ECE 3317
Applied Electromagnetic Waves

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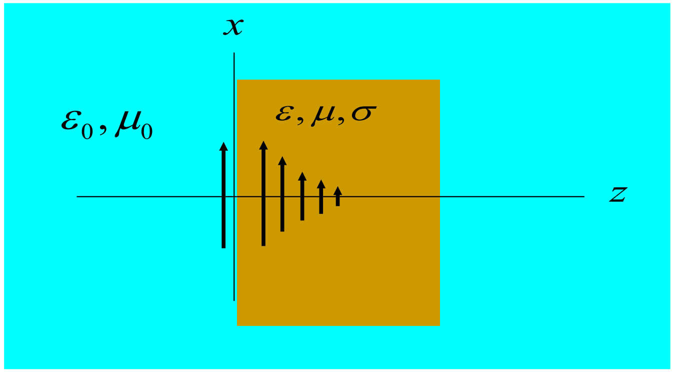
Notes 16

Plane Waves in Good Conductors



Good Conductor

Good conductor: $\left|\dfrac{\sigma}{\omega\varepsilon}\right| >> 1$

$$\varepsilon_c = \varepsilon - j\frac{\sigma}{\omega} = \varepsilon\left(1 - j\frac{\sigma}{\omega\varepsilon}\right)$$

$$\left|\frac{\sigma}{\omega\varepsilon}\right| >> 1 \quad \Rightarrow \quad \varepsilon_c \approx -j\frac{\sigma}{\omega}$$

$$\begin{aligned} k &= \omega\sqrt{\mu\varepsilon_c} \\ &\approx \omega\sqrt{\mu}\sqrt{-j\frac{\sigma}{\omega}} \\ &= \sqrt{\omega\mu\sigma}\,\sqrt{-j} \end{aligned}$$

Use

$$\sqrt{-j} = \sqrt{e^{-j\pi/2}} = e^{-j\pi/4} = \frac{1-j}{\sqrt{2}}$$

$$\underline{J} = \sigma\underline{E}$$

$$\left|\frac{\sigma}{\omega\varepsilon}\right| >> 1$$

Example: copper

Hence

$$k \approx (1-j)\sqrt{\frac{\omega\mu\sigma}{2}}$$

Recall $k \approx k' - jk''$

Therefore $k' \approx k'' \approx \sqrt{\dfrac{\omega\mu\sigma}{2}}$

Skin Depth

$$k' \approx k'' \approx \sqrt{\frac{\omega\mu\sigma}{2}}$$

Denote

$$\delta \equiv d_p = \frac{1}{k''}$$

"skin depth"

