ECE 3317
Applied Electromagnetic Waves

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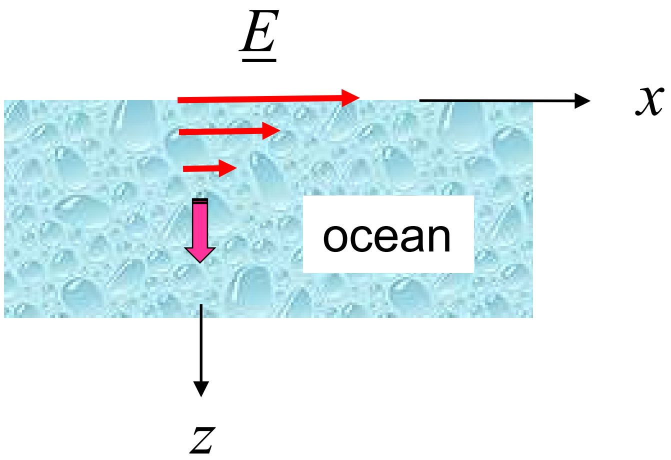
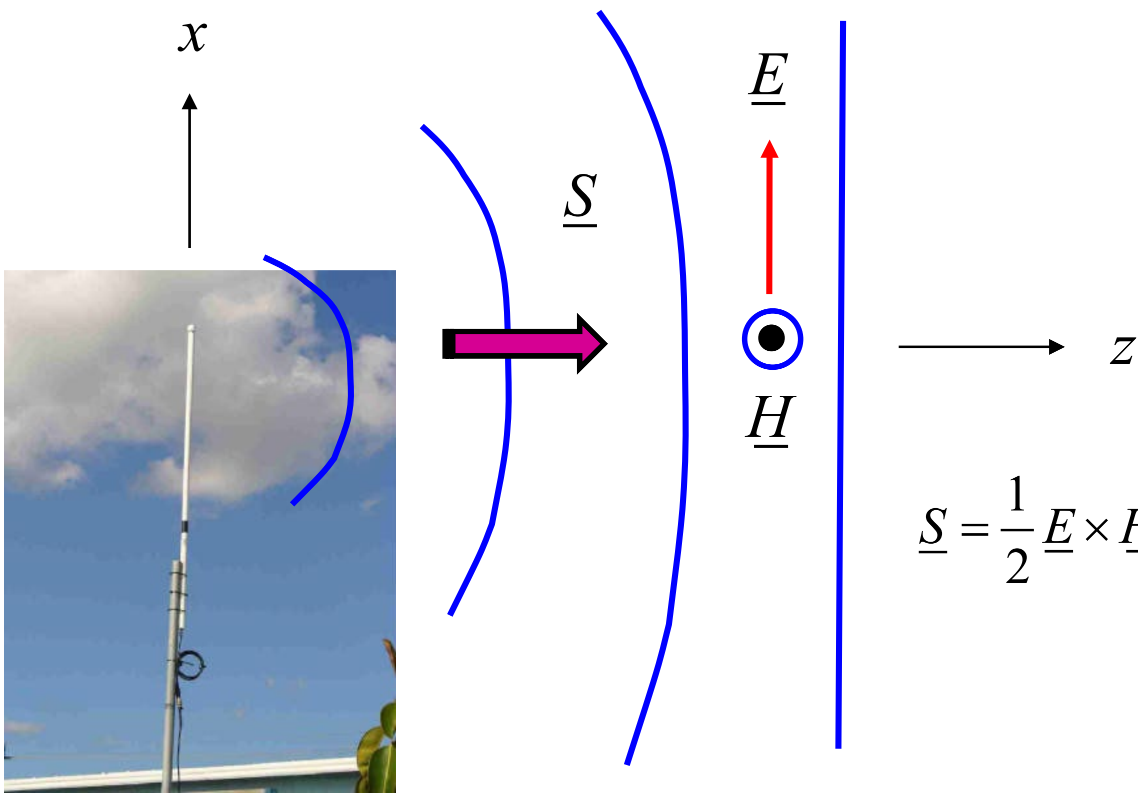
Notes 15
Plane Waves



Introduction to Plane Waves

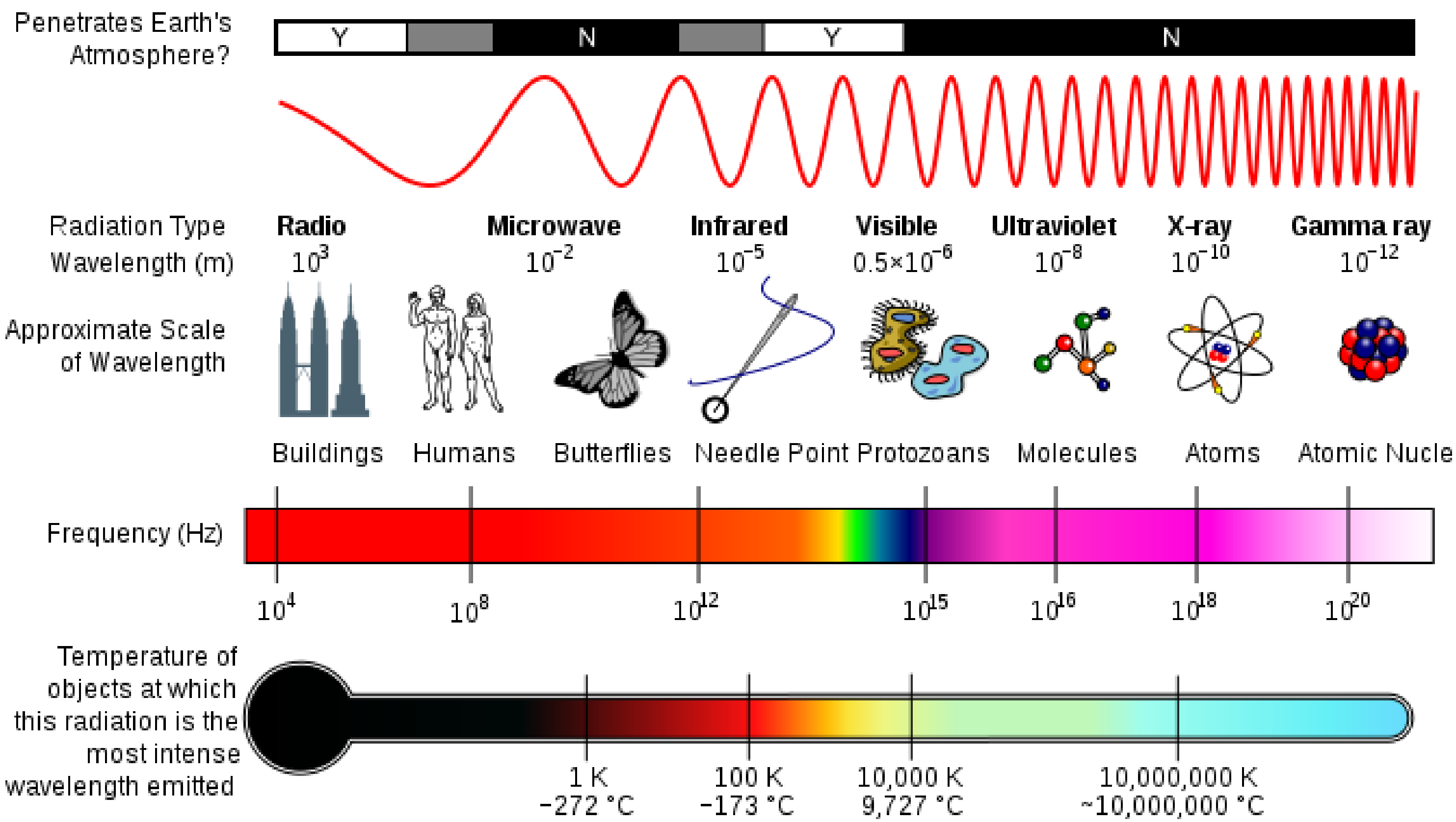
- A plane wave is the simplest solution to Maxwell's equations for a wave that travels through free space.
- The wave does not require any conductors – it exists in free space.
- A plane wave is a good model for radiation from an antenna, if we are far enough away from the antenna.



