ECE 3317
Applied Electromagnetic Waves

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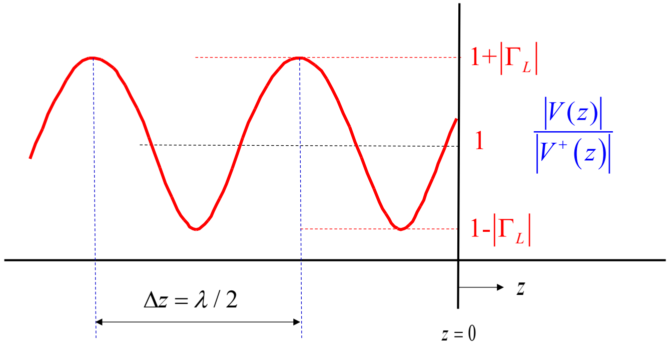
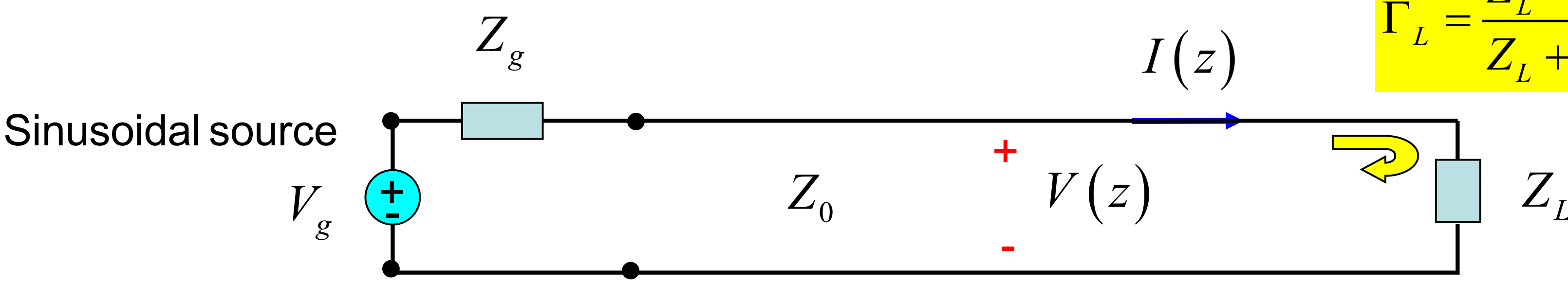
Notes 11
Transmission Lines
(Standing Wave Ratio (SWR) and Generalized Reflection Coefficient)

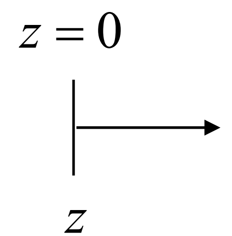


Standing Wave Ratio

Consider a <u>lossless</u> transmission line that is terminated with a load:

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$$





$$Z_0 = \sqrt{\frac{L}{C}} \qquad \Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$$

Lossless line:

$$\beta = \omega\sqrt{LC} = \omega\sqrt{\mu\varepsilon} = \omega\sqrt{\mu_0\varepsilon_0}\sqrt{\mu_r\varepsilon_r} = k_0\sqrt{\mu_r\varepsilon_r} = \frac{2\pi}{\lambda_d}, \qquad \lambda_d = \frac{\lambda_0}{\sqrt{\mu_r\varepsilon_r}}$$

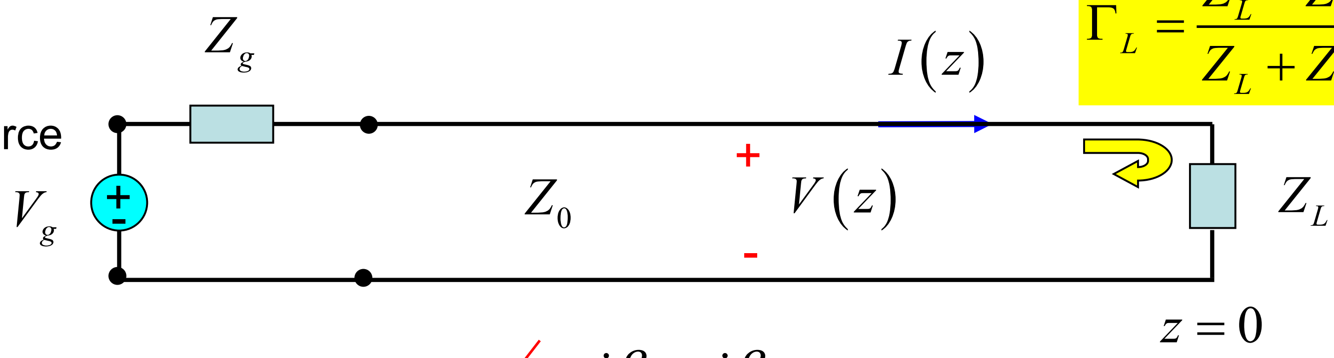
$$k_0 \equiv \omega\sqrt{\mu_0\varepsilon_0} = \frac{\omega}{c} = \frac{2\pi}{\lambda_0} \qquad \lambda_0 = \frac{c}{f} \qquad c = 2.99792458 \times 10^8 \ [\mathrm{m/s}]$$

Standing Wave Ratio (cont.)

