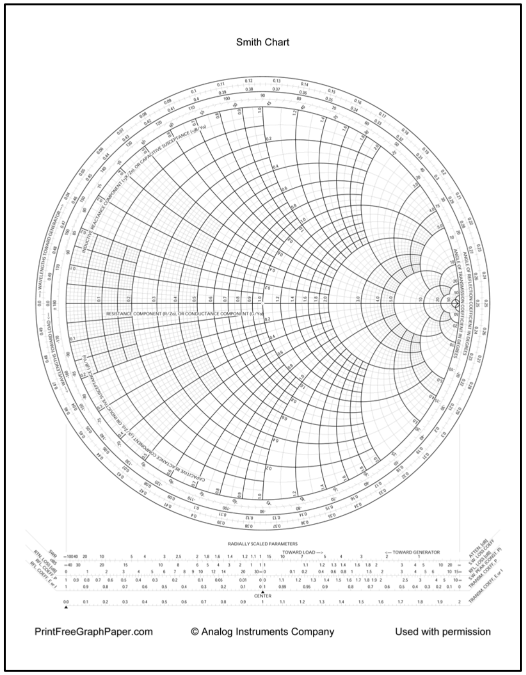
ECE 3317
Applied Electromagnetic Waves

Prof. David R. Jackson

Fall 2023



Notes 12
Transmission Lines
(Smith Chart)

Smith Chart

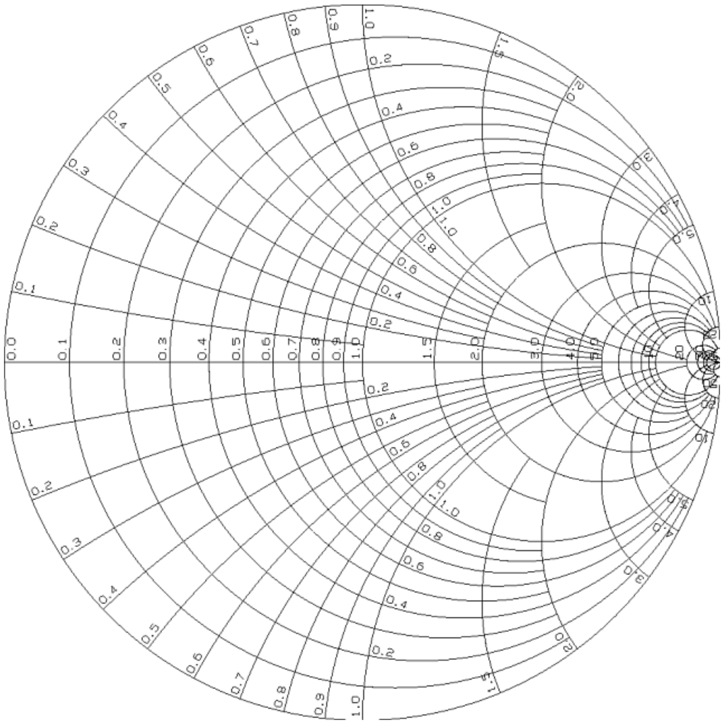
The Smith chart is a very convenient graphical tool for analyzing transmission lines and studying their behavior.

A network analyzer (Agilent N5245A PNA-X) showing a Smith chart.

Smith Chart (cont.)

From Wikipedia:

Phillip Hagar Smith
(April 29, 1905–August 29, 1987) was an electrical engineer, who became famous for his invention of the Smith chart. Smith graduated from Tufts College in 1928. While working for RCA, he invented his eponymous Smith chart. He retired from Bell Labs in 1970.

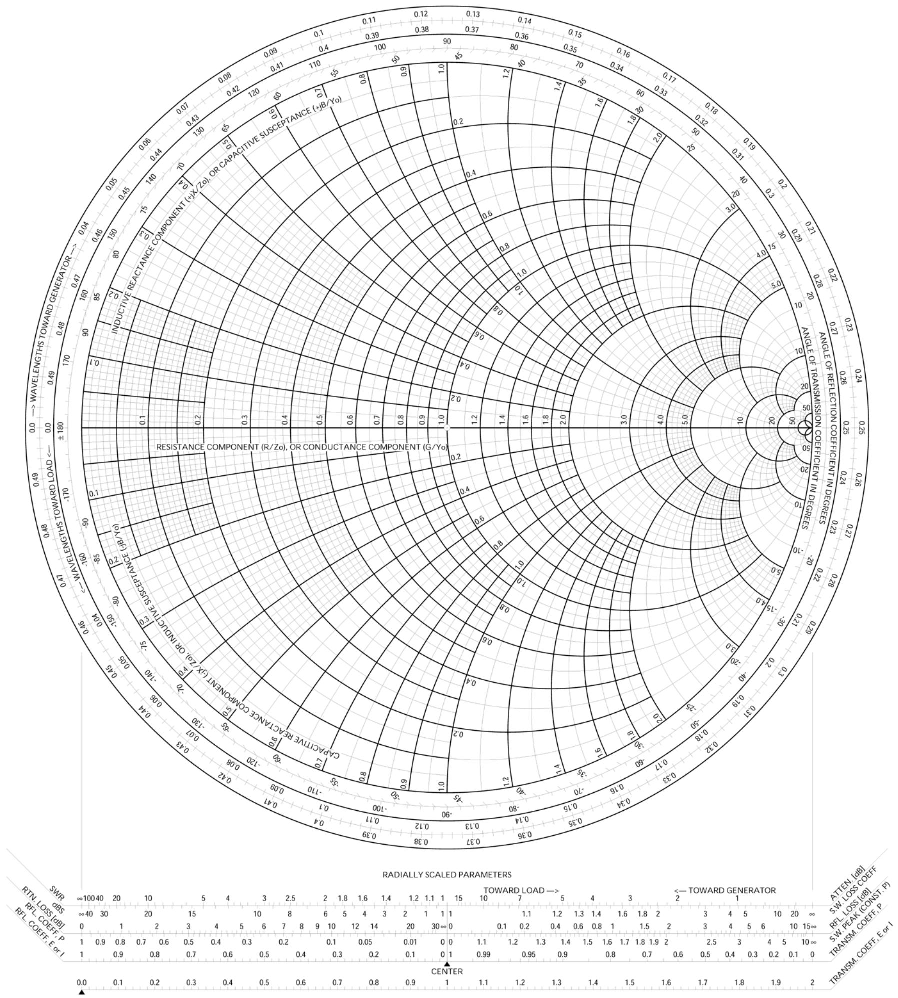


Phillip Smith invented the Smith Chart in 1939 while he was working for The Bell Telephone Laboratories. When asked why he invented this chart, Smith explained: “*From the time I could operate a slide rule, I've been interested in graphical representations of mathematical relationships.*”

In 1969 he published the book *Electronic Applications of the Smith Chart in Waveguide, Circuit, and Component Analysis*, a comprehensive work on the subject.

Smith Chart (cont.)

Smith Chart



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Smith Chart (cont.)

The Smith chart is really a complex Γ plane:

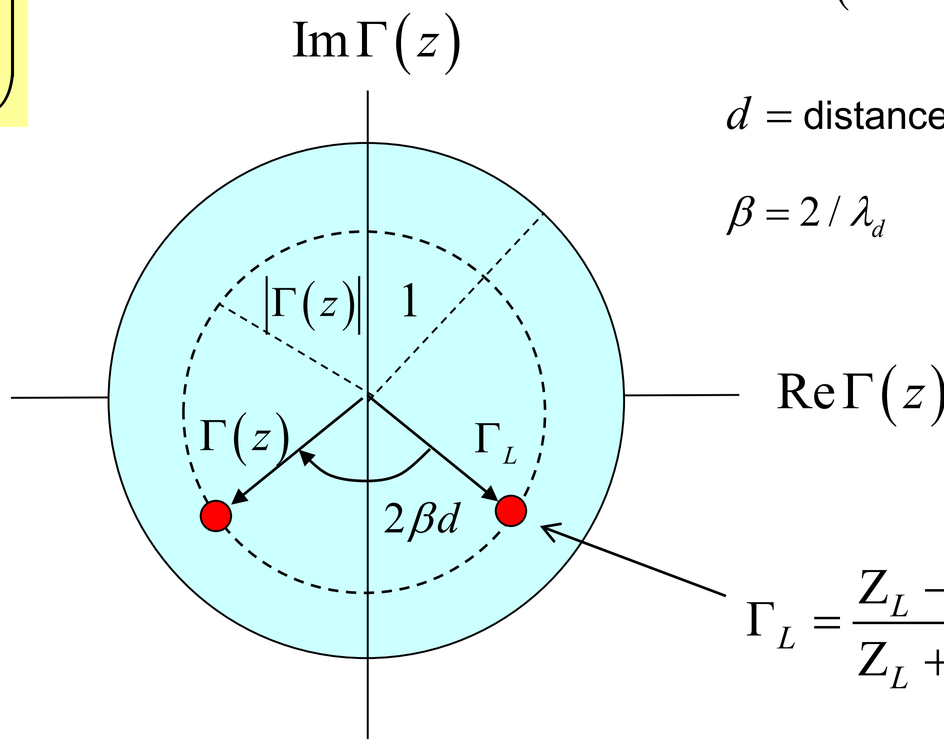
$$\Gamma(z) = \Gamma_L \, e^{j2\beta z}$$
$$= \Gamma_L \, e^{-j2\beta d}$$
$$(z = -d)$$

Recall:

$$Z_{in}^{N}(z) = \left(\frac{1+\Gamma(z)}{1-\Gamma(z)} \right)$$

d = distance from the load

$\beta = 2 / \lambda_d$



$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$$

Note:
We rotate clockwise as we go from the load towards the generator (z becomes more negative).

Smith Chart (cont.)

$$Z_{in}^{N} = R_{in}^{N} + jX_{in}^{N} = \left(\frac{1+\Gamma(z)}{1-\Gamma(z)} \right)$$

Denote:

$$\Gamma(z) = x + jy$$ (complex variable)

$$R_{in}^{N} + jX_{in}^{N} = \left(\frac{(1+x)+jy}{(1-x)-jy} \right)$$

so

$$\left[(1-x)-jy\right]\left(R_{in}^{N} + jX_{in}^{N}\right) = \left[(1+x)+jy\right]$$

Real part: $(1-x)R_{in}^{N} + yX_{in}^{N} = 1+x$

Imaginary part: $(1-x)X_{in}^{N} - yR_{in}^{N} = y$

Smith Chart (cont.)

Two equations:

$$\left(1-x\right)R_{in}^{N}+yX_{in}^{N}=1+x$$
$$\left(1-x\right)X_{in}^{N}-yR_{in}^{N}=y$$

Goal: Eliminate the reactance.

From the second one we have

$$X_{in}^{N}=\left(\frac{y}{1-x}\right)\left(1+R_{in}^{N}\right)$$

Substituting into the first one, and then multiplying by $(1\text{-}x)$, we have

$$\left(1-x\right)R_{in}^{N}+y\left[\left(\frac{y}{1-x}\right)\left(1+R_{in}^{N}\right)\right]=1+x$$

Multiply by $1\text{-}x$

$$\left(1-x\right)^{2}R_{in}^{N}+y^{2}\left(1+R_{in}^{N}\right)=\left(1+x\right)\left(1-x\right)$$

Smith Chart (cont.)