$$\underline{S} = \frac{1}{2}\underline{E} \times \underline{H}^* = \underline{\hat{z}}\frac{1}{2}\left(E_x H_y^*\right)$$

The Electromagnetic Spectrum

$$\lambda_0 = c / f$$



http://en.wikipedia.org/wiki/Electromagnetic_spectrum

The Electromagnetic Spectrum (cont.)

Source	Frequency	Wavelength
U.S. AC Power	60 Hz	5000 km
ELF Submarine Communications	500 Hz	600 km
AM radio (KTRH)	740 kHz	405 m
TV ch. 2 (VHF)	60 MHz	5 m
FM radio (Sunny 99.1)	99.1 MHz	3 m
TV PBS ch. 8 (VHF ch 8)	183 MHz	1.6 m
TV KPRC ch. 2 (UHF ch. 35)	599 MHz	50 cm
Cell Phone (4G)	1.9 GHz	16 cm
μ-wave oven	2.45 GHz	12 cm
Police radar (X-band)	10.5 GHz	2.85 cm
Cell Phone (Verizon 5G mmWave)	28 GHz	1.1 cm
THz	1000 GHz	0.3 mm
Light	5×10^{14} [Hz]	0.60 μm
X-ray	10^{18} [Hz]	3 Å

$$\lambda_0 = c / f$$

Note : $\lambda_0\,[\mathrm{cm}] \approx 30 / f\,[\mathrm{GHz}]$

TV and Radio Spectrum

AM Radio: 520-1610 kHz

VHF TV: 55-216 MHz (channels 2-13)

Band I : 55-88 MHz (channels 2-6)
Band III: 174-216 MHz (channels 7-13)

FM Radio: (Band II) 88-108 MHz

UHF TV: 470-806 MHz (channels 14-69)

Digital TV broadcast takes place primarily in UHF and VHF Bands I & III.

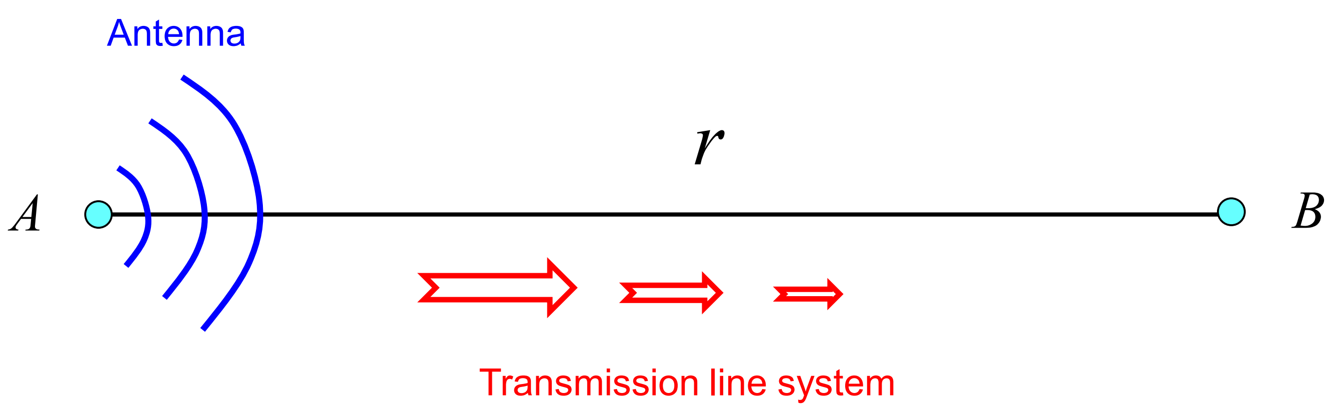
Note:
Monopole antenna are ideally about one-quarter of a wavelength in length.
Dipole antennas are ideally about one-half of a wavelength in length.