Consider a lossless transmission line that is terminated with a load:

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$$

Sinusoidal source



$$\gamma = \alpha + j\beta = j\beta$$

$$V(z) = A\left(e^{-j\beta z} + \Gamma_L e^{+j\beta z}\right)$$

$$I(z) = \frac{1}{Z_0} A\left(e^{-j\beta z} - \Gamma_L e^{+j\beta z}\right)$$

$$V(z) = Ae^{-j\beta z}\left(1 + \Gamma_L e^{+j2\beta z}\right)$$

$$I(z) = \frac{1}{Z_0} Ae^{-j\beta z}\left(1 - \Gamma_L e^{+j2\beta z}\right)$$

$$V^+(z) = Ae^{-j\beta z} \qquad V^-(z) = A\Gamma_L e^{+j\beta z} \qquad V^-(z) / V^+(z) = \Gamma_L e^{+j2\beta z}$$

Standing Wave Ratio (cont.)

$$V(z) = Ae^{-j\beta z}\left(1 + \Gamma_L e^{+j2\beta z}\right)$$

Denote $\Gamma_L = |\Gamma_L| e^{j\phi}$ (the polar form of Γ_L)

Then we have $V(z) = Ae^{-j\beta z}\left(1 + |\Gamma_L| e^{+j(\phi + 2\beta z)}\right)$

The magnitude is $|V(z)| = |A| \left|1 + |\Gamma_L| e^{+j(\phi + 2\beta z)}\right|$

Maximum voltage: $V_{\max} \equiv |V(z)|_{\max} = |A|\left(1 + |\Gamma_L|\right)$ $\phi + 2\beta z_{\max} = 2\pi m$

Minimum voltage: $V_{\min} \equiv |V(z)|_{\min} = |A|\left(1 - |\Gamma_L|\right)$ $\phi + 2\beta z_{\min} = \pi + 2\pi n$

$m, n = 0, \pm 1, \pm 2, \ldots$

Standing Wave Ratio (cont.)

The voltage standing wave ratio is the ratio of $V_{\max}$ to $V_{\min}$.

$$\text{VSWR} \equiv \frac{V_{\max}}{V_{\min}}$$

We then have:

$$\text{VSWR} = \frac{1+|\Gamma_L|}{1-|\Gamma_L|}$$

$$1 \le \text{VSWR} \le \infty$$

Perfect match: $\Gamma_L = 0$ (VSWR = 1)

Reactive load: $|\Gamma_L| = 1$ (VSWR = ∞)

Standing Wave Ratio (cont.)

For the current we have

$$\left|I(z)\right| = \left|A\right|\left(\frac{1}{Z_0}\right)\left|1-\left|\Gamma_L\right|e^{+j(\phi+2\beta z)}\right|$$

Hence we have:

$$I_{\max} \equiv \left|I(z)\right|_{\max} = \left|A\right|\left(\frac{1}{Z_0}\right)\left(1+\left|\Gamma_L\right|\right) \qquad \phi + 2\beta z_{\max} = \pi + 2\pi n$$

$$I_{\min} \equiv \left|I(z)\right|_{\min} = \left|A\right|\left(\frac{1}{Z_0}\right)\left(1-\left|\Gamma_L\right|\right) \qquad \phi + 2\beta z_{\min} = 2\pi m$$

Note:
The current is maximum where the voltage is minimum, and vice versa.