Algebraic simplification:

$$(1-x)^2 R_{in}^N + y^2 \left(1+R_{in}^N\right) = (1+x)(1-x)$$

Simplify the right-hand side.

$$(1-x)^2 R_{in}^N + y^2 \left(1+R_{in}^N\right) = 1-x^2$$

Expand the $(1\text{-}x)^2$ term, collect terms.

$$x^2 \left(1+R_{in}^N\right) - 2xR_{in}^N + \left(R_{in}^N - 1\right) + y^2 \left(1+R_{in}^N\right) = 0$$

Take the third term across the equal sign.

$$x^2 \left(1+R_{in}^N\right) - 2xR_{in}^N + y^2 \left(1+R_{in}^N\right) = 1-R_{in}^N$$

Divide by the term in parenthesis.

$$x^2 - 2x\left(\frac{R_{in}^N}{1+R_{in}^N}\right) + y^2 = \frac{1-R_{in}^N}{1+R_{in}^N}$$

Smith Chart (cont.)

$$x^2 - 2x\left(\frac{R_{in}^N}{1+R_{in}^N}\right) + y^2 = \frac{1-R_{in}^N}{1+R_{in}^N}$$

Complete the square.

$$\left(x - \frac{R_{in}^N}{1+R_{in}^N}\right)^2 + y^2 = \frac{1-R_{in}^N}{1+R_{in}^N} + \left(\frac{R_{in}^N}{1+R_{in}^N}\right)^2$$

Put the right-hand side terms over a common denominator.

$$\left(x - \frac{R_{in}^N}{1+R_{in}^N}\right)^2 + y^2 = \frac{\left(1-R_{in}^N\right)\left(1+R_{in}^N\right) + \left(R_{in}^N\right)^2}{\left(1+R_{in}^N\right)^2}$$

Simplify the right-hand side.

$$\left(x - \frac{R_{in}^N}{1+R_{in}^N}\right)^2 + y^2 = \frac{1}{\left(1+R_{in}^N\right)^2}$$

Smith Chart (cont.)

$$\left(x-\frac{R_{in}^N}{1+R_{in}^N}\right)^2+y^2=\frac{1}{\left(1+R_{in}^N\right)^2}$$

Recall : $\Gamma(z)=x+jy$

This defines the equation of a circle:

Center:

$$(x_c,y_c)=\left(\frac{R_{in}^N}{1+R_{in}^N},0\right)$$

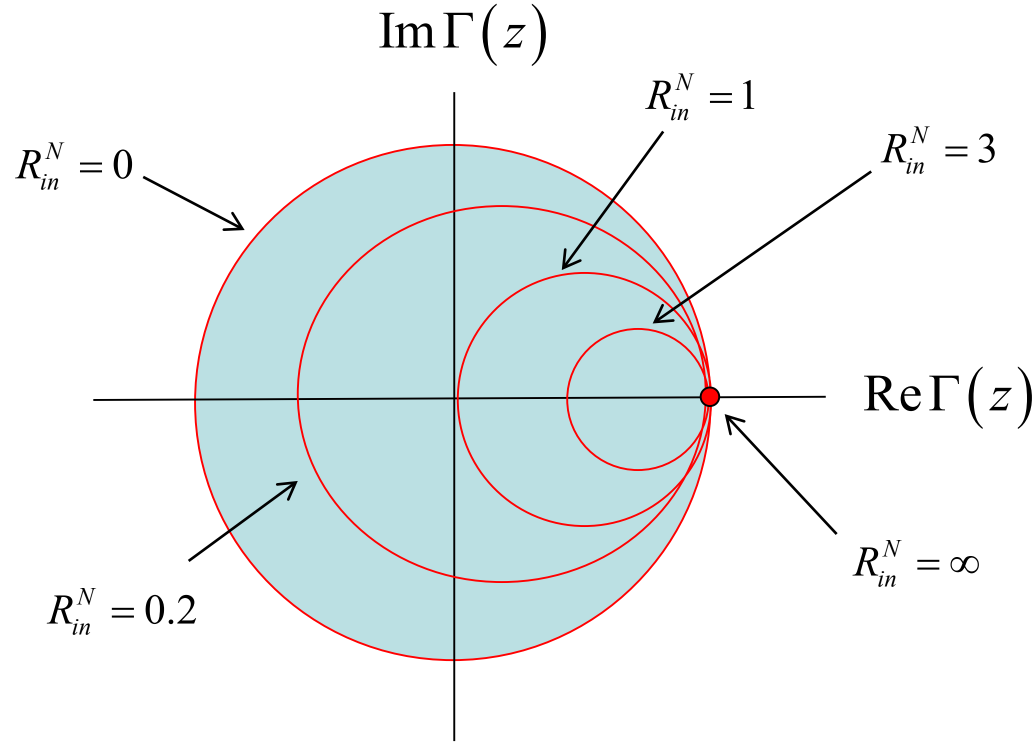
Radius:

$$R=\frac{1}{1+R_{in}^N}$$

Note:

$$x_c+R=1$$

(All circles go through $(1,0)$.)



Smith Chart (cont.)

$$\left(1-x\right)R_{in}^{N}+yX_{in}^{N}=1+x$$

$$\left(1-x\right)X_{in}^{N}-yR_{in}^{N}=y$$

Next, we eliminate the resistance from the two equations.

From the second one we have:

$$R_{in}^{N}=\frac{\left(1-x\right)X_{in}^{N}-y}{y}$$

Substituting into the first one, we have

$$\left(1-x\right)\left[\frac{\left(1-x\right)X_{in}^{N}-y}{y}\right]+yX_{in}^{N}=1+x$$

Smith Chart (cont.)

Algebraic simplification:

$$(1-x)\left[\frac{(1-x)X_{in}^{N}-y}{y}\right]+yX_{in}^{N}=1+x$$

Multiply by y.

$$(1-x)^2 X_{in}^{N}-y(1-x)+y^2X_{in}^{N}-y(1+x)=0$$

Cancel xy terms, collect y terms.

$$(1-x)^2 X_{in}^{N}-2y+y^2X_{in}^{N}=0$$

Divide by ${X_{in}}^{N}$

$$(x-1)^2-\left(\frac{2}{X_{in}^{N}}\right)y+y^2=0$$

Smith Chart (cont.)

$$(x-1)^2 - \left(\frac{2}{X_{in}^N}\right)y + y^2 = 0$$

Complete the square

$$(x-1)^2 + \left(y - \frac{1}{X_{in}^N}\right)^2 = \left(\frac{1}{X_{in}^N}\right)^2$$

Note:

(All circles go through $(1,0)$.)

This defines the equation of a circle:

center: $(x_c, y_c) = \left(1, \frac{1}{X_{in}^N}\right)$

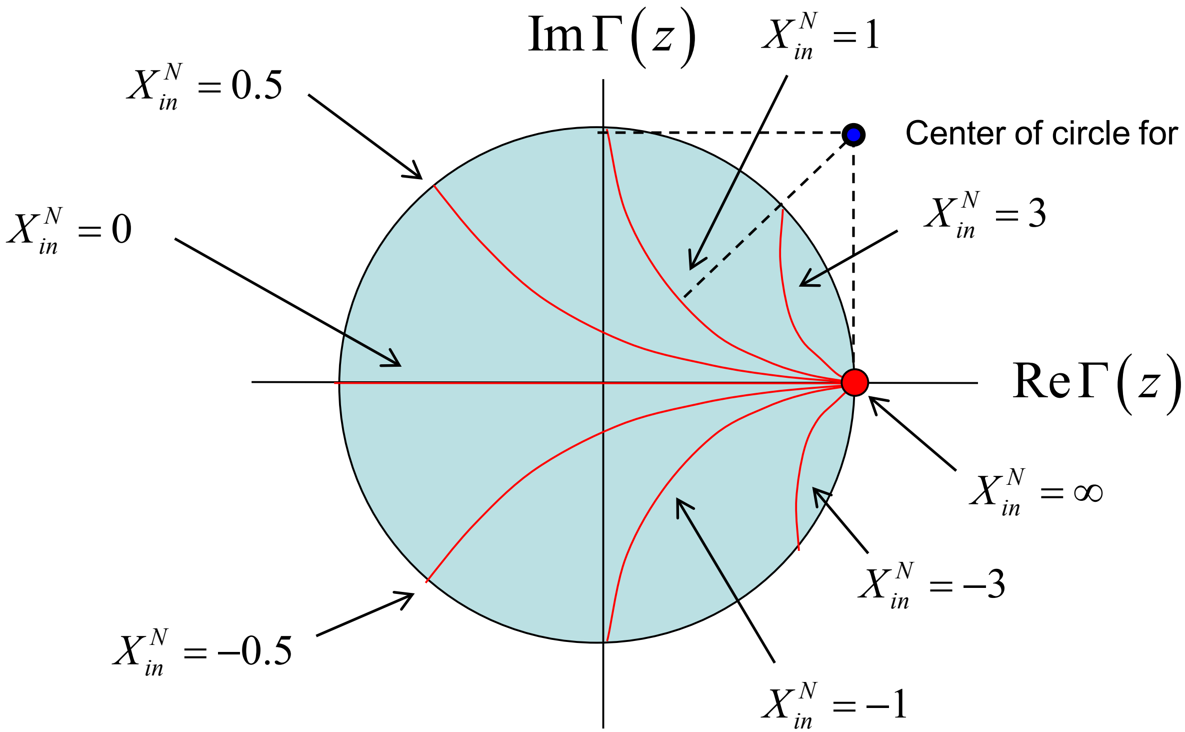
radius: $R = \frac{1}{\left|X_{in}^N\right|}$

Smith Chart (cont.)

$$\left(x_c, y_c\right) = \left(1, \frac{1}{X_{in}^N}\right) \qquad R = \frac{1}{\left|X_{in}^N\right|}$$