Comparison of Wired Systems with Wireless Systems

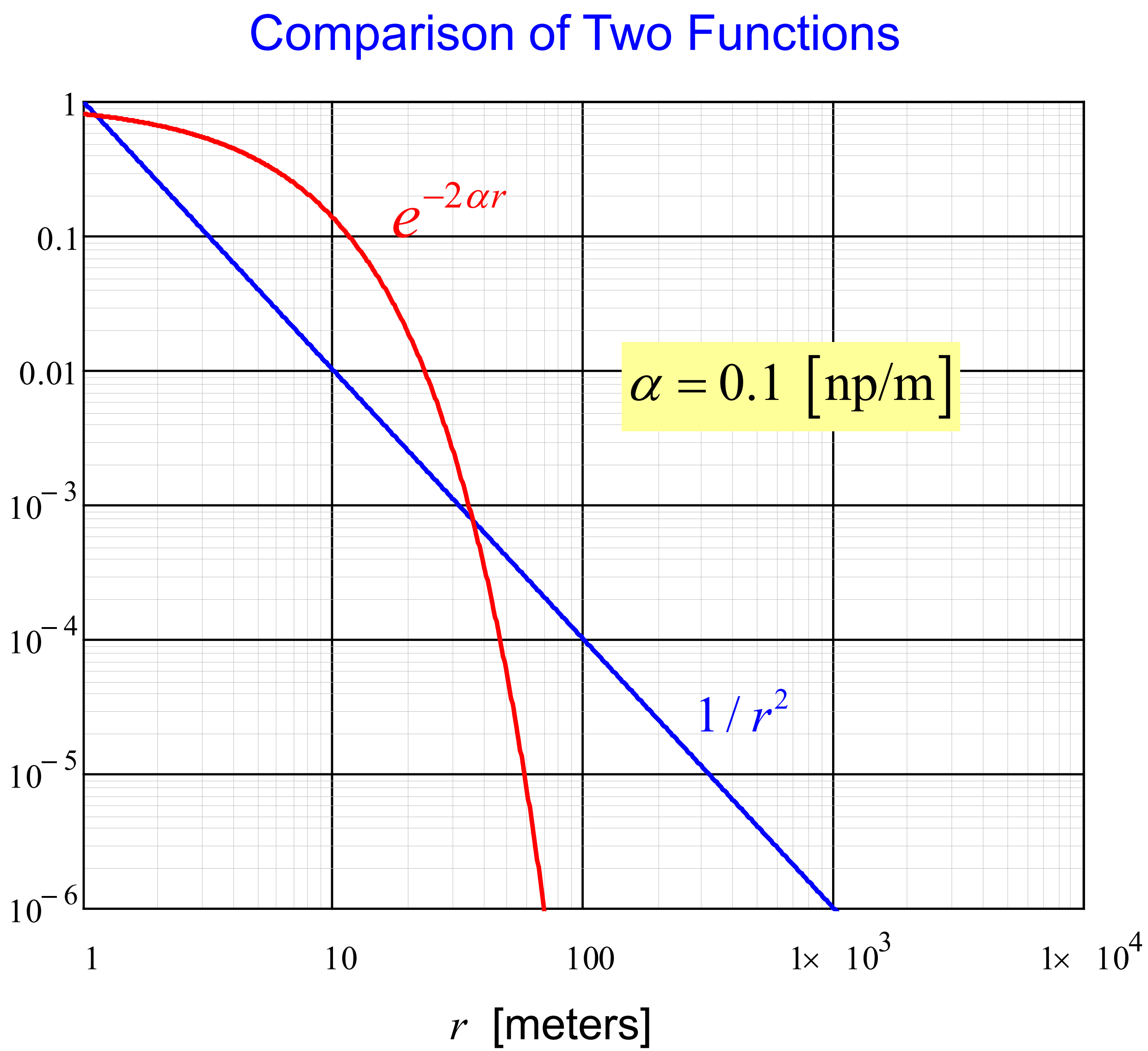
Wireless systems using antennas will always be better (lower loss) than wired (transmission line) systems for large distances.

Power loss from antenna broadcast: $1/r^2$ ← (always better for very large r)

Power loss from waveguiding system: $e^{-2\alpha r}$



Comparison of Wired Systems with Wireless Systems (cont.)



Comparison of Wired Systems with Wireless Systems (cont.)

Attenuation in dB **1 GHz**

	RG59	Single Mode	Two Dipoles	34m Dish+Dipole
Distance	Coax	Fiber	Wireless	Wireless
1 m	0.4	0.0003	28.2	-
10 m	4	0.003	48.2	-
100 m	40	0.03	68.2	-
1 km	400	0.3	88.2	39.3
10 km	4000	3	108.2	59.3
100 km	-	30	128.2	79.3
1000 km	-	300	148.2	99.3
10,000 km	-	3000	168.2	119.3
100,000 km	-	-	188.2	139.3
1,000,000 km	-	-	208.2	159.3
10,000,000 km	-	-	228.2	179.3
100,000,000 km	-	-	248.2	199.3

Vector Wave Equation

Start with Maxwell's equations in the phasor domain:

$$\nabla \times \underline{E} = -j\omega\mu \underline{H} \qquad \text{Faraday's law}$$

$$\nabla \times \underline{H} = \underline{J} + j\omega\varepsilon \underline{E} \qquad \text{Ampere's law}$$

Assume <u>free space</u>:

Ohm's law: $\underline{J} = \cancel{\sigma}\underline{E} = \underline{0} \qquad \varepsilon = \varepsilon_0,\ \mu = \mu_0$

We then have:

$$\nabla \times \underline{E} = -j\omega\mu_0 \underline{H}$$

$$\nabla \times \underline{H} = j\omega\varepsilon_0 \underline{E}$$

Vector Wave Equation (cont.)

$$\nabla \times \underline{E} = -j\omega\mu_0 \underline{H}$$

$$\nabla \times \underline{H} = j\omega\varepsilon_0 \underline{E}$$

Take the curl of the first equation and then substitute the second equation into the first one:

$$\nabla \times \left(\nabla \times \underline{E}\right) = -j\omega\mu_0 \left(\nabla \times \underline{H}\right)$$

$$= -j\omega\mu_0 \left(j\omega\varepsilon_0 \underline{E}\right)$$

Define:

$$\boxed{k_0 \equiv \omega\sqrt{\mu_0\varepsilon_0}}$$

Wavenumber of free space [rad/m]

Then

$$\boxed{\nabla \times \left(\nabla \times \underline{E}\right) - k_0^2 \underline{E} = \underline{0}}$$

"Vector wave equation"

Vector Helmholtz Equation

$$\nabla \times \left(\nabla \times \underline{E}\right) - k_0^2 \underline{E} = \underline{0}$$

Recall the vector Laplacian identity:

$$\nabla^2 \underline{V} \equiv \nabla\left(\nabla \cdot \underline{V}\right) - \nabla \times \left(\nabla \times \underline{V}\right)$$

Hence, we have

$$\nabla\left(\nabla \cdot \underline{E}\right) - \nabla^2 \underline{E} - k_0^2 \underline{E} = \underline{0}$$

Also, from the electric Gauss law we have (in free space):

$$\nabla \cdot \underline{E} = \frac{1}{\varepsilon_0} \nabla \cdot \underline{D} = \frac{1}{\varepsilon_0} \cancel{\rho_v} = 0$$

Vector Helmholtz Equation (cont.)

Hence, we have:

$$\nabla^2 \underline{E} + k_0^2 \underline{E} = \underline{0}$$

Vector Helmholtz equation

Recall the property of the vector Laplacian in rectangular coordinates:

$$\nabla^2 \underline{V} = \underline{\hat{x}} \nabla^2 V_x + \underline{\hat{y}} \nabla^2 V_y + \underline{\hat{z}} \nabla^2 V_z$$

Reminder:
This identity only holds in rectangular coordinates.