The current standing wave ratio is thus

$$\text{ISWR} \equiv \frac{1+\left|\Gamma_L\right|}{1-\left|\Gamma_L\right|}$$

Hence $\text{VSWR} = \text{ISWR} = \text{SWR}$

Standing Wave Pattern

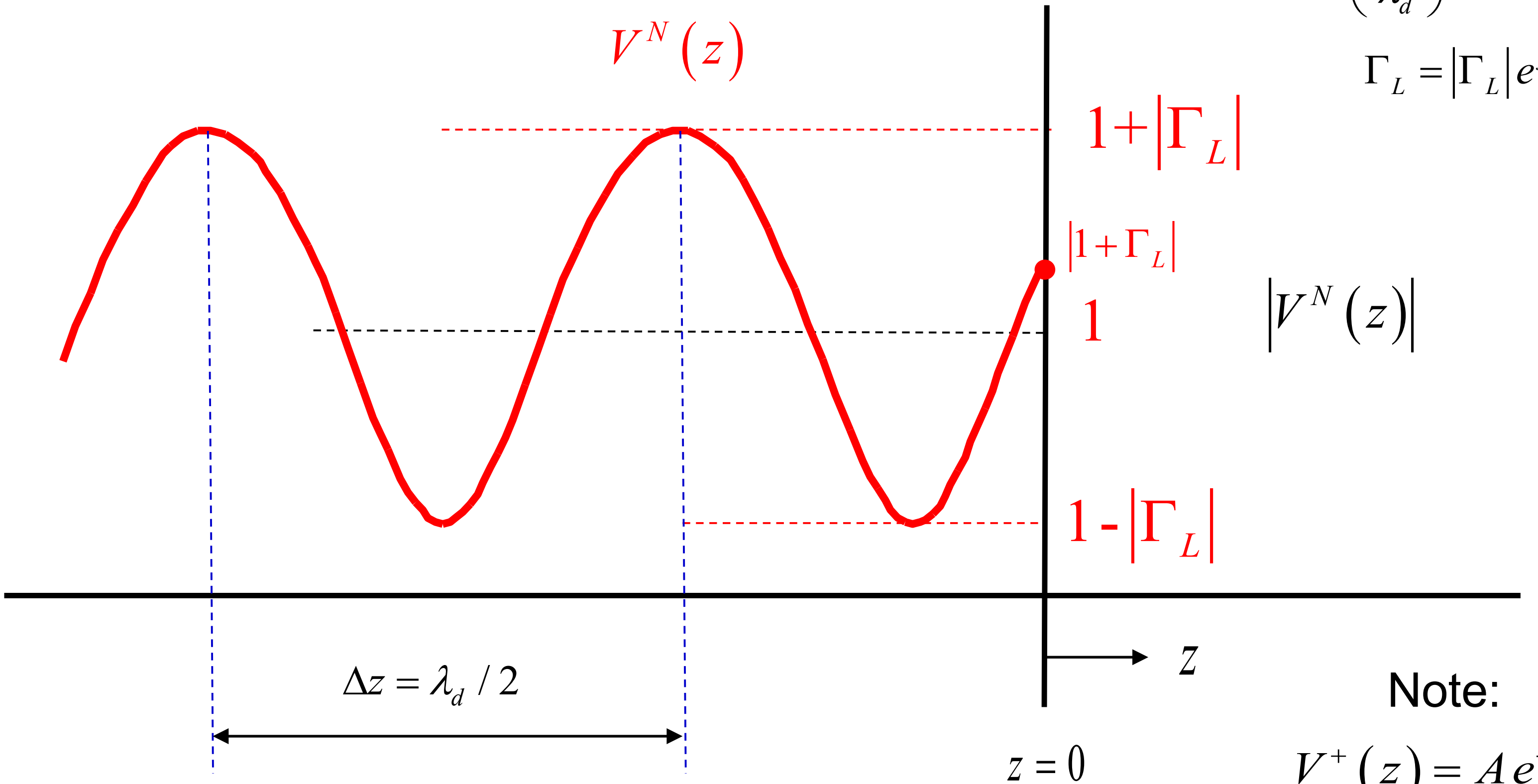
$$|V(z)| = |A| \left|1 + |\Gamma_L| e^{+j(\phi + 2\beta z)}\right|$$

Plot this part.

$$\left|V^N(z)\right| \equiv \left|\frac{V(z)}{V^+(z)}\right| = \left|1 + |\Gamma_L| e^{+j(\phi+2\beta z)}\right|$$

$$2\beta z = 2\left(\frac{2\pi}{\lambda_d}\right) z = 4\pi\left(\frac{z}{\lambda_d}\right)$$

$$\Gamma_L = |\Gamma_L| e^{j\phi}$$



Note:

$$V^+(z) = A e^{-j\beta z}$$

$$|V^+(z)| = |A|$$

(V^+ is the net wave going in the $+z$ direction.)

Standing Wave Pattern (cont.)

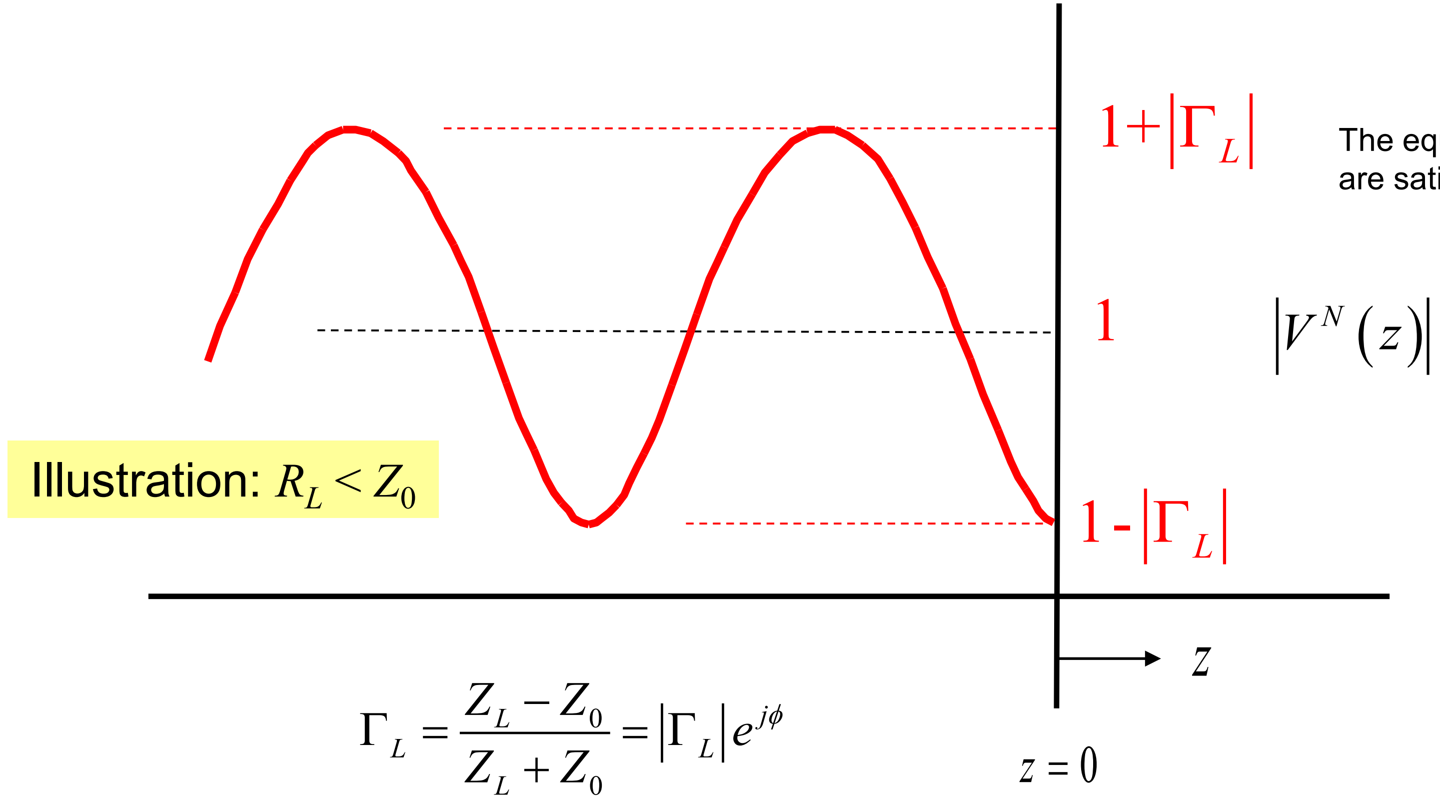
For a <u>real</u> load ($Z_L = R_L$):