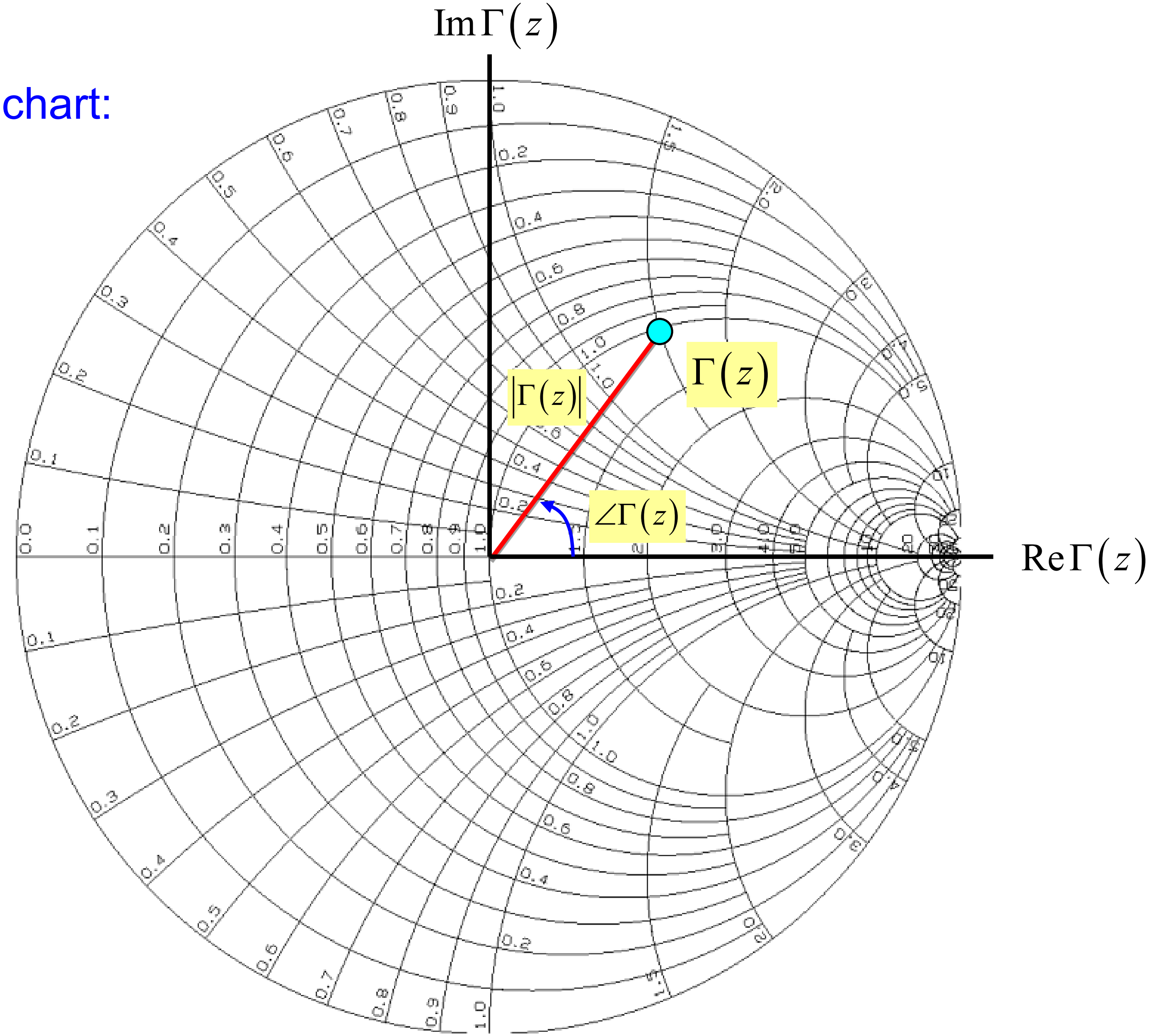
Note:

$$y_c = \pm R$$



Smith Chart (cont.)

Actual Smith chart:



Smith Chart (cont.)

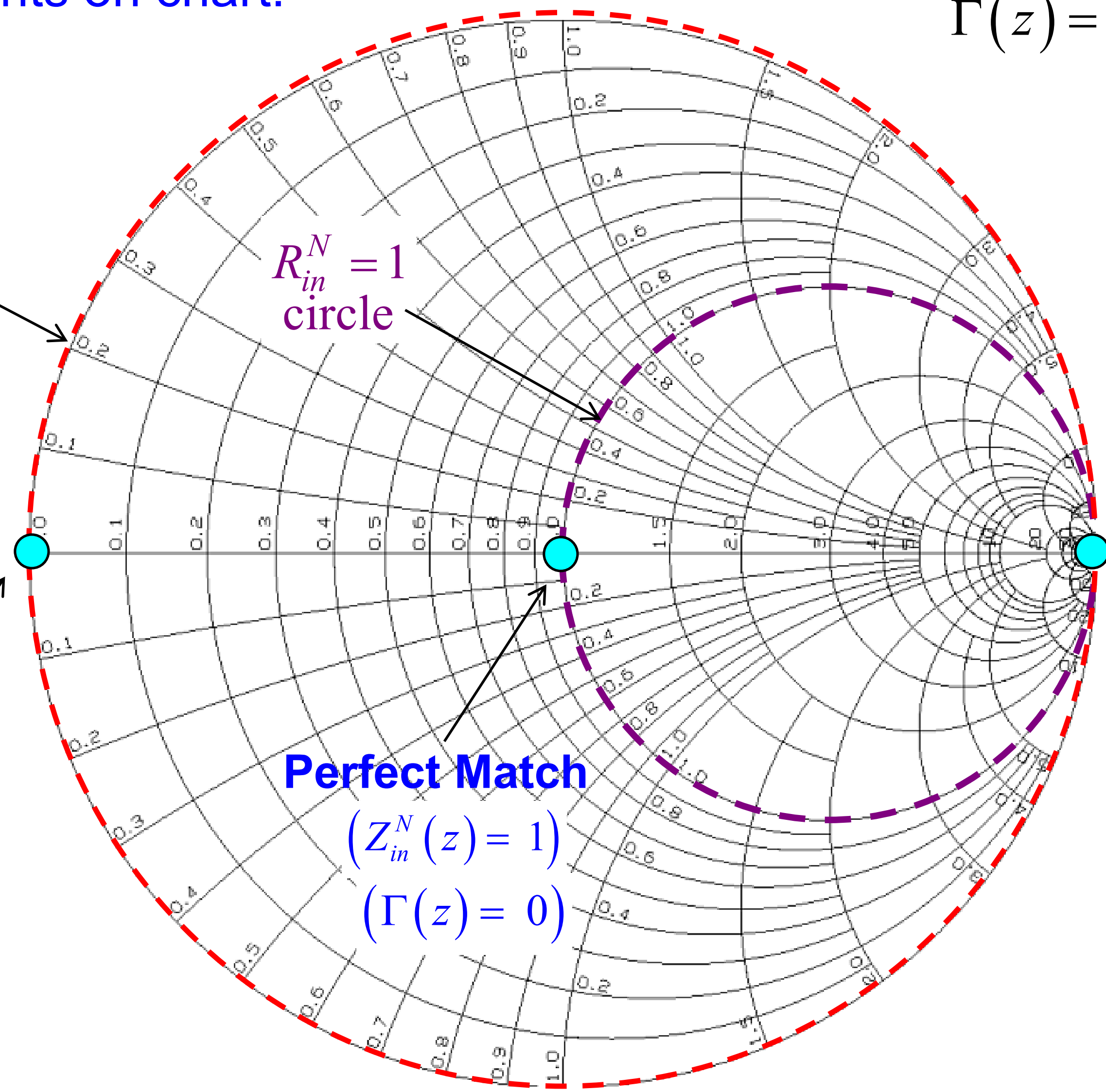
Important points on chart:

$$\Gamma(z) = \frac{Z_{in}^N(z) - 1}{Z_{in}^N(z) + 1}$$

$R_{in}^N = 0$

$\left(|\Gamma(z)| = 1\right)$

$R_{in}^N = 1$ circle



S/C

$\left(\Gamma(z) = -1\right)$

Perfect Match

$\left(Z_{in}^N(z) = 1\right)$

$\left(\Gamma(z) = 0\right)$

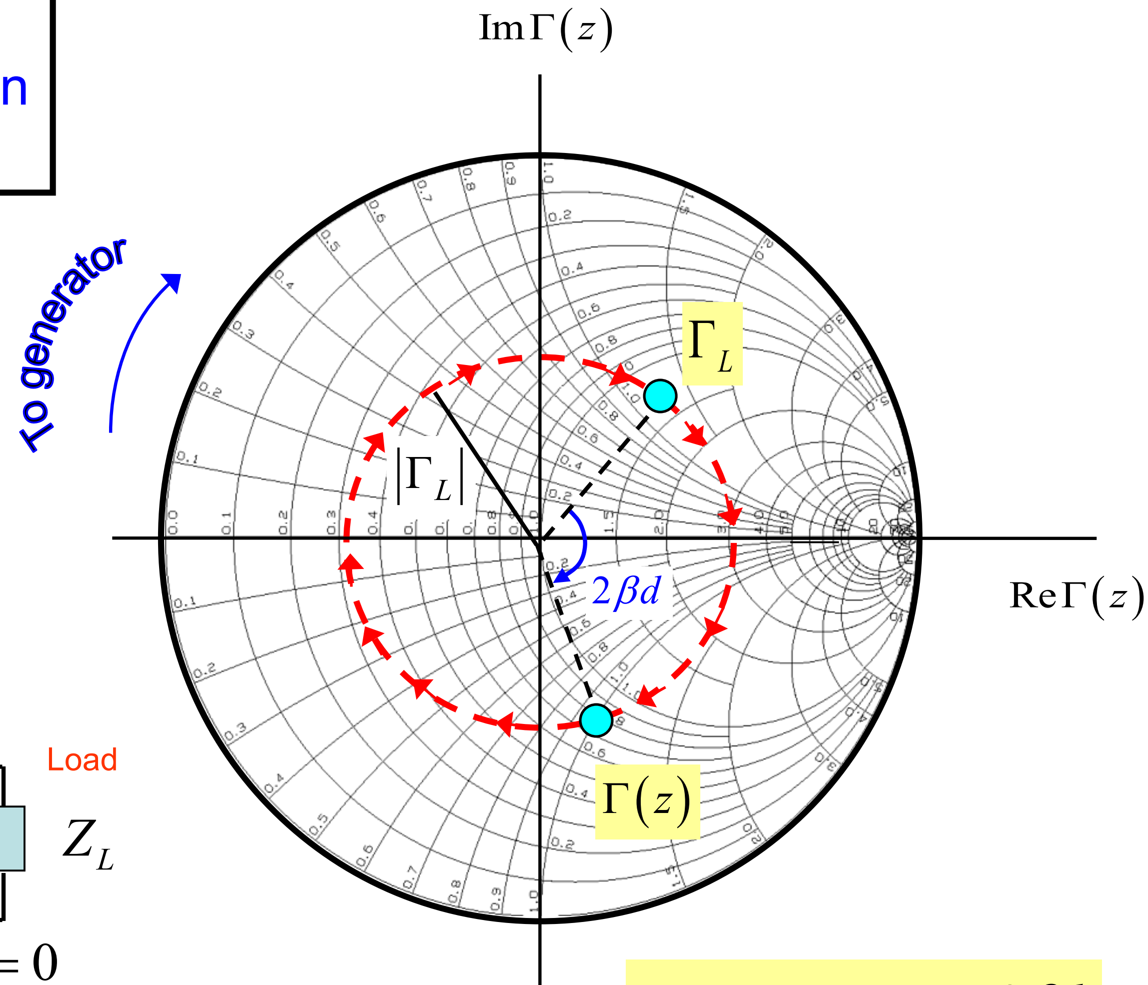
O/C

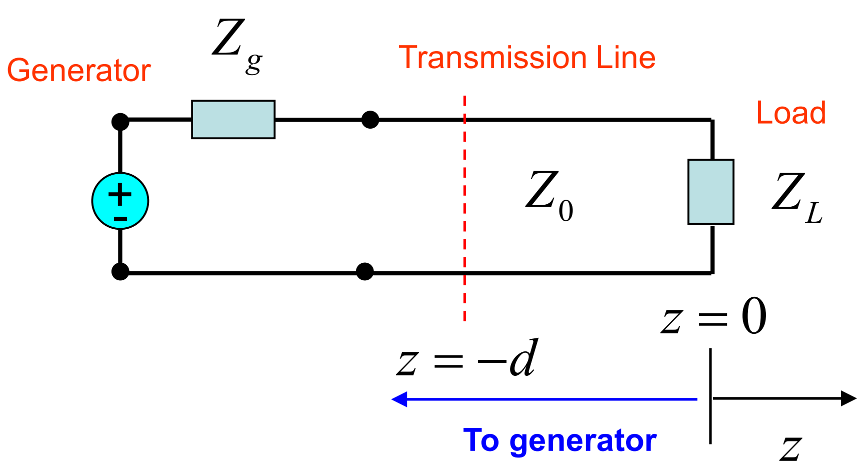
$\left(\Gamma(z) = 1\right)$

Smith Chart (cont.)