Taking the x component of the vector Helmholtz equation, we have

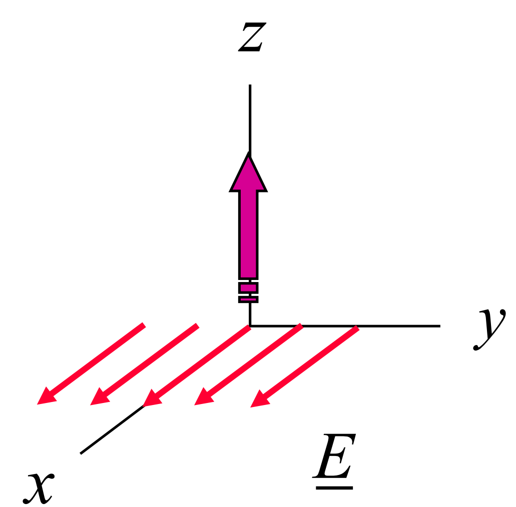
$$\nabla^2 E_x + k_0^2 E_x = 0$$

Scalar Helmholtz equation

Plane Wave Field

Assume $\underline{E} = \underline{\hat{x}}\, E_x(z)$

The electric field is polarized in the x direction, and the wave is propagating (traveling) in the z direction.



Then

$$\nabla^2 E_x + k_0^2 E_x = 0$$

or

$$\cancel{\frac{\partial^2 E_x}{\partial x^2}} + \cancel{\frac{\partial^2 E_x}{\partial y^2}} + \frac{\partial^2 E_x}{\partial z^2} + k_0^2 E_x = 0$$

Note:
The electric field is constant in the $z = 0$ plane.

$$\Rightarrow \quad \frac{d^2 E_x}{dz^2} + k_0^2 E_x = 0$$

For wave traveling in the negative z direction:

$$E_x(z) = E_0\, e^{+jk_0 z}$$

Solution: $\boxed{E_x(z) = E_0\, e^{-jk_0 z}}$ $\quad (k_0 = \omega\sqrt{\mu_0 \varepsilon_0})$

Plane Wave Field (cont.)

Lossless Dielectric Medium:

For a plane wave traveling in a lossless dielectric medium (does not have to be free space):

$$E_x(z) = E_0 \, e^{-jkz}$$

where

$$\boxed{k = \omega\sqrt{\mu\varepsilon} = k_0\sqrt{\mu_r \varepsilon_r}}$$ (wavenumber of dielectric medium)

$$\varepsilon = \varepsilon_0 \varepsilon_r$$

$$\mu = \mu_0 \mu_r$$

Plane Wave Field (cont.)

Comparison between plane wave and wave on a lossless transmission line

Plane Wave: $E_x(z) = E_0\, e^{-jkz}$

Transmission Line: $V(z) = V_0\, e^{-j\beta z}$

The electric field of a plane wave in a lossless medium propagates in z exactly as does the voltage on a lossless transmission line (filled with the same material):

$$\beta = \omega\sqrt{LC} = \omega\sqrt{\mu\varepsilon} = k$$

They have the same wavenumber!

Plane Wave Field (cont.)

$$E_x(z) = E_0\, e^{-jkz}$$

The $\underline{H}$ field is found from:

$$\nabla \times \underline{E} = -j\omega\mu\, \underline{H}$$

so

$$\begin{aligned}\underline{H} &= -\frac{1}{j\omega\mu}\nabla \times \left(\underline{\hat{x}}\, E_x(z)\right) \\ &= -\frac{1}{j\omega\mu}\left(\underline{\hat{y}}\frac{d\,E_x}{dz}\right) \\ &= -\frac{1}{j\omega\mu}\left(\underline{\hat{y}}\,(-jk)\,E_x\right)\end{aligned}$$

$$\nabla \times \underline{E} = \begin{vmatrix} \underline{\hat{x}} & \underline{\hat{y}} & \underline{\hat{z}} \\ \cancel{\dfrac{\partial}{\partial x}} & \cancel{\dfrac{\partial}{\partial y}} & \dfrac{\partial}{\partial z} \\ E_x & \cancel{E_y} & \cancel{E_z} \end{vmatrix}$$

Intrinsic Impedance

We then have $\underline{H} = \underline{\hat{y}} \left(\dfrac{k}{\omega\mu} \right) E_x$ so $H_y = \left(\dfrac{k}{\omega\mu} \right) E_x$

Hence

$$\frac{E_x}{H_y} = \frac{\omega\mu}{k} = \frac{\omega\mu}{\omega\sqrt{\mu\varepsilon}} = \sqrt{\frac{\mu}{\varepsilon}} = \eta$$

where

$$\eta \equiv \sqrt{\frac{\mu}{\varepsilon}} = \eta_0 \sqrt{\frac{\mu_r}{\varepsilon_r}}$$

(intrinsic impedance of the medium)

$\mu_0 = 4\pi \times 10^{-7}\ [\mathrm{H/m}]$ (exact before 2019)

$\varepsilon_0 = \dfrac{1}{\mu_0 c^2} \doteq 8.854 \times 10^{-12}\ [\mathrm{F/m}]$

$c \equiv 2.99792458 \times 10^{8}\ [\mathrm{m/s}]$ (exact)

Intrinsic impedance of free-space:

$$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \doteq 376.730313\ [\Omega]$$

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

Poynting Vector

For a wave traveling in a lossless dielectric medium:

$$\underline{E} = \underline{\hat{x}}\, E_0\, e^{-jkz}$$

$$\underline{H} = \underline{\hat{y}}\, \frac{1}{\eta} E_0\, e^{-jkz}$$

Note:
k and η are real here
(lossless medium).

The complex Poynting vector is given by $\underline{S} = \frac{1}{2}\left(\underline{E} \times \underline{H}^*\right)$

Hence, we have:

$$\underline{S} = \frac{1}{2}\underline{\hat{z}}\left(E_0\, e^{-jkz}\right)\left(\frac{E_0}{\eta} e^{-jkz}\right)^*$$

$$= \frac{1}{2\eta}\underline{\hat{z}}\left(E_0\, e^{-jkz}\right)\left(E_0^* e^{+jkz}\right)$$

$$= \frac{1}{2\eta}\underline{\hat{z}}\, \left|E_0\right|^2$$

$$\underline{S} = \underline{\hat{z}}\frac{\left|E_0\right|^2}{2\eta} \quad [\mathrm{VA/m^2}]$$ (no VARS)

$$\left\langle \underline{\mathscr{S}}(t) \right\rangle = \mathrm{Re}\left(\underline{S}\right) = \underline{\hat{z}}\frac{\left|E_0\right|^2}{2\eta} \quad [\mathrm{W/m^2}]$$

Phase Velocity

From our previous discussion on phase velocity for transmission lines, we know that for a wave on a transmission line:

$$v_p = \frac{\omega}{\beta}$$

Hence, for a plane wave in a lossless dielectric medium (letting $\beta = k$):

$$v_p = \frac{\omega}{k} = \frac{\omega}{\omega\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{\mu\varepsilon}}$$

so

$$\boxed{v_p = c_d = \frac{1}{\sqrt{\mu\varepsilon}}}$$

(speed of light in the dielectric material)

Notes:

- All plane waves travel at the same speed in a lossless medium, regardless of the frequency.
- This implies that there is no dispersion in a lossless medium, which in turn implies that there is no distortion of the signal.
- The phase velocity of a plane wave in a lossless medium is the same as that of a wave on a lossless transmission line that is filled with the same material.