Γ_L is real, and there is always a voltage maximum or minimum at the load ($z = 0$).

$R_L < Z_0 \;\Rightarrow\; \Gamma_L < 0 \;\Rightarrow\; \phi = \pi \;\Rightarrow\;$ voltage minimum at $z = 0$ $\quad (\phi + 2\beta z_{\min} = \pi + 2\pi n)$

$R_L > Z_0 \;\Rightarrow\; \Gamma_L > 0 \;\Rightarrow\; \phi = 0 \;\Rightarrow\;$ voltage maximum at $z = 0$ $\quad (\phi + 2\beta z_{\max} = 2\pi m)$

The equations in parenthesis are satisfied for $z_{\min,\max} = 0$.



$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = |\Gamma_L| e^{j\phi}$$

Standing Wave Ratio: Real Load

Special case of a real load impedance $Z_L = R_L$

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{R_L - Z_0}{R_L + Z_0}$$

$$\text{SWR} = \frac{1+|\Gamma_L|}{1-|\Gamma_L|} = \frac{1+\left|\dfrac{R_L - Z_0}{R_L + Z_0}\right|}{1-\left|\dfrac{R_L - Z_0}{R_L + Z_0}\right|}$$

Case a: $R_L \geq Z_0$

$$\text{SWR} = \frac{1+\left(\dfrac{R_L - Z_0}{R_L + Z_0}\right)}{1-\left(\dfrac{R_L - Z_0}{R_L + Z_0}\right)} = \frac{(R_L + \cancel{Z_0}) + (R_L - \cancel{Z_0})}{(\cancel{R_L} + Z_0) - (\cancel{R_L} - Z_0)}$$

Standing Wave Ratio: Real Load (cont.)

Hence

$$\text{SWR} = \frac{R_L}{Z_0} \quad \left(R_L \geq Z_0\right)$$

Case b: $R_L \leq Z_0$

$$\text{SWR} = \frac{1+|\Gamma_L|}{1-|\Gamma_L|} = \frac{1-\left(\dfrac{R_L - Z_0}{R_L + Z_0}\right)}{1+\left(\dfrac{R_L - Z_0}{R_L + Z_0}\right)} = \frac{\left(R_L + Z_0\right) - \left(R_L - Z_0\right)}{\left(R_L + Z_0\right) + \left(R_L - Z_0\right)}$$

Hence

$$\text{SWR} = \frac{Z_0}{R_L} \quad \left(R_L \leq Z_0\right)$$

Standing Wave Ratio: Real Load (cont.)

Hence, for a real load impedance we have:

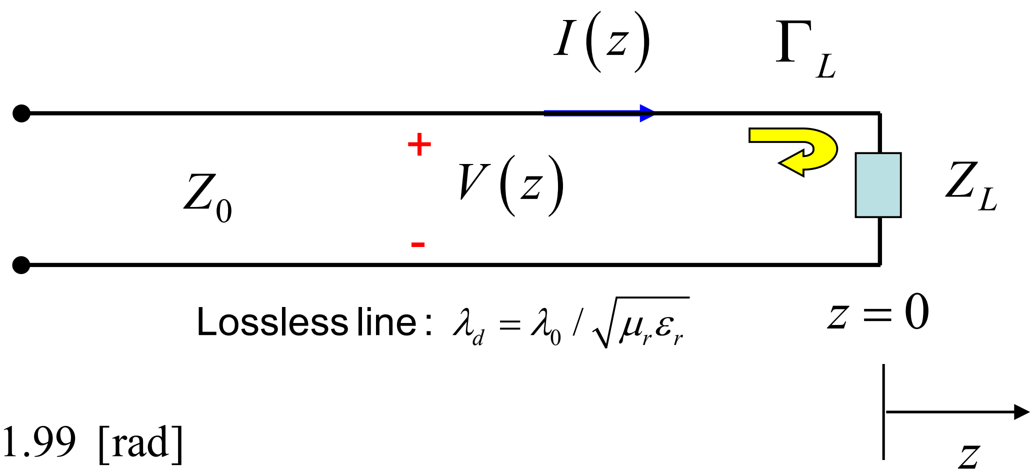
$$\mathrm{SWR} = \max\left(\frac{R_L}{Z_0}, \frac{Z_0}{R_L}\right)$$

(We choose whichever one is greater than 1.)

