$$\begin{aligned} X_{in} &\approx -Z_0 / (\beta l) \\ &= -\sqrt{\frac{L}{C}} \left(\frac{1}{\omega \sqrt{LC}\, l} \right) \\ &= \frac{-1}{\omega (Cl)} \end{aligned}$$



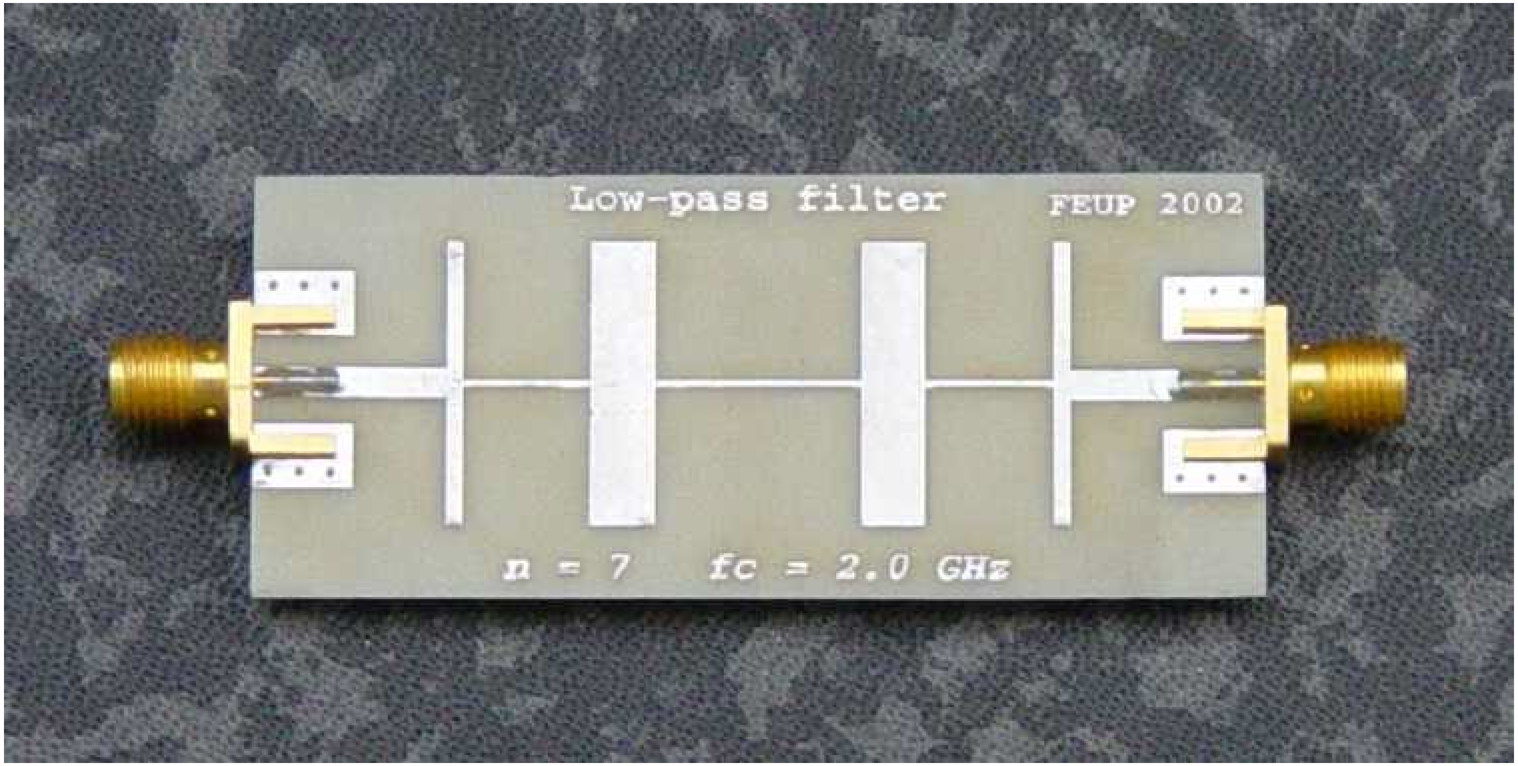
$B_{in} = -1 / X_{in}$ (for reactive element)

$$B_{in} \approx \omega (Cl)$$

Filter Application

Microstrip Filter

(Here is an application with microstrip lines being used to realize L and C elements.)



Appendix: Summary of Formulas

General Lossy Case

$$Z_{in} = Z_0 \left(\frac{Z_L + Z_0 \tanh(\gamma d)}{Z_0 + Z_L \tanh(\gamma d)} \right)$$

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$$

$$Z_{in} = Z_0 \left(\frac{1 + \Gamma_L e^{-2\gamma d}}{1 - \Gamma_L e^{-2\gamma d}} \right)$$

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

$$V(z) = A\left(e^{-\gamma z} + \Gamma_L e^{+\gamma z}\right)$$

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)}$$

$$\gamma = \alpha + j\beta$$

$$I(z) = \frac{1}{Z_0} A\left(e^{-\gamma z} - \Gamma_L e^{+\gamma z}\right)$$

$$\lambda_g = 2\pi / \beta$$

Appendix: Summary of Formulas

Lossless Case

$$Z_{in} = Z_0 \left(\frac{Z_L + jZ_0 \tan(\beta d)}{Z_0 + jZ_L \tan(\beta d)} \right)$$

$$Z_{in} = Z_0 \left(\frac{1 + \Gamma_L e^{-j2\beta d}}{1 - \Gamma_L e^{-j2\beta d}} \right)$$

$$V(z) = A\left(e^{-j\beta z} + \Gamma_L e^{+j\beta z}\right)$$

$$I(z) = \frac{1}{Z_0} A\left(e^{-j\beta z} - \Gamma_L e^{+j\beta z}\right)$$

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$$

$$Z_0 = \sqrt{\frac{L}{C}}$$

$$\beta = \omega\sqrt{LC} = \omega\sqrt{\mu\varepsilon} = \frac{2\pi f}{c_d} = \frac{2\pi}{\lambda_d}$$

$$\lambda_d = \lambda_0 / \sqrt{\varepsilon_r \mu_r}$$

$$Z_{in}^{\text{short}} = jZ_0 \tan(\beta l)$$

$$Z_{in}^{\text{open}} = -jZ_0 \cot(\beta l)$$