Wavelength

From our previous discussion on wavelength for transmission lines, we know that

$$\lambda_g = \frac{2\pi}{\beta}$$

Hence, for a plane wave in a lossless dielectric medium (letting $\beta = k$):

$$\lambda_g = \frac{2\pi}{k} = \frac{2\pi}{\omega\sqrt{\mu\varepsilon}} = \frac{2\pi}{2\pi f\sqrt{\mu\varepsilon}} = \frac{1}{f\sqrt{\mu\varepsilon}} = \frac{c_d}{f} = \frac{c/f}{\sqrt{\mu_r\varepsilon_r}} = \frac{\lambda_0}{\sqrt{\mu_r\varepsilon_r}} = \lambda_d$$

Hence, we have

$$\lambda_g = \lambda_d = \frac{\lambda_0}{\sqrt{\mu_r\varepsilon_r}}$$

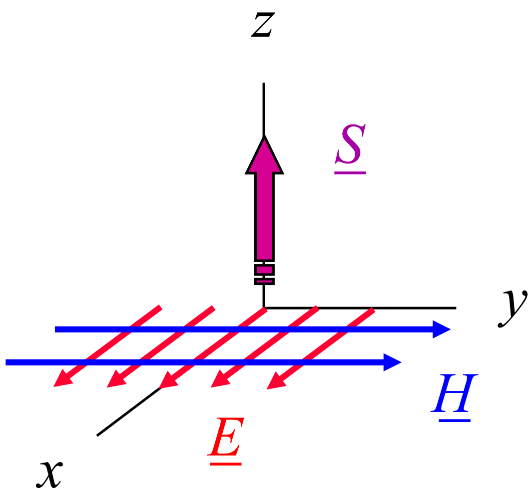
For free space: $\lambda_0 = c/f$

Summary (Lossless Case)

$$E_x = E_0 \, e^{-jkz}$$

$$H_y = \frac{1}{\eta} E_0 \, e^{-jkz}$$

$$S_z = \frac{\left|E_0\right|^2}{2\eta}$$



$$k = \omega\sqrt{\mu\varepsilon} \qquad v_p = c_d = \frac{c}{\sqrt{\mu_r \varepsilon_r}} \qquad \lambda_g = \lambda_d = \frac{\lambda_0}{\sqrt{\varepsilon_r \mu_r}}$$

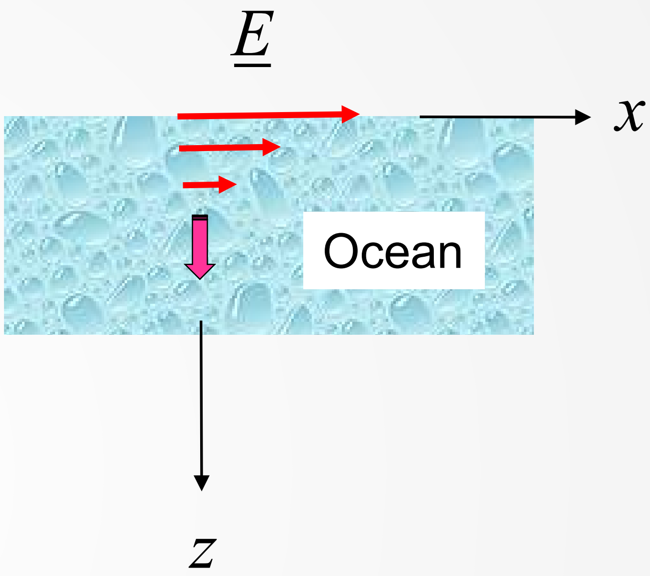
$$\eta = \sqrt{\frac{\mu}{\varepsilon}} = \eta_0 \sqrt{\frac{\mu_r}{\varepsilon_r}} \qquad \eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \doteq 376.730313 \; [\Omega]$$

Lossy Medium

Return to Maxwell's equations:

$$\nabla \times \underline{E} = -j\omega\mu\underline{H}$$

$$\nabla \times \underline{H} = \underline{J} + j\omega\varepsilon\underline{E}$$



Assume Ohm's law: $\underline{J} = \sigma\underline{E}$

Ampere's law:

$$\nabla \times \underline{H} = \sigma\underline{E} + j\omega\varepsilon\underline{E}$$

$$= (\sigma + j\omega\varepsilon)\underline{E}$$

We define an effective (complex) permittivity ε_c that accounts for conductivity:

Set $j\omega\varepsilon_c = \sigma + j\omega\varepsilon$ $\Rightarrow$ $\boxed{\varepsilon_c \equiv \varepsilon - j\left(\dfrac{\sigma}{\omega}\right)}$

Lossy Medium (cont.)

Maxwell's equations then become:

$$\nabla \times \underline{E} = -j\omega\mu\underline{H}$$
$$\nabla \times \underline{H} = j\omega\varepsilon_c\underline{E}$$

Lossy

$$\nabla \times \underline{E} = -j\omega\mu\underline{H}$$
$$\nabla \times \underline{H} = j\omega\varepsilon\underline{E}$$

Lossless

The lossy form is exactly the same as we have for the lossless case, with

$$\varepsilon \rightarrow \varepsilon_c$$

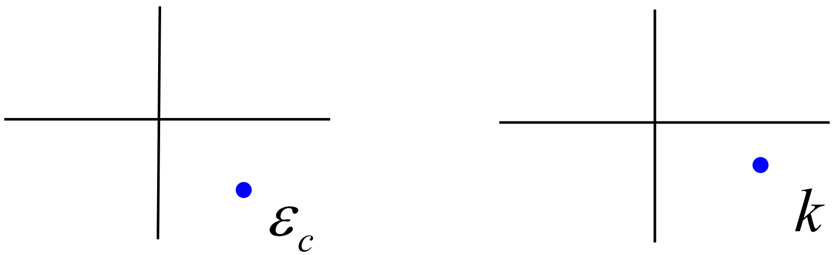
Hence, we have for a lossy medium:

$$E_x = E_0 e^{-jkz}$$
$$H_y = \frac{1}{\eta} E_0 e^{-jkz}$$

$$k = \omega\sqrt{\mu\varepsilon_c} \quad \text{(complex)}$$
$$\eta = \sqrt{\frac{\mu}{\varepsilon_c}} \quad \text{(complex)}$$

Lossy Medium (cont.)