As we move along the transmission line, we stay on a circle of constant radius.

$$\Gamma(z) = \Gamma_L \, e^{-j2\beta d}$$





Angle change $= 2\beta d$

$\beta = 2 / \lambda_d$

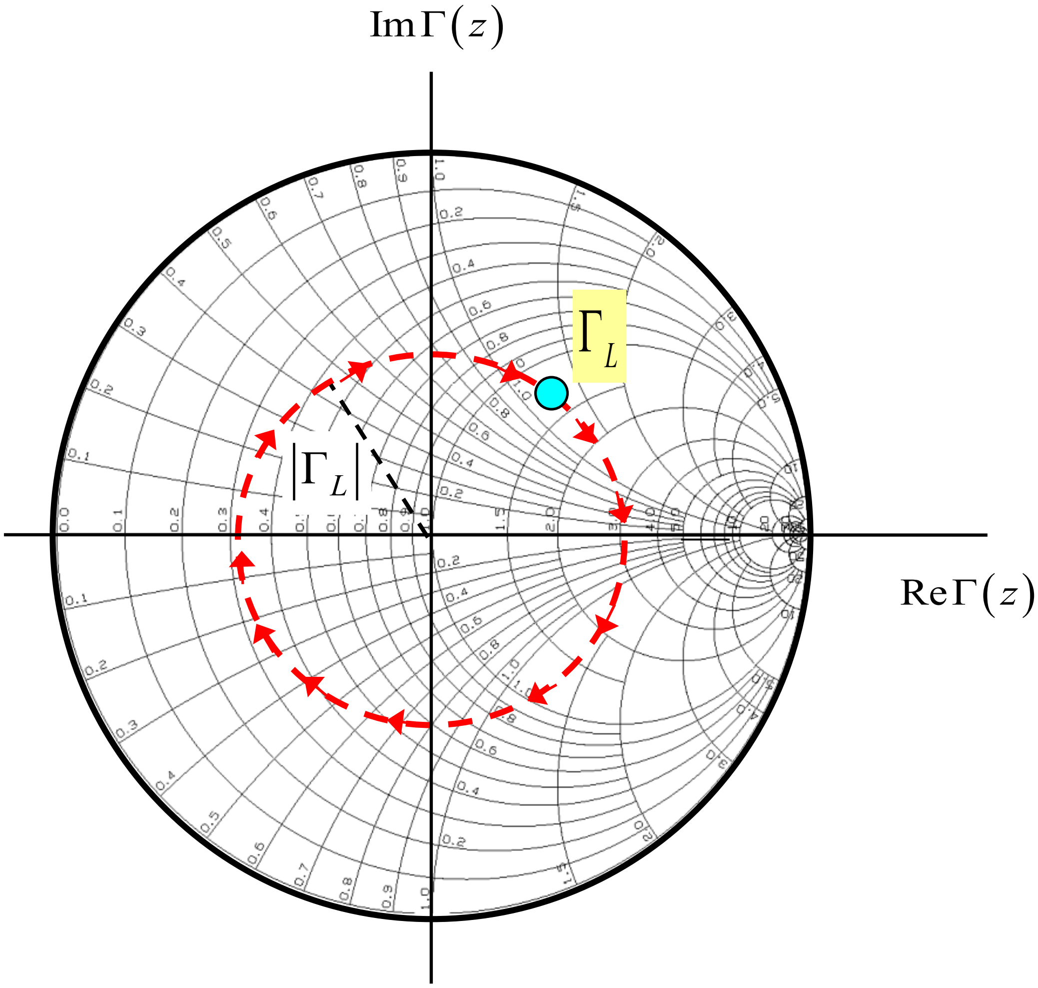
Smith Chart (cont.)

$$Z_{in}^{N}(z)=\left(\frac{1+\Gamma_L e^{-j2\beta d}}{1-\Gamma_L e^{-j2\beta d}}\right)$$

We go completely around the Smith chart when

$$d=\lambda_d/2$$

$$2\beta d=2\left(\frac{2\pi}{\lambda_d}\right)\left(\frac{\lambda_d}{2}\right)=2\pi$$



Recall: The impedance repeats every half of a wavelength.

Smith Chart (cont.)

In general, the angle change on the Smith chart as we go towards the generator is:

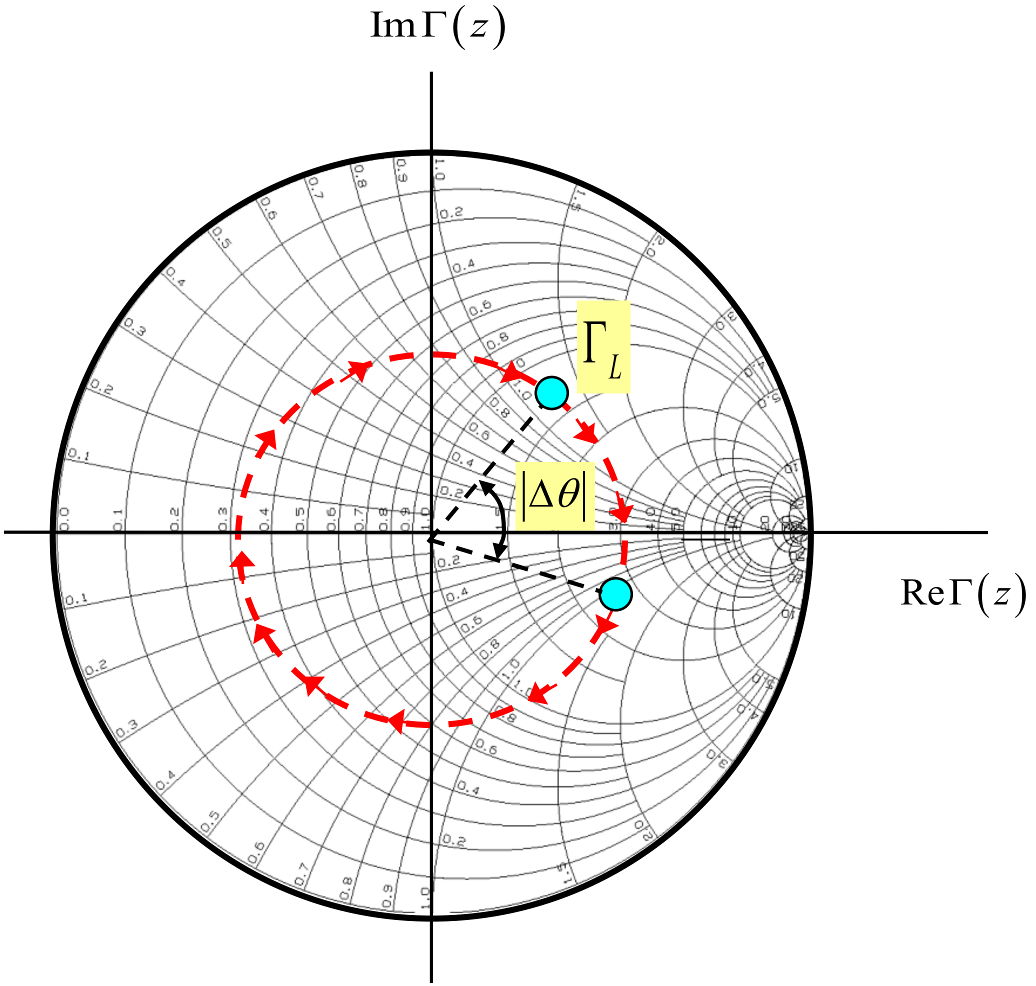
$$\Delta\theta = 2(\beta d)$$

or

$$\Delta\theta = 2(\beta d) = 2\left(\frac{2\pi}{\lambda_d}\right)d = 4\pi\left(\frac{d}{\lambda_d}\right)$$

so

$$\Delta\theta = 4\pi\left(\frac{d}{\lambda_d}\right)$$



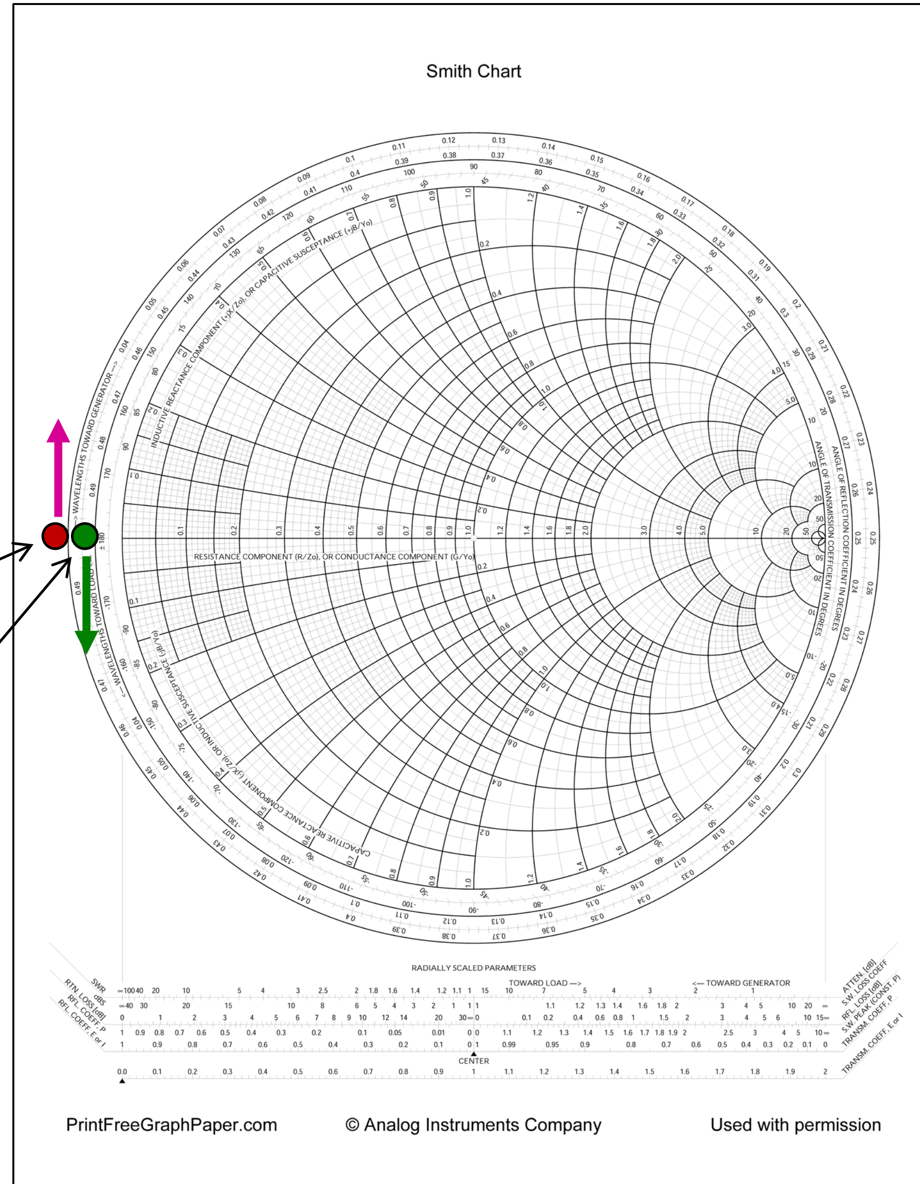
$$\Gamma(z) = \Gamma_L\, e^{j2\beta z} = \Gamma_L\, e^{-j2\beta d} = \Gamma_L e^{-j\Delta\theta}$$

The angle change is <u>twice</u> the electrical length change on the line: $\Delta\theta = 2(\beta d)$.