Example (6.6, Shen and Kong)

Given: $Z_L = 17.4 - j30\ [\Omega]$ and $Z_0 = 50\ [\Omega]$

Find: Γ_L, SWR, $z_{\min}$, $V^N_{\max}$, $V^N_{\min}$



Lossless line : $\lambda_d = \lambda_0 / \sqrt{\mu_r \varepsilon_r}$

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$$

⇩

$$\Gamma_L = -0.24 - j0.55 = 0.6\, e^{-j(1.99)}$$

$$|\Gamma_L| = 0.6$$

$$\phi = \angle \Gamma_L = -1.99\ [\text{rad}]$$

$$\text{SWR} = \frac{V_{\max}}{V_{\min}} = \frac{1+|\Gamma_L|}{1-|\Gamma_L|} = \frac{1+0.6}{1-0.6} = 4.0$$

⇨

$$V^N_{\max} = V_{\max} / |V^+| = 1 + |\Gamma_L| = 1.6$$

$$V^N_{\min} = V_{\min} / |V^+| = 1 - |\Gamma_L| = 0.4$$

$V_{\max}$ when $\phi + 2\beta z_{\max} = 0, \pm 2\pi, \ldots$

$V_{\min}$ when $\phi + 2\beta z_{\min} = \pm\pi, \pm 3\pi, \ldots$

$$z_{\min} = \frac{-\pi - \phi}{2\beta} = \frac{(-\pi + 1.99)}{2(2\pi / \lambda_d)} = -0.092\lambda_d$$

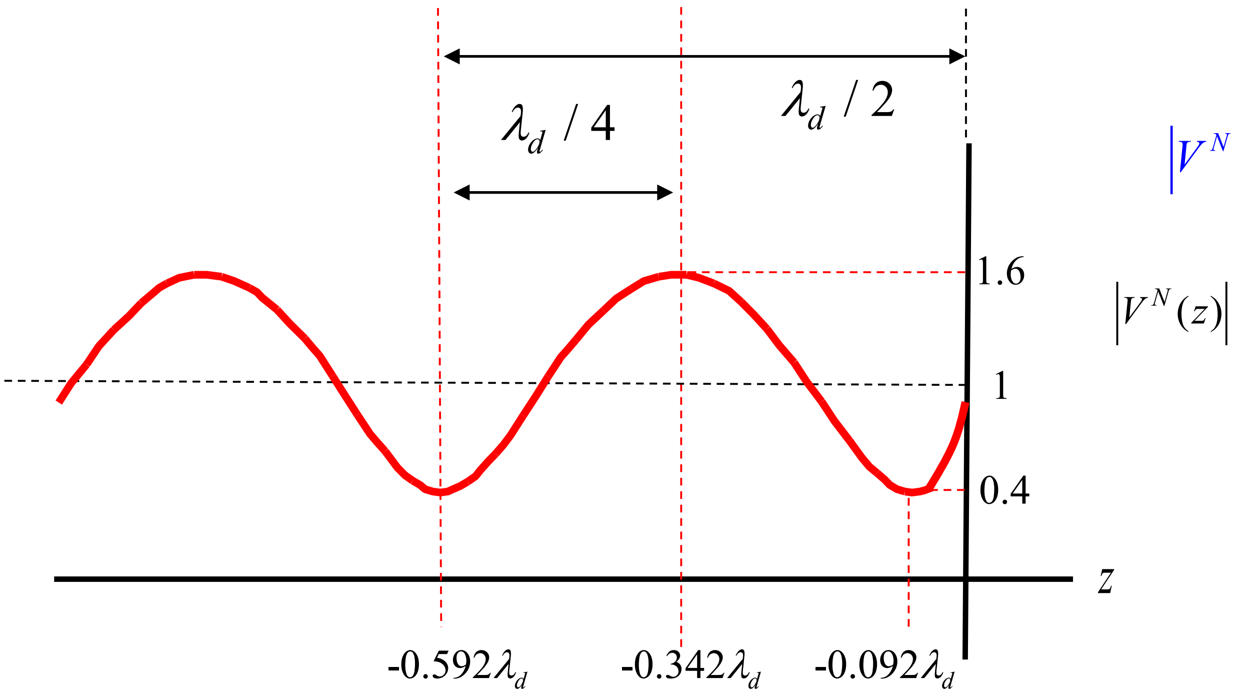
(Choose $-\pi$: pick the value of $z_{\min}$ closest to the load.)

Example (6.6, Shen and Kong) (cont.)

$$\left|V^N(z)\right| = \left|1 + |\Gamma_L| e^{+j(\phi + 2\beta z)}\right|$$

⇩

$$\left|V^N(z)\right| = \left|1 + (0.6) e^{+j(-1.99 + 4\pi(z/\lambda_d))}\right|$$



$$\Gamma_L = -0.24 - j0.55 = 0.6\, e^{-j(1.99)}$$

$$\text{SWR} = 4.0$$

$$z_{\min} = -0.092\lambda_d$$

$$V_{\max} / \left|V^+\right| = 1.6$$

$$V_{\min} / \left|V^+\right| = 0.4$$

Example (6.6, Shen and Kong) (cont.)