Examine the wavenumber: $k = \omega\sqrt{\mu\varepsilon_c}$ $\quad \varepsilon_c = \varepsilon - j\left(\dfrac{\sigma}{\omega}\right)$



$k' \geq 0$

$k'' \geq 0$

Reminder about principal branch:

$$\sqrt{z} = \sqrt{|z|e^{j\theta}} = \sqrt{|z|}\, e^{j\theta/2}$$

$$-\pi < \theta \leq \pi$$

Denote: $k = k' - jk''$

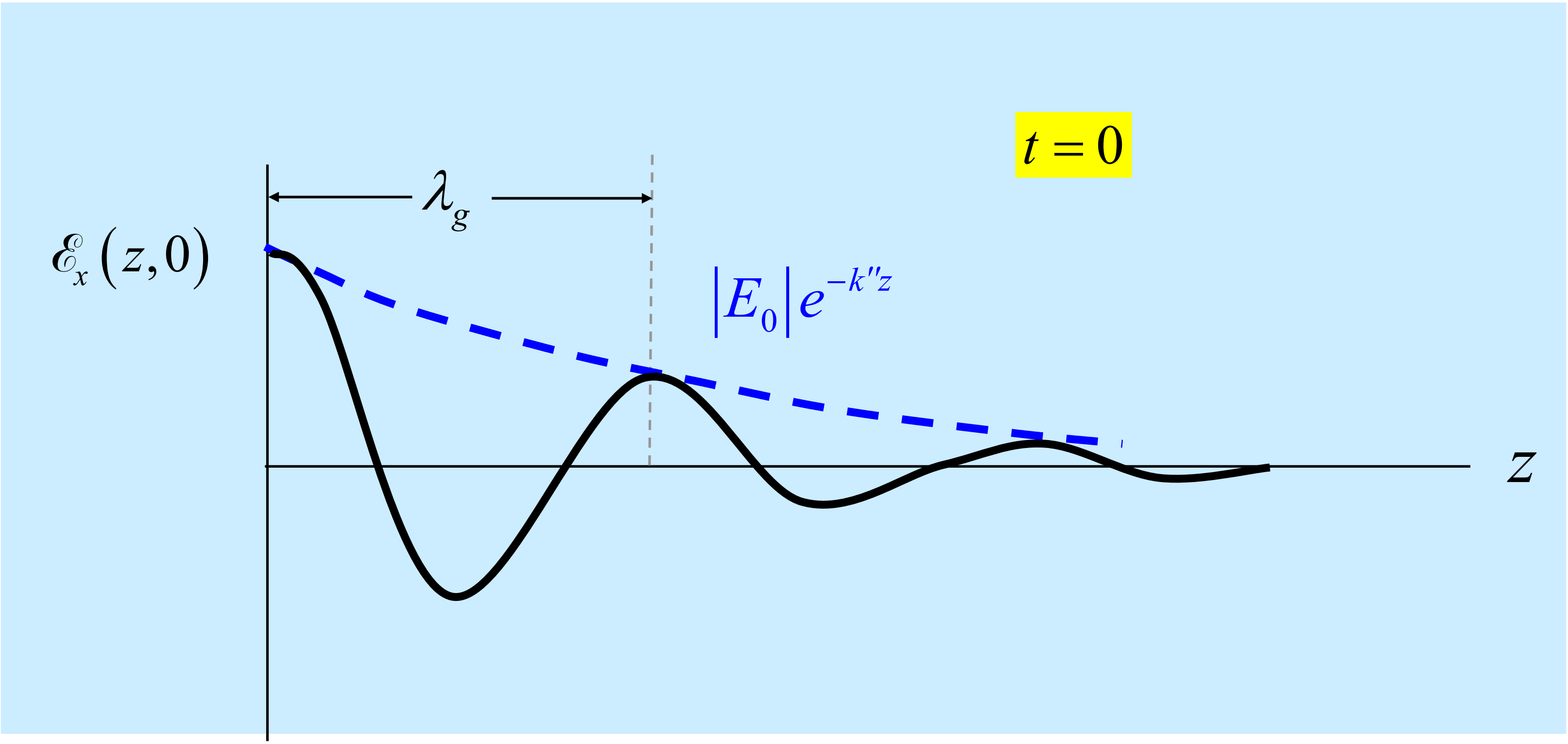
$$E_x = E_0\, e^{-jkz} = E_0\, e^{-jk'z} e^{-k''z}$$

Compare with lossy TL: $k' \leftrightarrow \beta$ $\quad k'' \leftrightarrow \alpha$

Lossy Medium (cont.)

$$E_x(z) = E_0\, e^{-jk'z} e^{-k''z} \quad \Rightarrow \quad \mathscr{E}_x(z,t) = |E_0| \cos(\omega t - k'z + \phi_0)\, e^{-k''z}$$

$$E_0 = |E_0|\, e^{j\phi_0}$$

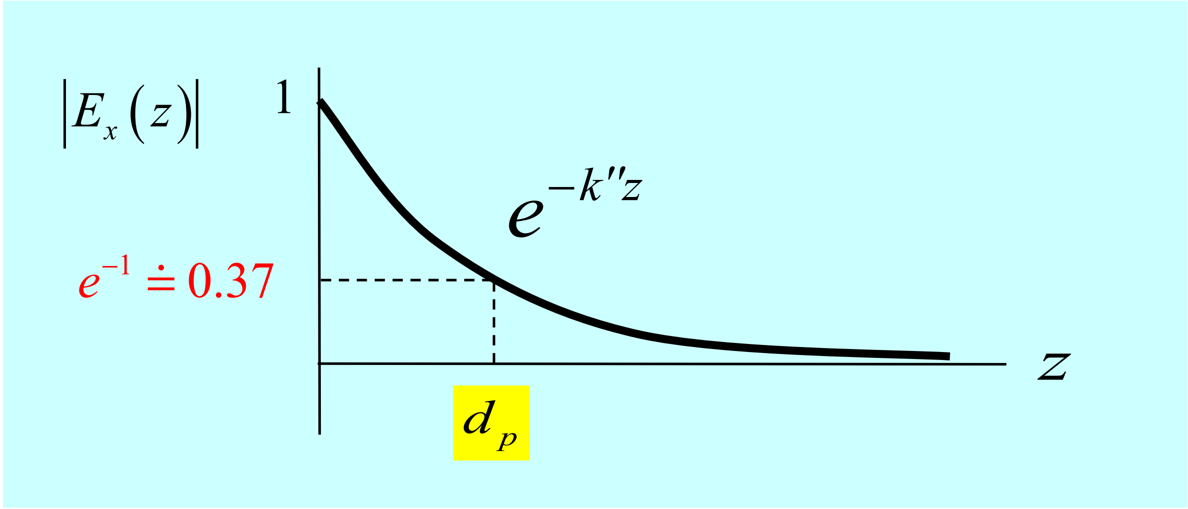


$$\lambda_g = \frac{2\pi}{k'}$$

Lossy Medium (cont.)