Smith Chart (cont.)

Scales:

The Smith chart already has wavelength scales on the perimeter for your convenience (so you don't need to measure angles).



The "wavelengths towards generator" scale is measured <u>clockwise</u>, starting (arbitrarily) at zero here.

The "wavelengths towards load" scale is measured <u>counterclockwise</u>, starting (arbitrarily) at zero here.

Reciprocal Property

$$Z_{in}^{N}(z)=\left(\frac{1+\Gamma_L e^{-j2\beta d}}{1-\Gamma_L e^{-j2\beta d}}\right)$$

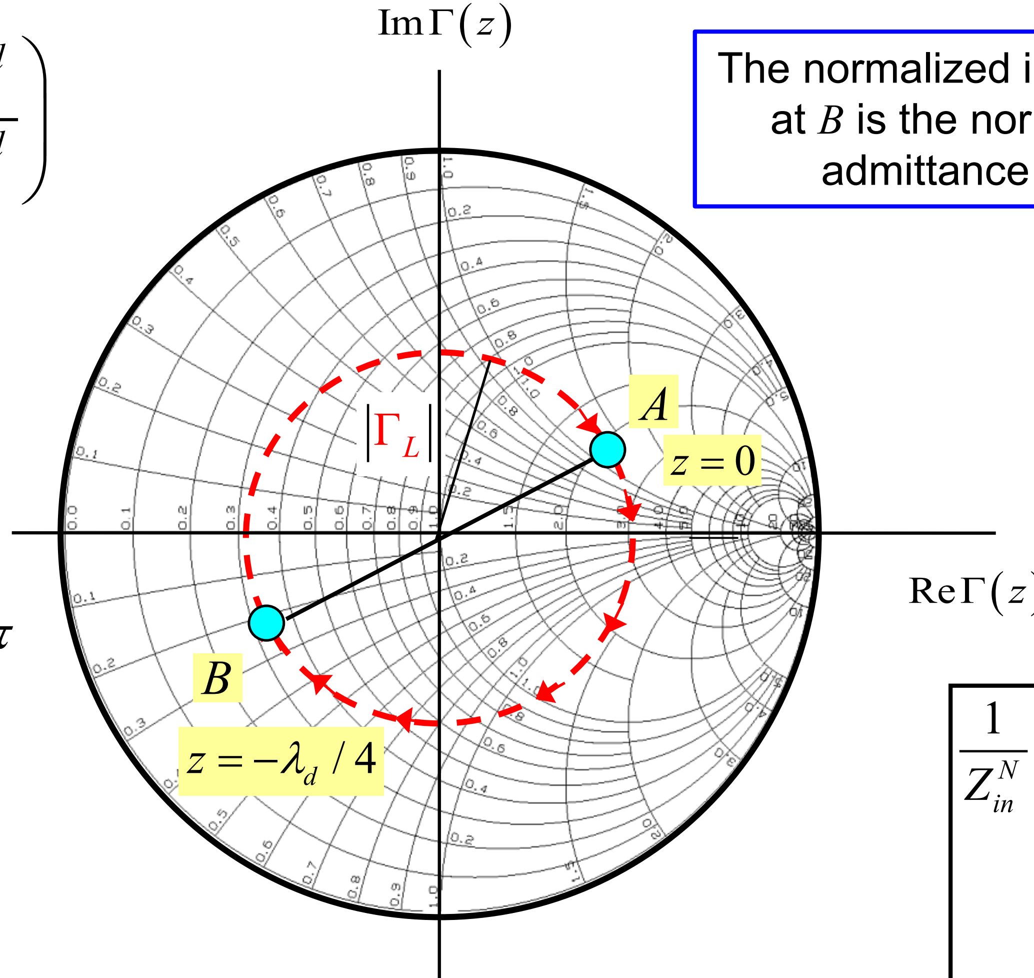
Go half-way around the Smith chart:

$$d=\lambda_d/4$$

$$2\beta d=2\left(\frac{2\pi}{\lambda_d}\right)\left(\frac{\lambda_d}{4}\right)=\pi$$

$$A:\ Z_{in}^{N}(0)=\left(\frac{1+\Gamma_L}{1-\Gamma_L}\right)$$

$$B:\ Z_{in}^{N}(-d)=\left(\frac{1-\Gamma_L}{1+\Gamma_L}\right)$$



$$Z_{in}^{N}(B)=\frac{1}{Z_{in}^{N}(A)}$$

The normalized impedance at B is the normalized admittance at A.

$$\frac{1}{Z_{in}^{N}}=\frac{1}{Z_{in}/Z_0}=\frac{1/Z_{in}}{1/Z_0}=\frac{Y_{in}}{Y_0}=Y_{in}^{N}$$

Normalized Voltage

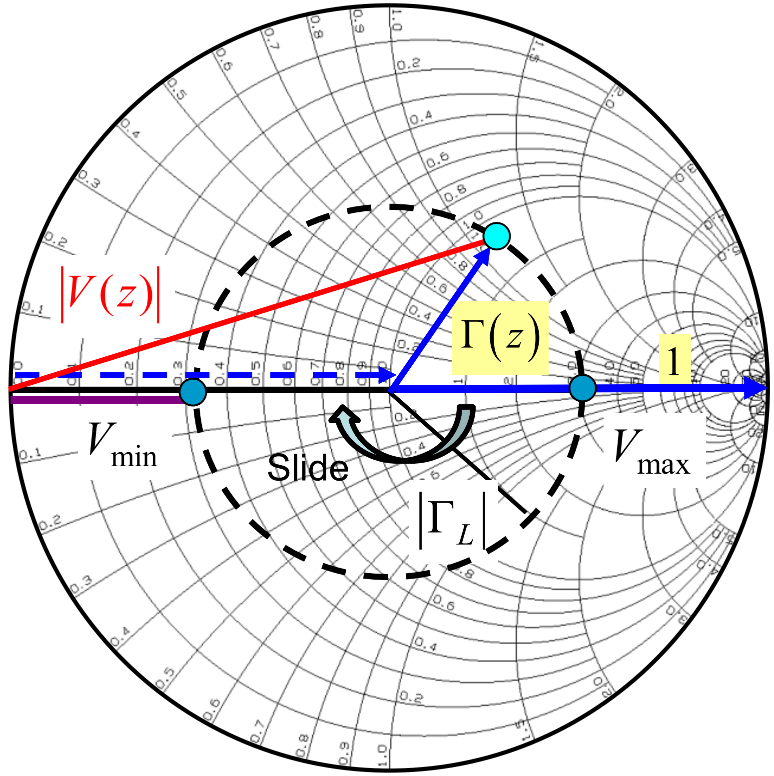
We slide the complex number "1" to the left so we can add the two complex numbers using the "tip to tail rule".

Normalized voltage

$$\left|V^N(z)\right| = \left|1 + \Gamma_L e^{j2\beta z}\right| = \left|1 + \Gamma(z)\right|$$

$$V_{\max} = \left|V(z)\right|_{\max} = 1 + \left|\Gamma_L\right|$$

$$V_{\min} = \left|V(z)\right|_{\min} = 1 - \left|\Gamma_L\right|$$



We can use the Smith chart as a crank diagram!

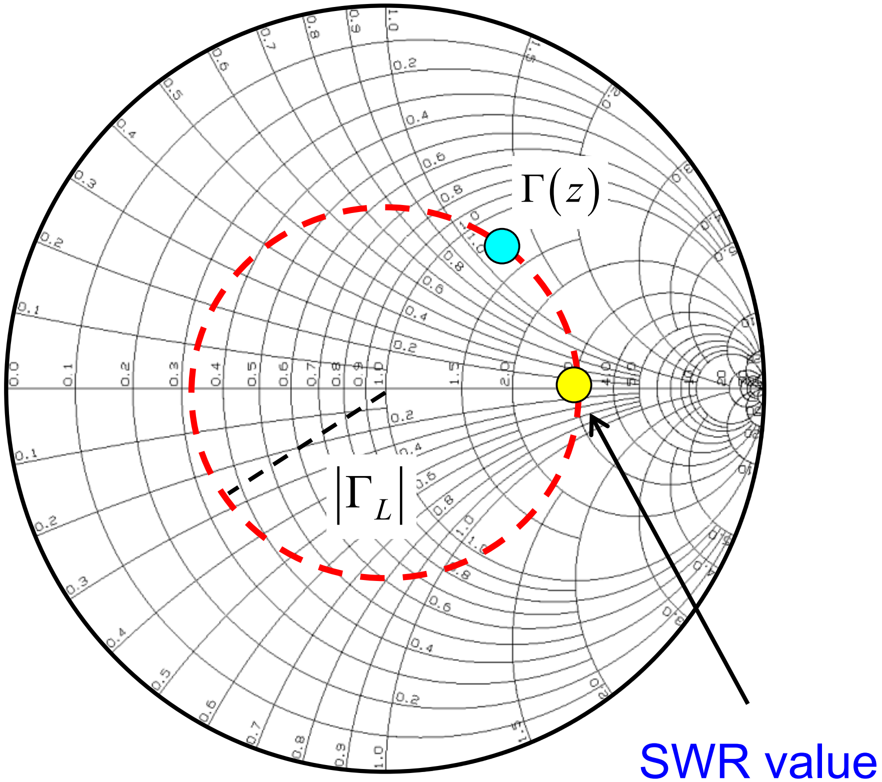
SWR

Recall:

If $Z_{in} = R_{in}$ (real)

$$\Rightarrow \text{SWR} = \max\left(\frac{R_{in}}{Z_0}, \frac{Z_0}{R_{in}}\right)$$