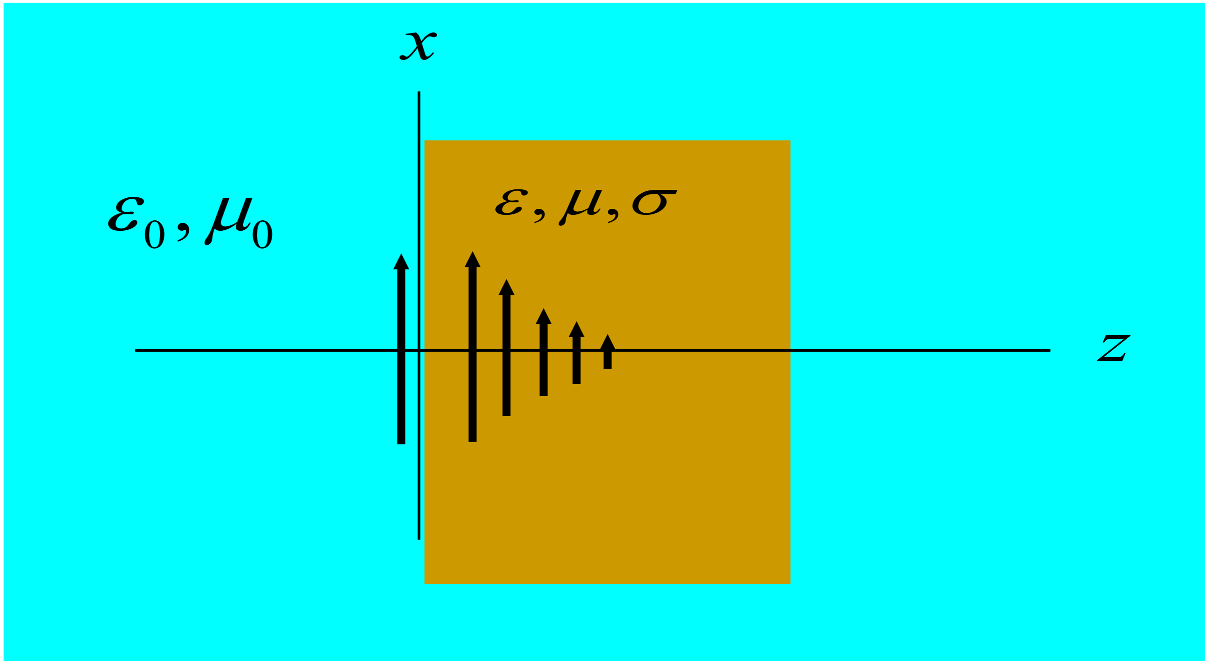
Then we have

$$\delta = d_p = \sqrt{\frac{2}{\omega\mu\sigma}}$$

$$k' \approx k'' \approx \frac{1}{\delta}$$

For a good conductor, the depth of penetration is called the "skin depth". The symbol is δ.

Skin Depth (cont.)



$$E_x(z) = E_{x0}\, e^{-jkz} = E_{x0}\, e^{-jk'z} e^{-k''z}$$

Hence

$$k' \approx k'' \approx \frac{1}{\delta}$$

$$E_x(z) = E_{x0}\, e^{-j(z/\delta)}\, e^{-(z/\delta)}$$

Controls the magnitude

Skin Depth (cont.)

$$\delta = \sqrt{\frac{2}{\omega\mu\sigma}}$$

Example: copper

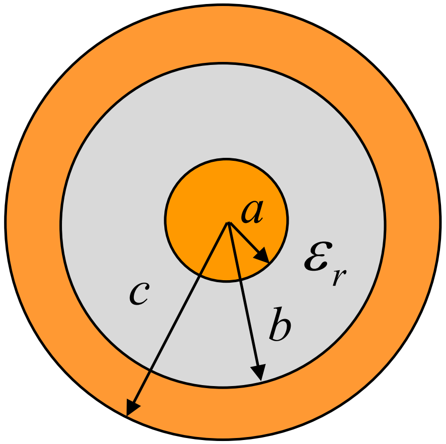
$$\mu = \mu_0 = 4\pi \times 10^{-7}\,[\text{H/m}]$$

$$\sigma = 5.8 \times 10^{7}\,[\text{S/m}]$$

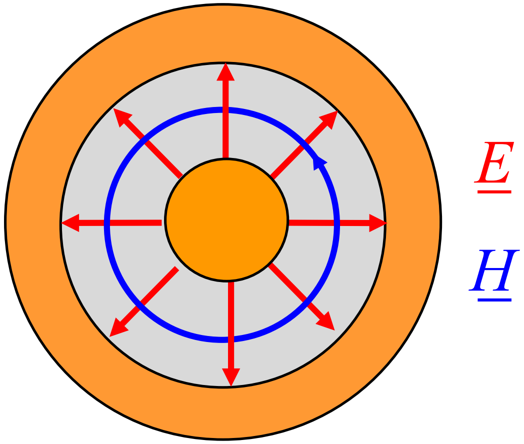
Frequency	δ
1 [Hz]	6.6 [cm]
10 [Hz]	2.1 [cm]
100 [Hz]	6.6 [mm]
1 [kHz]	2.1 [mm]
10 [kHz]	0.66 [mm]
100 [kHz]	0.21 [mm]
1 [MHz]	66 [μm]
10 [MHz]	21 [μm]
100 [MHz]	6.6 [μm]
1 [GHz]	2.1 [μm]
10 [GHz]	0.66 [μm]
100 [GHz]	0.21 [μm]

Skin Depth (cont.)