$$E_x(z) = E_0\, e^{-jk'z} e^{-k''z}$$

(choose $E_0 = 1$)



The "depth of penetration" d_p is defined.

$$k'' d_p = 1 \quad \Rightarrow \quad \boxed{d_p \equiv 1 / k''}$$

Lossy Medium (cont.)

$$E_x = E_0 \, e^{-jk'z} e^{-k''z}$$

$$H_y = \frac{1}{\eta} E_0 \, e^{-jk'z} e^{-k''z}$$

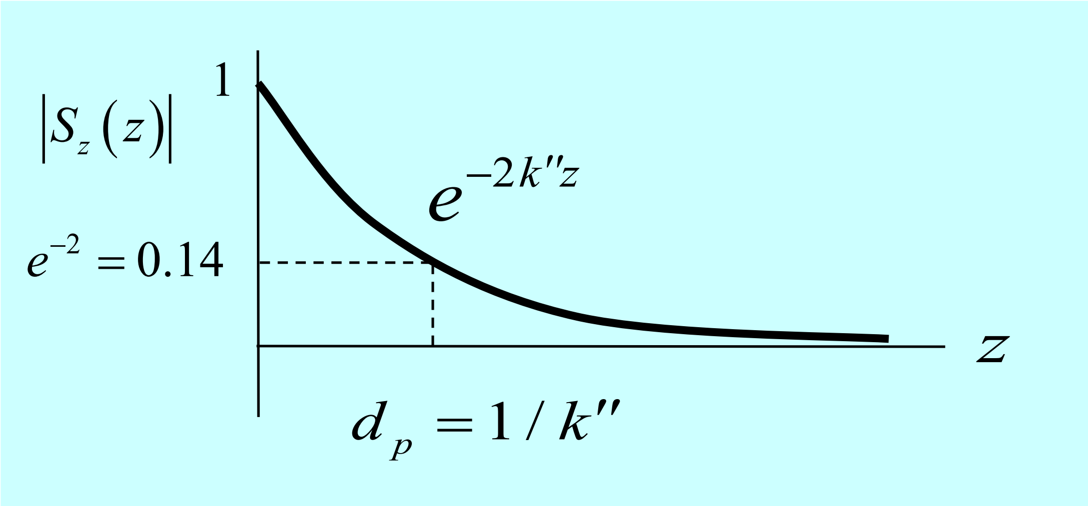
$$\eta = \sqrt{\frac{\mu}{\varepsilon_c}} = |\eta| e^{j\phi}$$

Note: The angle between the E_x and H_y phasors is ϕ.

The complex Poynting vector is

$$\underline{S} = \frac{1}{2}\left(\underline{E} \times \underline{H}^*\right) = \underline{\hat{z}} \frac{1}{2} E_x H_y^* = \underline{\hat{z}} \frac{|E_0|^2}{2\eta^*} e^{-2k''z} = \underline{\hat{z}} \frac{|E_0|^2}{2|\eta|} e^{j\phi} e^{-2k''z}$$

$$\langle S_z(t) \rangle = \operatorname{Re} S_z = \frac{|E_0|^2}{2|\eta|} \cos\phi \, e^{-2k''z}$$



Lossy Medium (cont.)

Summary for Depth of Penetration Formula:

$$\boxed{d_p \equiv 1 / k''}$$

$$k = k' - jk'' \quad \left(k'' = -\operatorname{Im}(k)\right)$$

$$k = \omega\sqrt{\mu\varepsilon_c} = k_0\sqrt{\varepsilon_c / \varepsilon_0} = k_0\sqrt{\varepsilon_{rc}}$$

$$\varepsilon_c \equiv \varepsilon - j\left(\frac{\sigma}{\omega}\right), \qquad \varepsilon_{rc} = \frac{\varepsilon_c}{\varepsilon_0} = \varepsilon_r - j\left(\frac{\sigma}{\omega\varepsilon_0}\right)$$

Example

Ocean water:

$$\varepsilon_r = 81$$

$$\sigma = 4 \;\; [\mathrm{S/m}]$$

$$\mu = \mu_0$$

(These values are fairly constant up through low microwave frequencies.)

Assume $f = 2.0$ GHz:

$$\varepsilon_c = \varepsilon - j\left(\frac{\sigma}{\omega}\right) = \varepsilon_0\left[\varepsilon_r - j\left(\frac{\sigma}{\omega\varepsilon_0}\right)\right]$$

$$\varepsilon_c = \varepsilon_0\big(81 - j(35.95)\big)\;[\mathrm{F/m}]$$

$$\varepsilon_{rc} = 81 - j(35.95)$$

$$k = \omega\sqrt{\mu_0\varepsilon_c} = \omega\sqrt{\mu_0\varepsilon_0\varepsilon_{rc}} = k_0\sqrt{\varepsilon_{rc}}$$

$$k = 386.022 - j(81.816) \;\; [1/\mathrm{m}]$$

$$k' = 386.022 \;\; [\mathrm{rad/m}]$$

$$k'' = 81.816 \;\; [\mathrm{nepers/m}]$$

$$d_p = 1/k'' \qquad d_p = 0.01222 \;\; [\mathrm{m}]$$

$$\lambda_g = 2\pi/k' \qquad \lambda_g = 0.01628\;[\mathrm{m}]$$

Example (cont.)

The depth of penetration into ocean water is shown for various frequencies.

$$\varepsilon_r = 81$$
$$\sigma = 4 \;\; [\text{S/m}]$$
$$\mu = \mu_0$$

$$d_p = 1/k''$$

f	d_p [m]
1 [Hz]	251.6
10 [Hz]	79.6
100 [Hz]	25.2
1 [kHz]	7.96
10 [kHz]	2.52
100 [kHz]	0.796
1 [MHz]	0.262
10 [MHz]	0.080
100 [MHz]	0.0262
1.0 [GHz]	0.013
10.0 [GHz]	0.012
100 [GHz]	0.012

Note:
The relative permittivity of water starts changing at very high frequencies (above about 2GHz), but this is ignored here.

Loss Tangent

Recall:

$$\varepsilon_c = \varepsilon - j\left(\frac{\sigma}{\omega}\right) \qquad \sigma = \sigma_d = \text{conductivity of dielectric material}$$

To be more general:

$\sigma = \sigma_d = \sigma_{\text{eff}} =$ **effective** conductivity of dielectric material

(accounts for actual conductivity + atomic and molecular loss effects)

$$\tan\delta \equiv \frac{\sigma}{\omega\varepsilon} = \frac{\sigma}{\omega\varepsilon_0\varepsilon_r}$$

Sometimes we write:

$$\varepsilon_c = \varepsilon' - j\varepsilon'' \qquad \varepsilon_{rc} = \varepsilon_c / \varepsilon_0 = \varepsilon_r' - j\varepsilon_r''$$

$$\varepsilon' = \varepsilon,\;\; \varepsilon'' = \frac{\sigma}{\omega} \qquad \varepsilon_r' = \varepsilon_r,\;\; \varepsilon_r'' = \frac{\sigma}{\omega\varepsilon_0} \qquad \Rightarrow \qquad \tan\delta = \frac{\varepsilon''}{\varepsilon'}$$

Loss Tangent (cont.)