(whichever one is larger than one)



On the positive real axis:

$$Z_{in}^N = R_{in}^N = R_{in} / Z_0 = \text{real} \geq 1$$

$$\text{SWR} = R_{in}^N \Big|_{\text{positive real axis}}$$

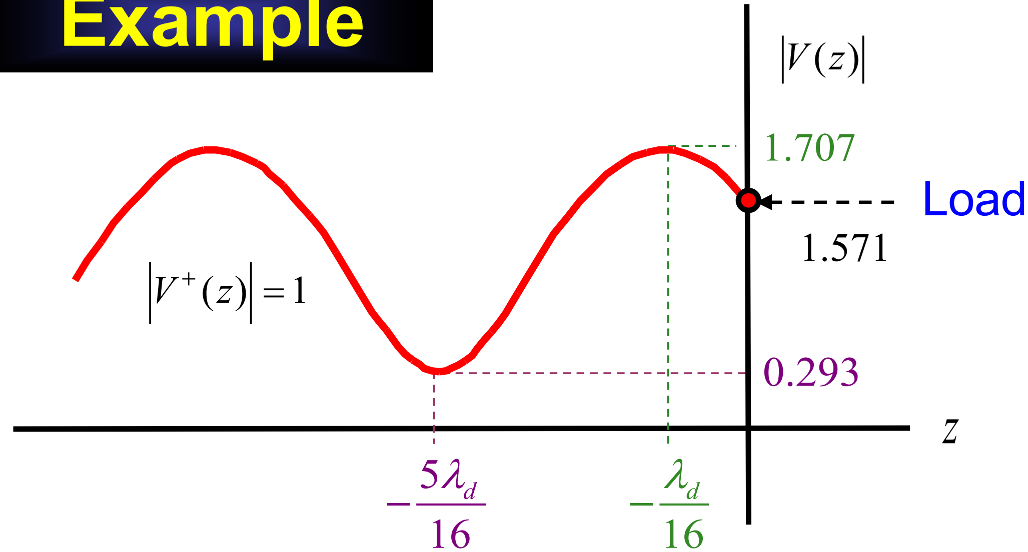
The SWR is read off from the normalized resistance value on the positive real axis.

Example

Given:

$\Gamma_L = 0.707\angle 45°$

Use the Smith chart to plot the magnitude of the normalized voltage, find the SWR, and find the normalized load admittance.



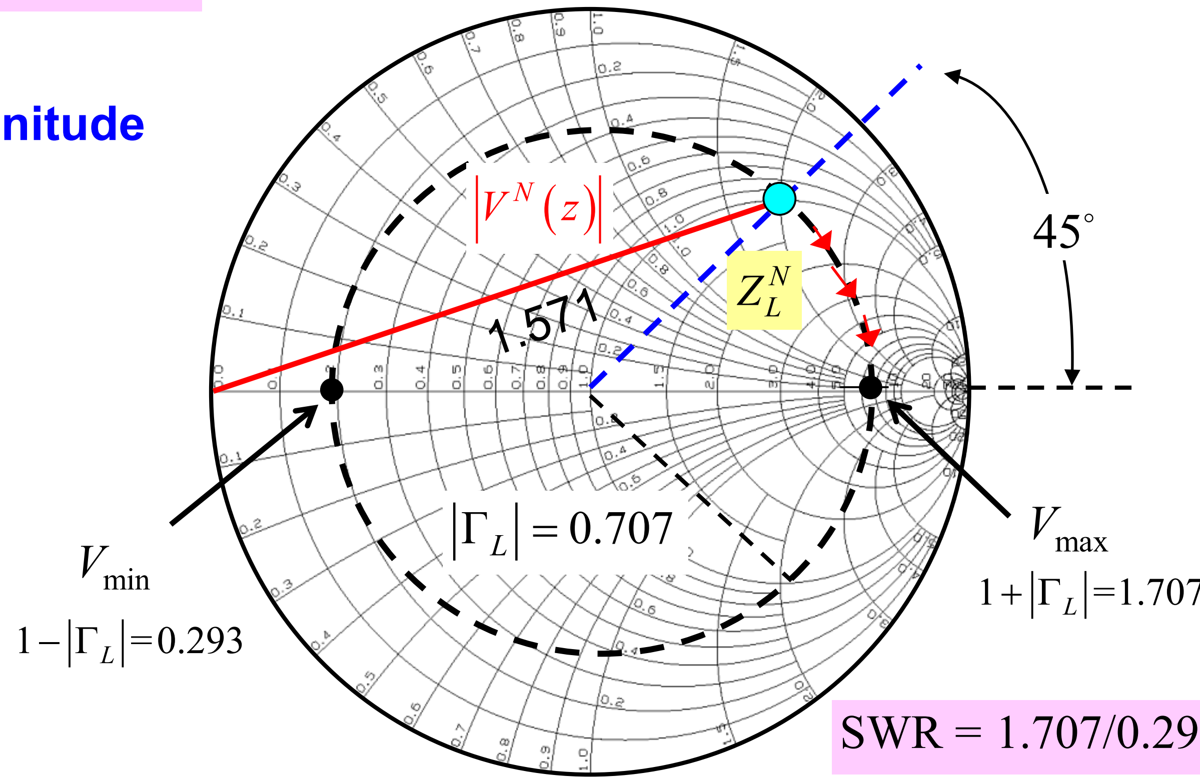
(a) Plot voltage magnitude

Find $V_{\max}$ position:

$$45° \Leftrightarrow \pi / 4 \;[\text{rad}]$$

$$\Rightarrow 2\beta d = 2\left(\frac{2\pi}{\lambda_d}\right)d = \frac{\pi}{4}$$

$$d = \frac{\lambda_d}{16} \quad \left(\text{or } 0.0625\lambda_d\right)$$

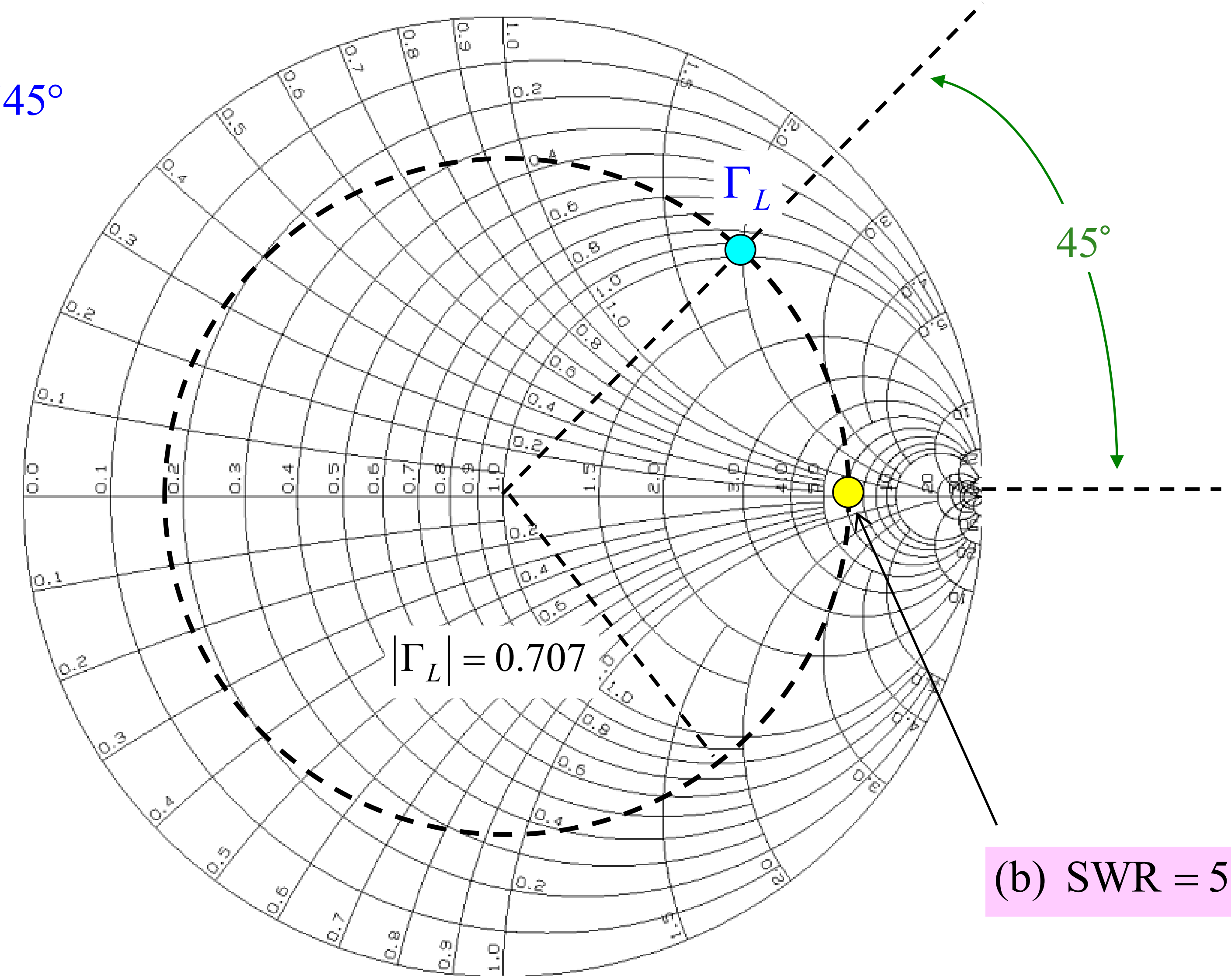


SWR = 1.707/0.293 = 5.8

Example (cont.)