The same penetration principle holds for curved conductors, as long as the radius of curvature is large compared with the skin depth.

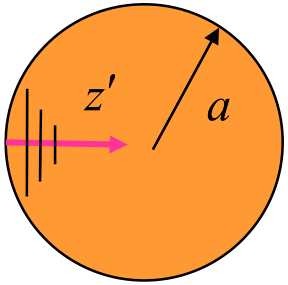


Coax

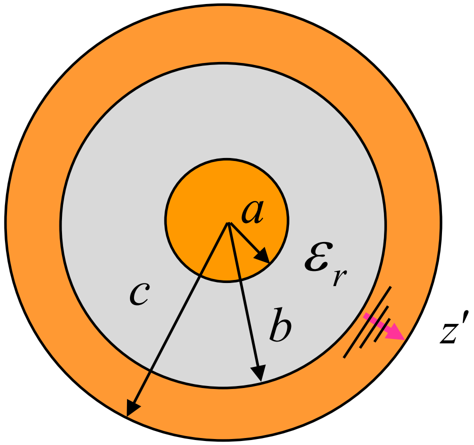


$$E_x(z) = E_{x0}\, e^{-j(z'/\delta)}\, e^{-(z'/\delta)}$$

The distance z' is now measured from the boundary of the conductor.



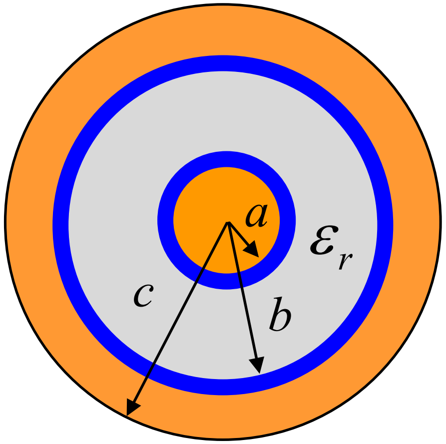
Penetration into inner conductor



Penetration into outer conductor

Skin Depth (cont.)

Coax

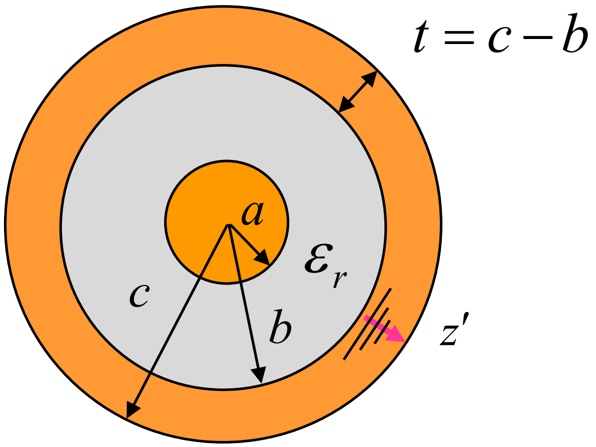


Regions of strong currents

(outer surface of inner conductor and inner surface of outer conductor)

Skin Depth (cont.)