Practical notes on loss tangent:

- For some materials (mostly good conductors), it is the conductivity that is approximately constant with frequency.

Ocean water: $\sigma \approx 4$ [S/m]

- For other materials (mostly good insulators), it is the loss tangent that is approximately constant with frequency. In this case the effective permittivity is mainly due to molecular loss effects.

Teflon: $\tan\delta \approx 0.001$

Low-Loss Limit: $\tan\delta << 1$

We approximate the wavenumber for <u>small</u> loss tangent:

(The derivation is omitted.)

$$d_p \approx \sqrt{\frac{\varepsilon}{\mu_0}}\left(\frac{2}{\sigma}\right) \qquad \left(\tan\delta << 1\right)$$

In the low-loss limit, the depth of penetration is independent of frequency.

Ocean Water

Ocean water

$$\tan\delta = \frac{\sigma}{\omega\varepsilon_0\varepsilon_r}$$

$$\varepsilon_r = 81$$

$$\sigma = 4 \;\; [\mathrm{S/m}]$$

$$\mu = \mu_0$$

"Low-loss" region $\tan\delta << 1$

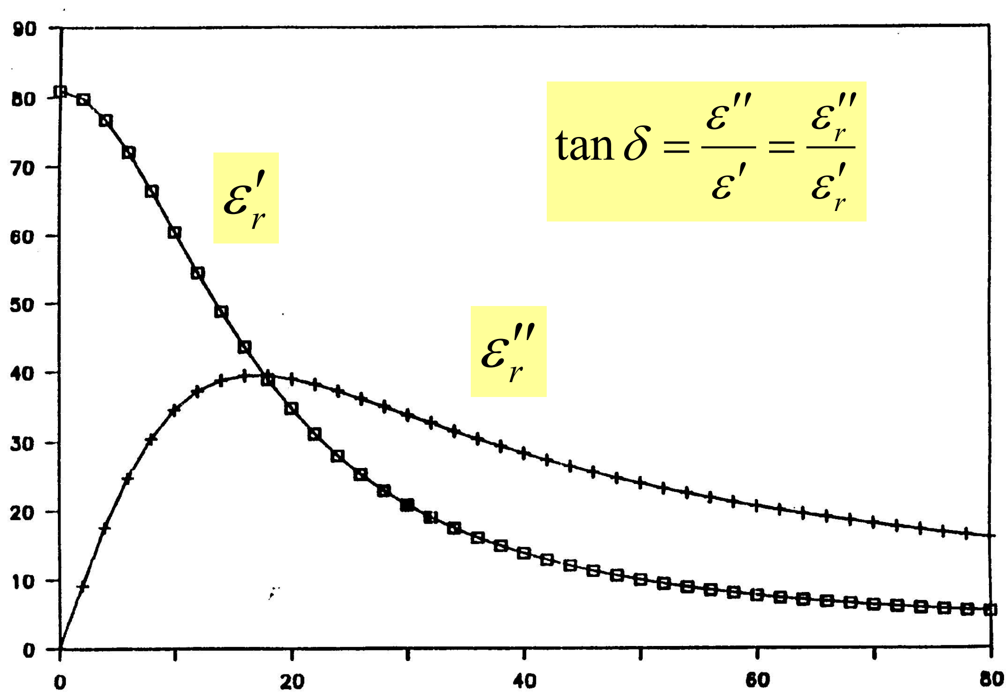
f	d_p [m]	$\tan\delta$
1 [Hz]	251.6	8.88×10^{8}
10 [Hz]	79.6	8.88×10^{7}
100 [Hz]	25.2	8.88×10^{6}
1 [kHz]	7.96	8.88×10^{5}
10 [kHz]	2.52	8.88×10^{4}
100 [kHz]	0.796	8.88×10^{3}
1 [MHz]	0.262	888
10 [MHz]	0.080	88.8
100 [MHz]	0.0262	8.88
1.0 [GHz]	0.013	0.888
10.0 [GHz]	0.012	0.0888
100 [GHz]	0.012	0.00888

Distilled Water

Complex Relative Permittivity for Pure (Distilled) Water

$$\varepsilon_{rc} = \frac{\varepsilon_c}{\varepsilon_0} = \frac{\varepsilon' - j\varepsilon''}{\varepsilon_0} = \varepsilon_r' - j\varepsilon_r''$$

$$\left(\varepsilon_r' = \varepsilon_r, \;\; \varepsilon_r'' = \frac{\sigma_{\text{eff}}}{\omega \varepsilon_0} \right)$$



Frequency [GHz]

Note:
For pure distilled water, the effective conductivity σ is due entirely to molecular loss effects, since pure water is almost a perfect insulator (no ions to carry current as for ocean water).

Appendix: Summary of Formulas

Lossless

$$E_x(z) = E_0\, e^{-jkz}$$

$$\lambda_g = \lambda_d = \frac{\lambda_0}{\sqrt{\varepsilon_r \mu_r}}$$

$$k = \omega\sqrt{\mu\varepsilon}$$

$$\frac{E_x}{H_y} = \eta$$

$$c_d = \frac{c}{\sqrt{\mu_r \varepsilon_r}}$$

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} = \eta_0\sqrt{\frac{\mu_r}{\varepsilon_r}}$$

$$\lambda_0 = \frac{c}{f}$$

$$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \doteq 376.730313\ [\Omega]$$

$$v_p = c_d$$

Appendix: Summary of Formulas (cont.)

Lossy

$$E_x(z) = E_0\, e^{-jkz}$$

$$k = \omega\sqrt{\mu\varepsilon_c}$$

$$k = k' - jk''$$

$$\varepsilon_c = \varepsilon - j\left(\frac{\sigma}{\omega}\right)$$

$$\frac{E_x}{H_y} = \eta$$

$$d_p \equiv 1/k''$$

$$\varepsilon_c = \varepsilon' - j\varepsilon''$$

$$\eta = \sqrt{\frac{\mu}{\varepsilon_c}}$$

$$\lambda_g = \frac{2\pi}{k'}$$

$$\tan\delta \equiv \frac{\varepsilon''}{\varepsilon'} = \frac{\sigma}{\omega\varepsilon} = \frac{\sigma}{\omega\varepsilon_0\varepsilon_r}$$