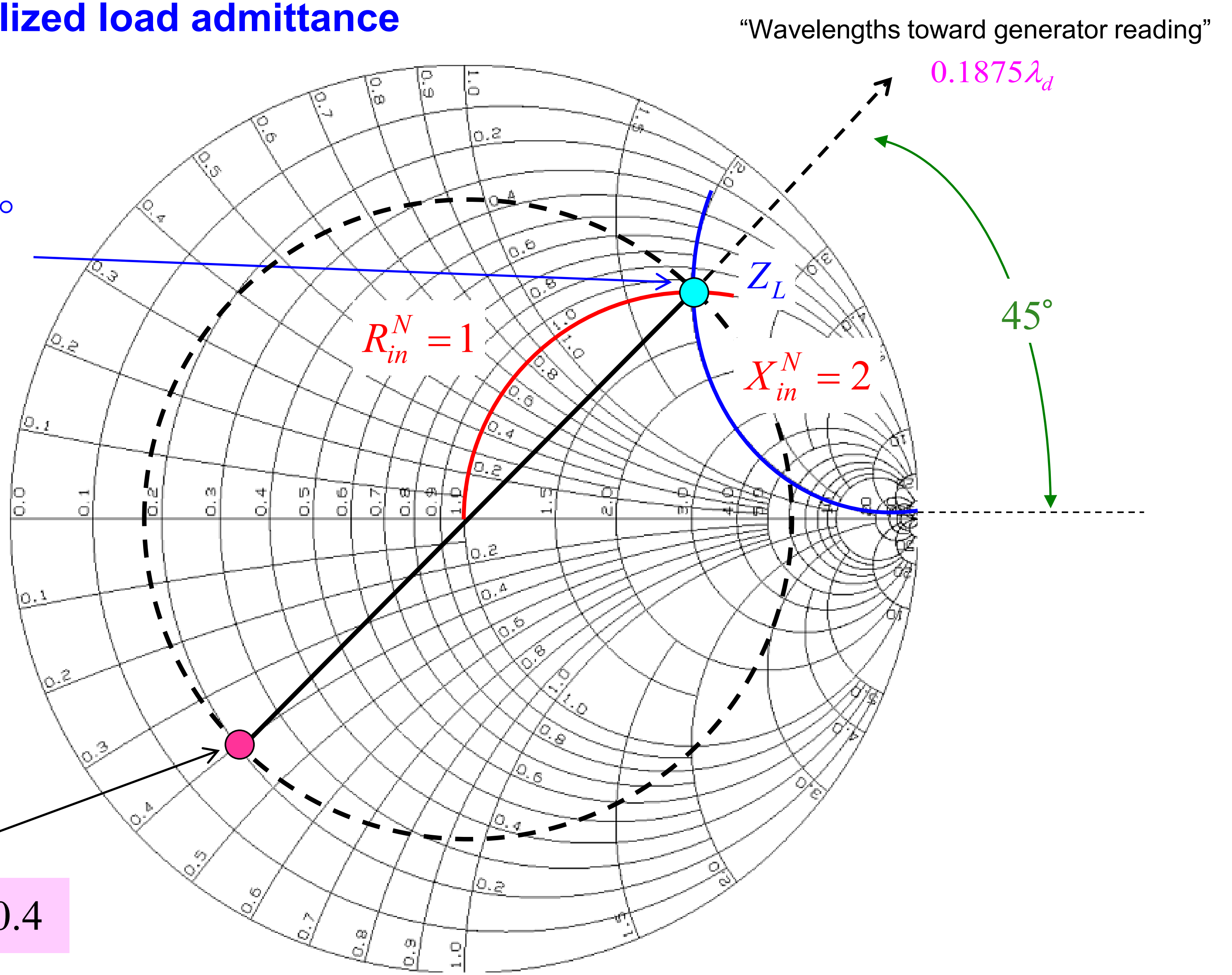
(b) Find SWR

$\Gamma_L = 0.707\angle 45°$



Example (cont.)

(c) Find normalized load admittance



$\Gamma_L = 0.707\angle 45°$

$Z_L^N = 1 + j2$

$(c)\ Y_L^N = 0.2 - j0.4$

Smith Chart as an Admittance Calculator

The Smith chart can also be used as an admittance calculator instead of an impedance calculator.

$$Z_{in}^{N}(z) = \left(\frac{1+\Gamma(z)}{1-\Gamma(z)} \right)$$

$$Y_{in}^{N}(z) = \frac{Y_{in}(z)}{Y_0} = \frac{1/Z_{in}(z)}{1/Z_0} = \frac{1}{Z_{in}(z)/Z_0} = \frac{1}{Z_{in}^{N}(z)}$$

Hence

$$Y_{in}^{N}(z) = \left(\frac{1-\Gamma(z)}{1+\Gamma(z)} \right)$$

or

$$Y_{in}^{N}(z) = \left(\frac{1+\Gamma'(z)}{1-\Gamma'(z)} \right)$$ where $\Gamma'(z) = -\Gamma(z)$

Admittance Calculator (cont.)

Compare:

$$Z_{in}^{N}(z) = R_{in}^{N} + jX_{n}^{N} = \left(\frac{1+\Gamma(z)}{1-\Gamma(z)}\right)$$

$$Y_{in}^{N}(z) = G_{in}^{N} + jB_{n}^{N} = \left(\frac{1+\Gamma'(z)}{1-\Gamma'(z)}\right) \qquad \left(\Gamma'(z) = -\Gamma(z)\right)$$

They have the same mathematical form!

Conclusion: If we interpret the Smith chart as being the complex Γ' plane instead of the complex Γ plane, then the R circles become G circles and the X circles become B circles.

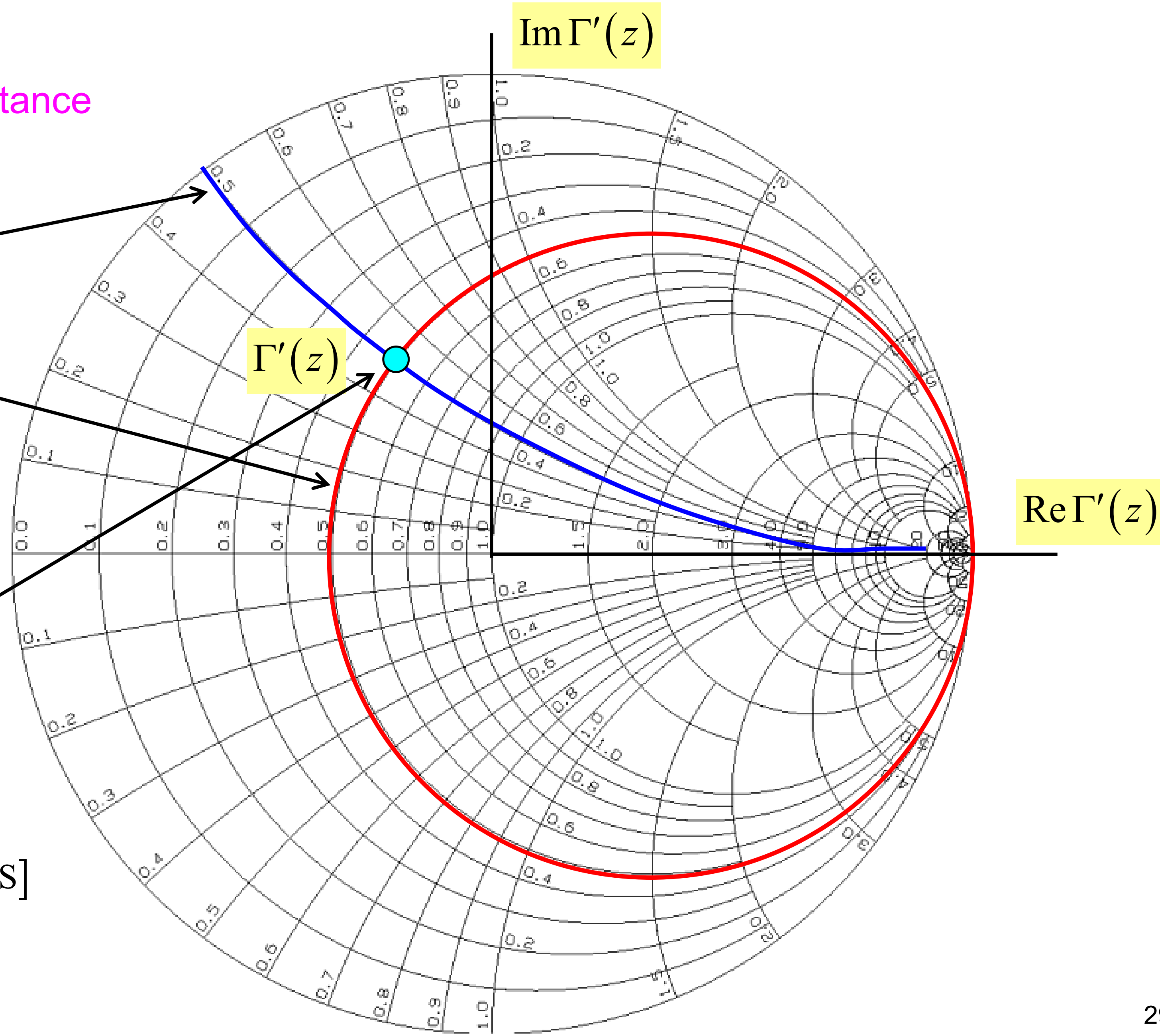
Admittance Calculator (cont.)

Example

What is the admittance at the blue dot?

$B_{in}^{N} = 0.5$

$G_{in}^{N} = 0.5$



At the given blue dot:

$$Y_{in}^{N} = 0.5 + j0.5$$

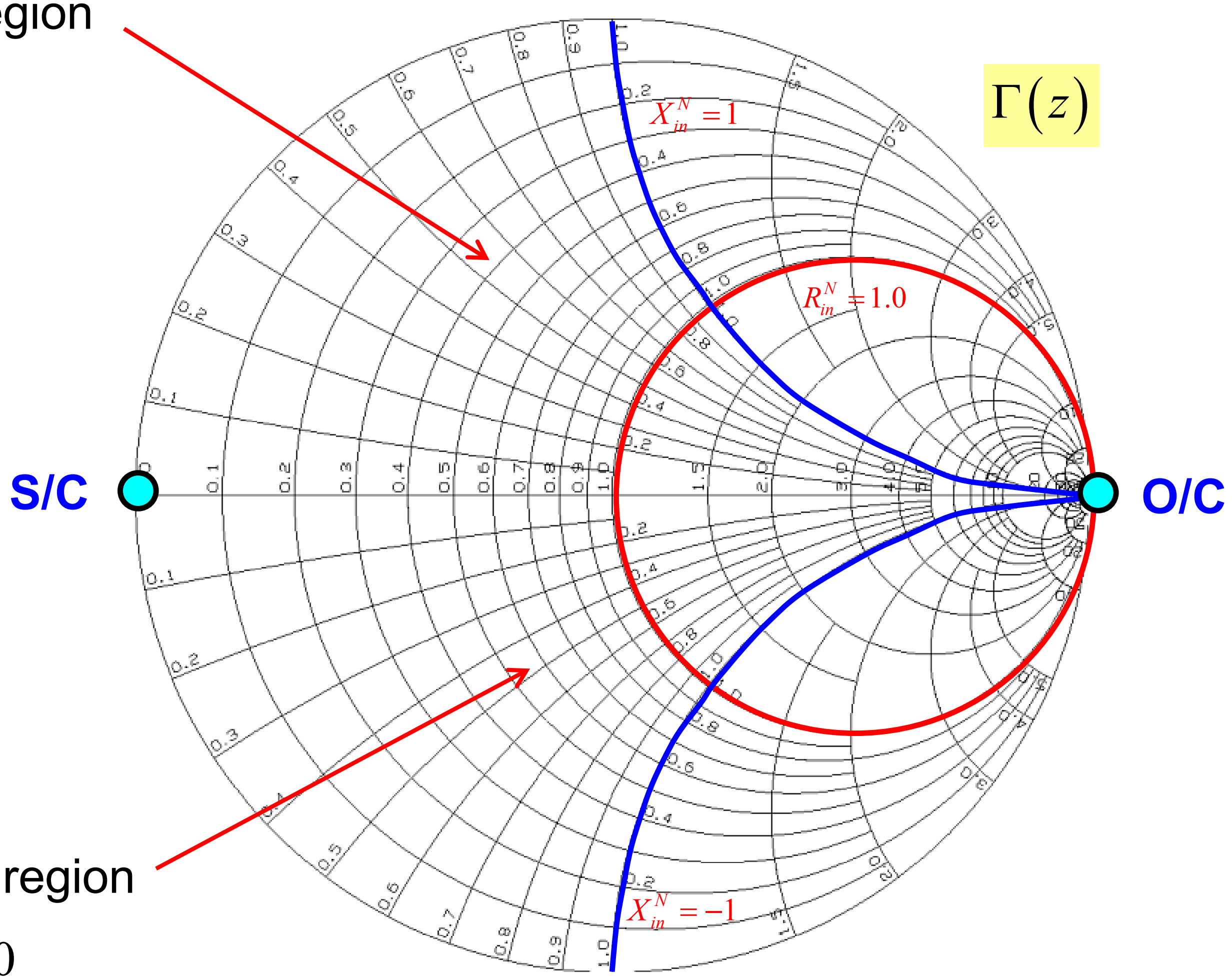
$$Y_{in} = \left(0.5 + j0.5\right)\frac{1}{Z_0} \ [\mathrm{S}]$$