Coax

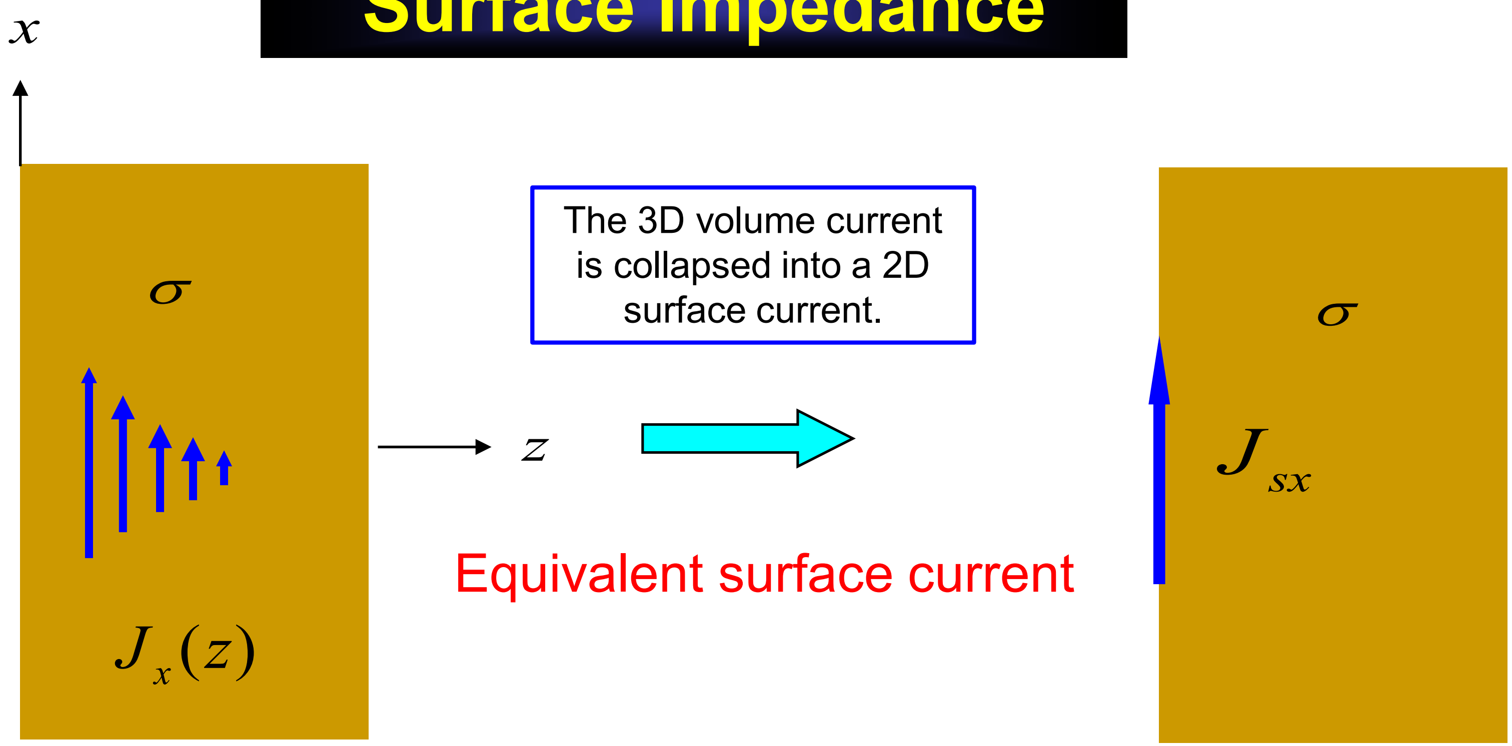


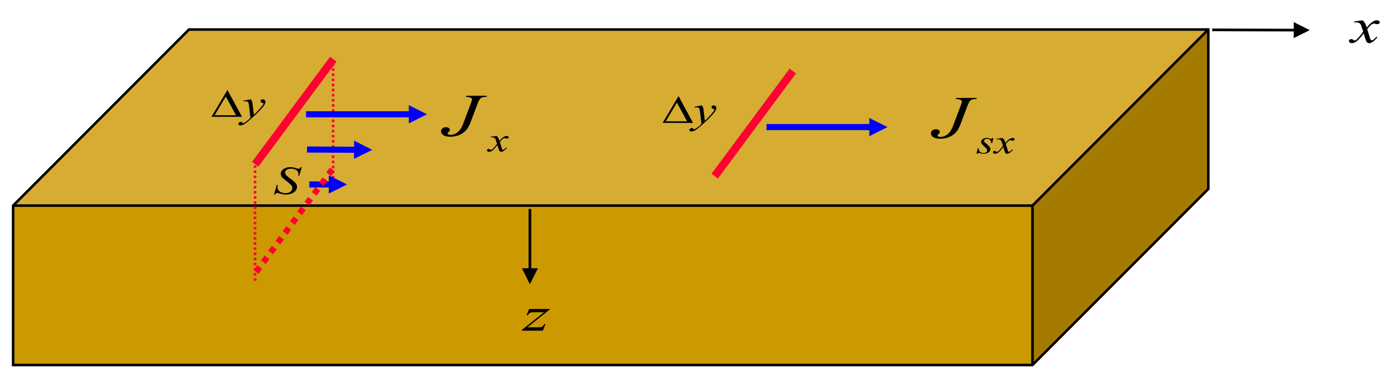
Penetration into outer conductor

The fields are <u>confined</u> inside the coax if

$$\delta << t$$

Surface Impedance





Surface Impedance (cont.)

$$I = \int_S J_x(z)\,dS = \Delta y \int_0^{\infty} J_x(z)\,dz$$ Actual current

$$I = J_{sx}\Delta y$$ Surface current model

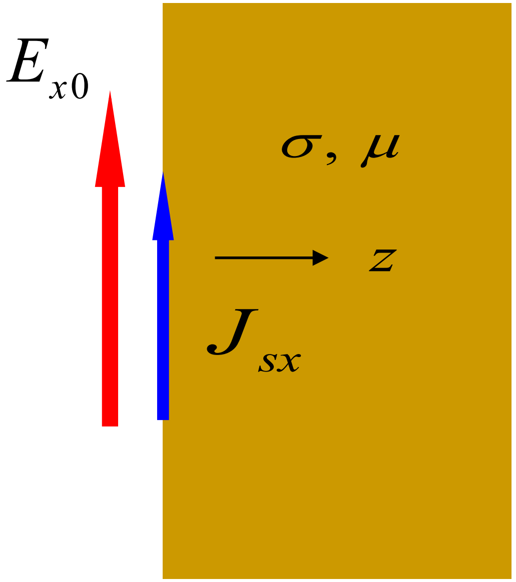
Hence

$$J_{sx} = \int_0^{\infty} J_x(z)\,dz$$

Surface Impedance (cont.)

Define the surface impedance:

$$Z_s \equiv \frac{E_{x0}}{J_{sx}}$$



$$E_x(z) = E_{x0}\, e^{-jkz}$$

$$\begin{aligned} J_{sx} &= \int_0^\infty J_x(z)\, dz \\ &= \int_0^\infty \sigma E_{x0}\, e^{-jkz} dz \\ &= \sigma E_{x0} \int_0^\infty e^{-jkz} dz \\ &= \sigma E_{x0} \left(-\frac{1}{jk} e^{-jkz} \right) \Bigg|_0^\infty \end{aligned}$$