Reverse problem:

This problem has practical significance: often we are interested in figuring out what an unknown load is.

Given: $\text{SWR} = 4.0$ $\quad z_{\min} = -0.092\lambda_d$

What is the unknown load impedance?

$$\text{SWR} = \frac{1+|\Gamma_L|}{1-|\Gamma_L|} = 4.0 \Rightarrow \text{SWR}\left(1-|\Gamma_L|\right) = 1+|\Gamma_L| \Rightarrow |\Gamma_L| = \frac{\text{SWR}-1}{\text{SWR}+1}$$

so

$$|\Gamma_L| = \frac{4.0-1}{4.0+1} = 0.6$$

$$|\Gamma_L| = 0.6$$

$$z_{\min} = \frac{-\pi-\phi}{2\beta} = \frac{(-\pi-\phi)}{2(2\pi/\lambda_d)} = -0.092\lambda_d$$

Solve for ϕ.

$$\Rightarrow \phi = \angle\Gamma_L = -1.99 \text{ [rad]}$$

$$\Gamma_L = 0.6\, e^{-j(1.99)}$$

(Note: Any multiple of 2π can be added to ϕ.)

Example (6.6, Shen and Kong) (cont.)

$$\Gamma_L = 0.6\, e^{-j(1.99)}$$

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} \Rightarrow (Z_L + Z_0)\Gamma_L = Z_L - Z_0$$

$$\Rightarrow Z_L = Z_0\left(\frac{1+\Gamma_L}{1-\Gamma_L}\right)$$

Hence, we have $Z_L = 50\left(\frac{1+0.6\, e^{-j(1.99)}}{1-0.6\, e^{-j(1.99)}}\right)$

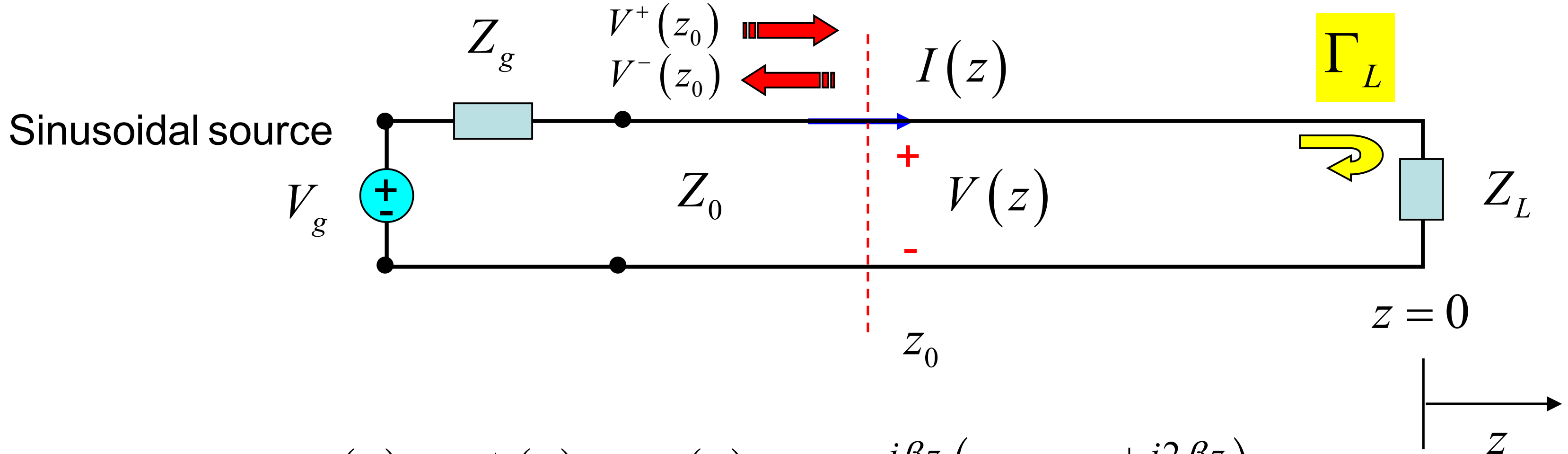
The calculation yields:

$$Z_L = 17.4 - j30\ [\Omega]$$

We have solved for the unknown load impedance!

Generalized Reflection Coefficient

Define the "generalized reflection coefficient" at a point z_0 on the line:



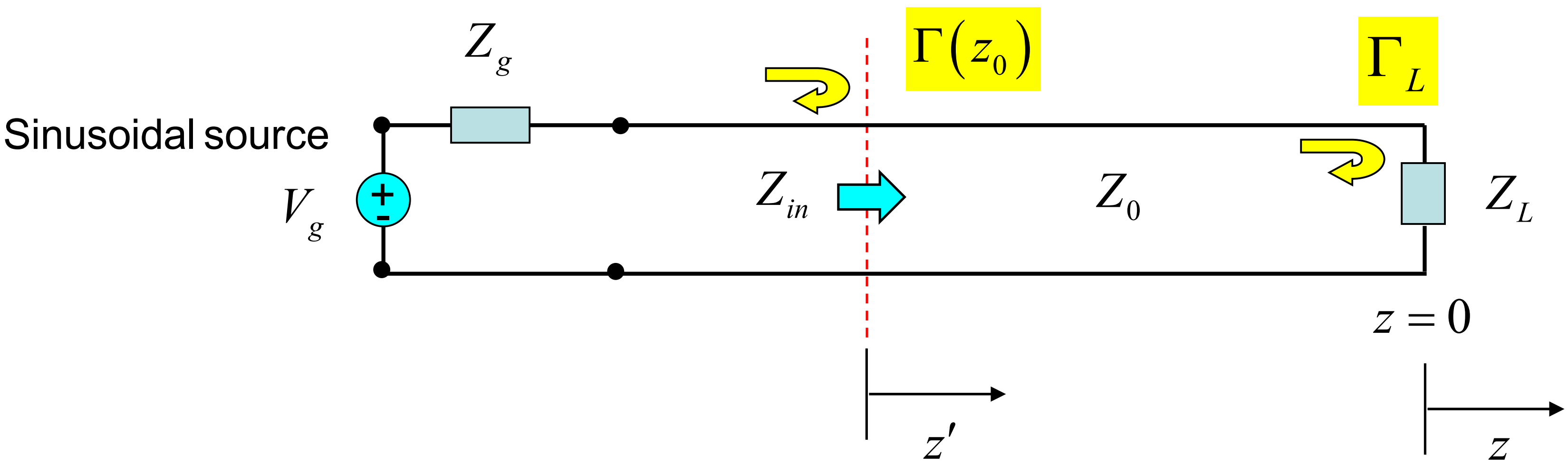
$$V(z) = V^+(z) + V^-(z) = Ae^{-j\beta z}\left(1 + \Gamma_L e^{+j2\beta z}\right)$$

$$V^+(z) = Ae^{-j\beta z}, \quad V^-(z) = A\Gamma_L e^{+j\beta z}$$

$$\Gamma(z_0) \equiv \frac{V^-(z_0)}{V^+(z_0)} \quad \Rightarrow \quad \Gamma(z_0) = \Gamma_L e^{+j2\beta z_0}$$

Generalized Reflection Coefficient (cont.)

We identify $\Gamma(z_0)$ as the reflection coefficient at the point z_0, with Z_{in} acting as the load impedance.



Hence

$$\Gamma(z_0) = \frac{Z_{in}(z_0) - Z_0}{Z_{in}(z_0) + Z_0}$$

Generalized Reflection Coefficient (cont.)

$$\Gamma(z_0) = \frac{Z_{in}(z_0) - Z_0}{Z_{in}(z_0) + Z_0}$$

Hence

$$\left(Z_{in}(z_0) + Z_0\right)\Gamma(z_0) = Z_{in}(z_0) - Z_0$$

$$\Downarrow$$

$$Z_{in}(z_0)\left[1 - \Gamma(z_0)\right] = Z_0\left[1 + \Gamma(z_0)\right]$$

$$\Downarrow$$

$$Z_{in}(z_0) = Z_0\left(\frac{1 + \Gamma(z_0)}{1 - \Gamma(z_0)}\right)$$

Generalized Reflection Coefficient (cont.)

Define a normalized input impedance at point z_0:

$$Z_{in}^{N}(z_0) \equiv \frac{Z_{in}(z_0)}{Z_0}$$

We then have, from the last slide

$$Z_{in}^{N}(z_0) = \left(\frac{1+\Gamma(z_0)}{1-\Gamma(z_0)} \right)$$

Note:
This can be used as an alternative formula to the "tangent formula" for calculating input impedance at $z_0 = -d$.

$$\Gamma(z_0) = \Gamma_L \, e^{+j2\beta z_0}$$

(This is the starting point for the Smith chart discussion.)