Comparison of Charts

Impedance Calculator

Inductive region

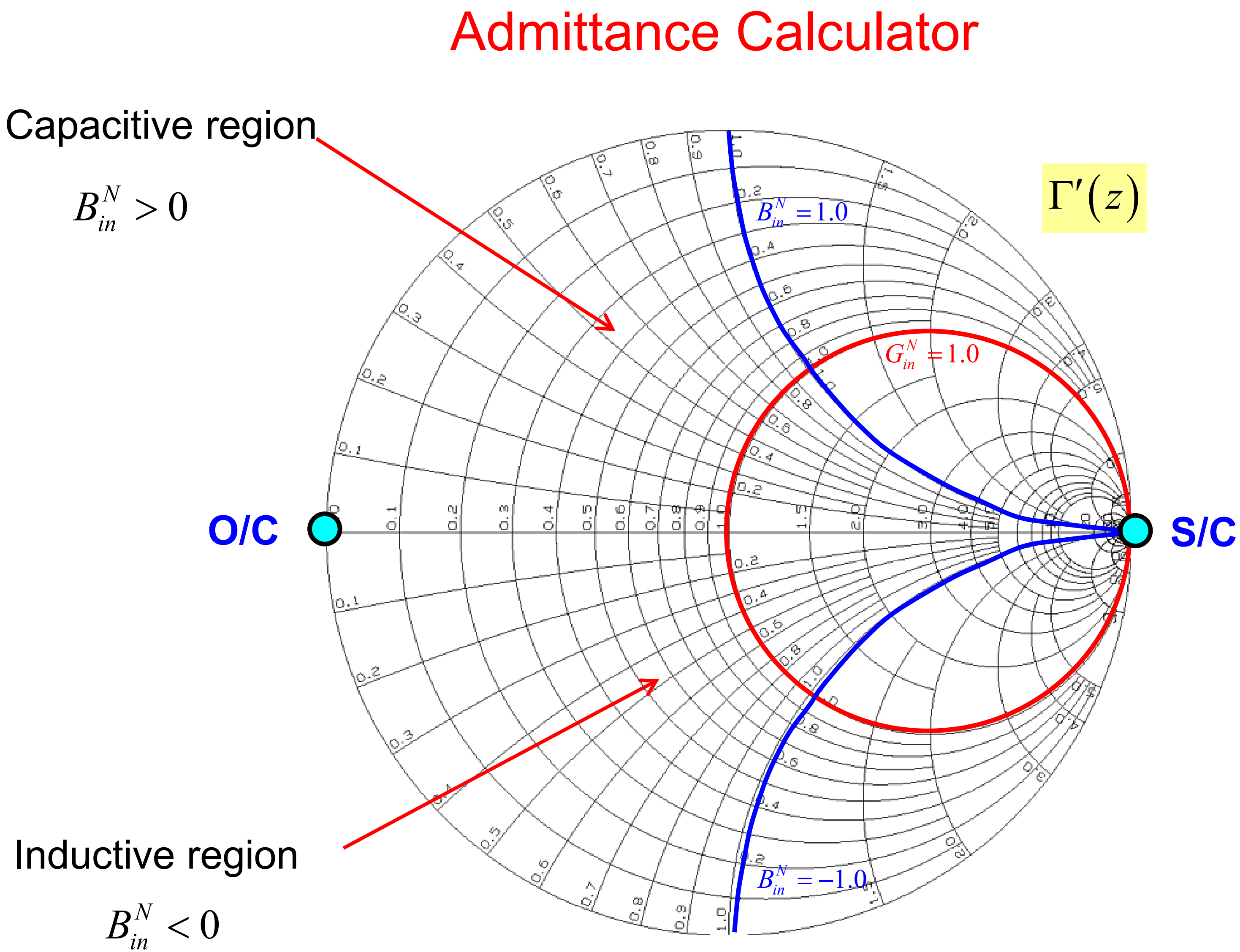
$X_{in}^{N} > 0$



Capacitive region

$X_{in}^{N} < 0$

Comparison of Charts (cont.)



Using the Smith chart for Impedance and Admittance Calculations

- We can use the same Smith chart for both impedance and admittance calculations.
- The Smith chart is then either the Γ plane or the Γ' plane, depending on which type of calculation we are doing.

For example:

- We can convert from normalized impedance to normalized admittance, using the reciprocal property (go half-way around the smith chart).
- We can then continue to use the Smith chart on an admittance basis.

Example

Given: $Z_L^N = 1 + j2$

Find the normalized admittance $d = \lambda_d / 8$ away from the load.

Example (cont.)

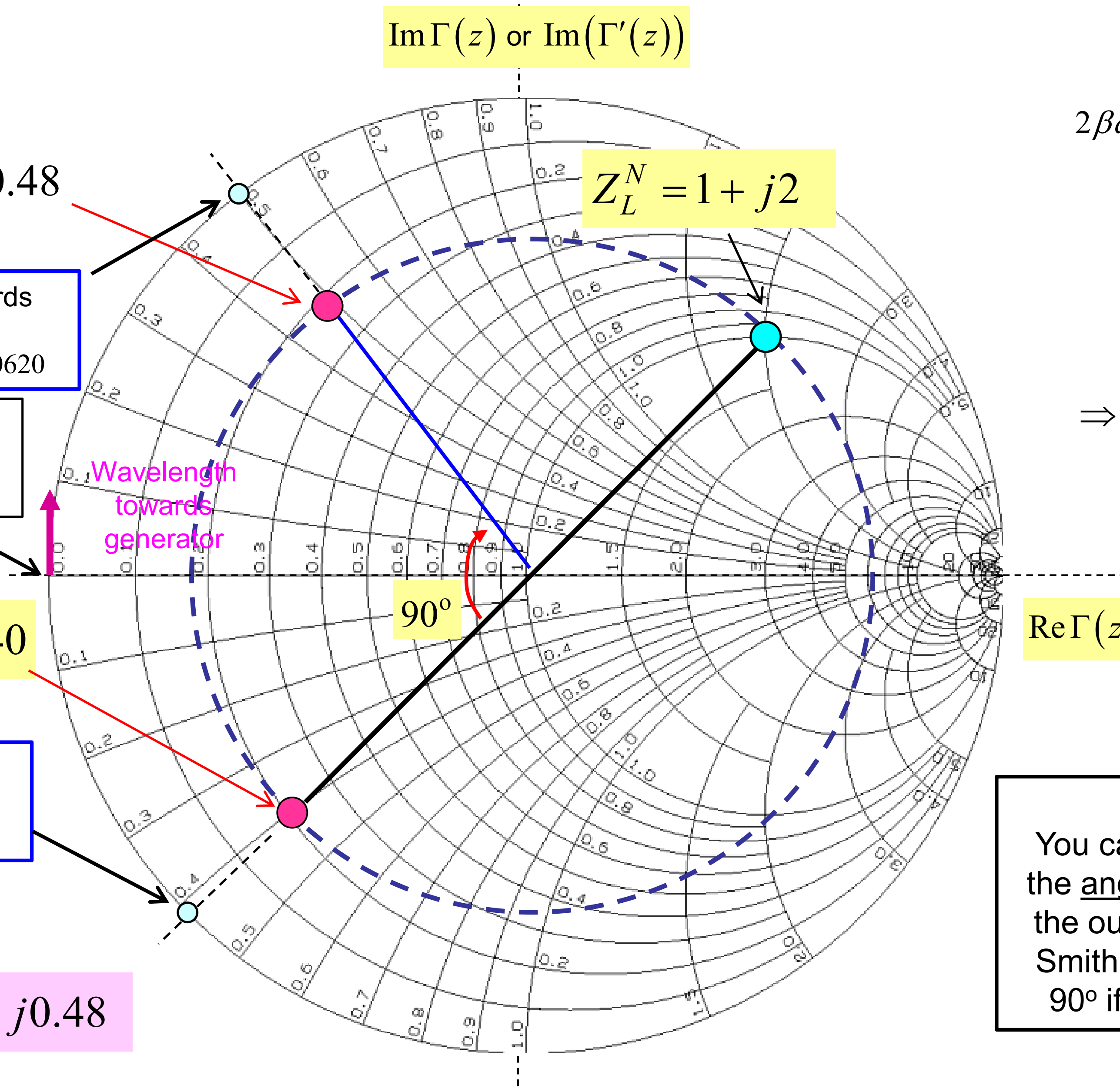
$Y_{in}^{N} = 0.23 + j0.48$

Wavelengths towards generator scale:
0.437+ 1/8 – 0.5 = 0.0620

Note:
The scale jumps by 0.5 here.

$Y_{L}^{N} = 0.20 - j0.40$

Wavelengths towards generator scale: 0.437



$$2\beta d = 2\left(\frac{2\pi}{\lambda}\right)d = 4\pi\left(\frac{1}{8}\right) = \frac{\pi}{2}$$

$$\Rightarrow \Delta\theta = 90^{\mathrm{o}}$$

Note:
You can also use the <u>angle</u> scale on the outside of the Smith chart to go 90º if you wish.

Answer:

$Y_{in}^{N} = 0.23 + j0.48$

Appendix: Summary of Formulas

$$Z_{in}^{N}\left(z\right)=\left(\frac{1+\Gamma\left(z\right)}{1-\Gamma\left(z\right)}\right)$$

$$\Gamma\left(z\right)=\Gamma_L\,e^{+j2\beta z}=\Gamma_L\,e^{-j2\beta d}$$

$$Z_{in}^{N}\left(z\right)=\left(\frac{1+\Gamma_L\,e^{-j2\beta d}}{1-\Gamma_L\,e^{-j2\beta d}}\right)$$

$$\Gamma_L=\frac{Z_L-Z_0}{Z_L+Z_0}$$

$$Z_{in}^{N}(B)=\frac{1}{Z_{in}^{N}(A)}$$

$$\Gamma'\left(z\right)=-\Gamma\left(z\right)$$

$$\Delta\theta=4\pi\left(\frac{d}{\lambda}\right)$$

$$\mathrm{SWR}=R_{in}^{N}\Big|_{\text{postive real axis}}$$