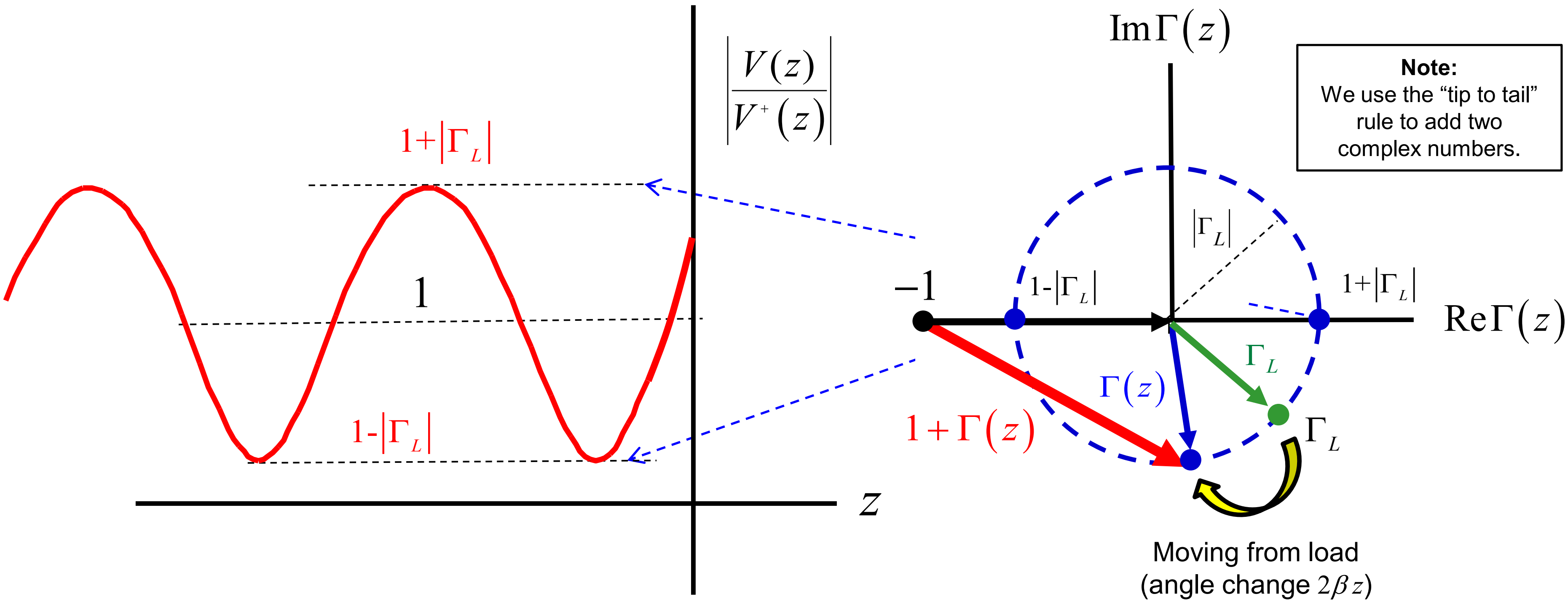
Crank Diagram

$$\left|V^N(z)\right| = \left|1+\left|\Gamma_L\right|e^{+j(\phi+2\beta z)}\right| = \left|1+\Gamma(z)\right|$$

where $\Gamma(z) \equiv \Gamma_L e^{+j(2\beta z)}$

$$\Gamma_L = \left|\Gamma_L\right| e^{+j\phi}$$

Note: $\Gamma(z) = \dfrac{V^-(z)}{V^+(z)}$



Note: We go all the way around the crank diagram when $z = -\lambda_d / 2$:

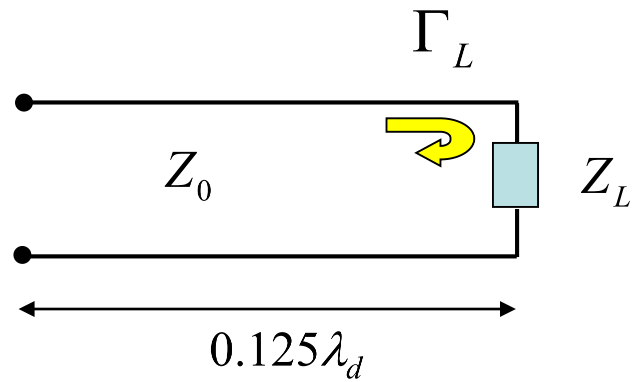
$$2\beta z = 2\left(\frac{2\pi}{\lambda_d}\right)\left(-\frac{\lambda_d}{2}\right) = -2\pi$$

Example

Given: $Z_L = 100\ [\Omega]$ and $Z_0 = 50\ [\Omega]$

Calculate the reflection coefficient and the input impedance at $z_0 = -0.125\lambda_d$

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{1}{3}$$



Lossless line: $\lambda_d = \lambda_0 / \sqrt{\mu_r \varepsilon_r}$

$$\Gamma(z_0) = \Gamma_L\, e^{j2\beta z_0}$$

$$= \left(\frac{1}{3}\right) e^{j2\left(\frac{2\pi}{\lambda_d}\right)(-0.125\lambda_d)} = \left(\frac{1}{3}\right) e^{-j\pi/2}$$

so

$$\Gamma(z_0) = -j/3$$

$$Z_{in}^N(z_0) = \left(\frac{1+\Gamma(z_0)}{1-\Gamma(z_0)}\right) = \frac{1-j/3}{1+j/3} = 0.8 + j(0.6)$$

$$Z_{in}(z_0) = (50)\, Z_{in}^N(z_0)$$

so

$$Z_{in}(z_0) = 40 + j(30)\ \ [\Omega]$$

Appendix: Summary of Formulas

$$\text{SWR} \equiv \frac{V_{\max}}{V_{\min}}$$

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$$

$$\text{SWR} = \frac{1+|\Gamma_L|}{1-|\Gamma_L|}$$

$$\Gamma(z_0) \equiv \frac{V^-(z_0)}{V^+(z_0)}$$

$$\text{SWR} = \max\left(\frac{R_L}{Z_0}, \frac{Z_0}{R_L}\right)$$

$$\Gamma(z_0) = \Gamma_L \, e^{+j2\beta z_0}$$

$$|\Gamma_L| = \frac{\text{SWR}-1}{\text{SWR}+1}$$

$$\Gamma(z_0) = \frac{Z_{in}(z_0) - Z_0}{Z_{in}(z_0) + Z_0}$$

$$Z_{in}(z_0) = Z_0\left(\frac{1+\Gamma(z_0)}{1-\Gamma(z_0)}\right)$$

$$Z_L = Z_0\left(\frac{1+\Gamma_L}{1-\Gamma_L}\right)$$

$$Z_{in}^N(z_0) \equiv \frac{Z_{in}(z_0)}{Z_0}$$