$$\Rightarrow \quad J_{sx} = \sigma E_{x0} \left(\frac{1}{jk} \right)$$

Surface Impedance (cont.)

Hence

$$J_{sx} = \sigma E_{x0}\left[\frac{1}{jk}\right]$$

$$= \sigma E_{x0}\left[\frac{1}{j(k' - jk'')}\right]$$

$$= \sigma E_{x0}\left[\frac{1}{(k'' + jk')}\right]$$

$$\approx \sigma E_{x0}\left[\frac{1}{k''(1+j)}\right]$$

$$= \sigma\delta E_{x0}\left[\frac{1}{1+j}\right]$$

We then have

$$Z_s = \frac{E_{x0}}{J_{sx}} = \left(\frac{1}{\sigma\delta}\right)(1+j)$$

Surface Impedance (cont.)

$$Z_s = \left(\frac{1}{\sigma\delta}\right)(1+j) \quad [\Omega]$$

Define "surface resistance" and "surface reactance" from

$$Z_s = R_s + jX_s$$

We then have:

$$R_s = \frac{1}{\sigma\delta} = \sqrt{\frac{\omega\mu}{2\sigma}}$$

$$X_s = R_s$$

$$Z_s = R_s(1+j) \quad [\Omega]$$

Surface Impedance (cont.)

$$R_s = \sqrt{\frac{\omega\mu}{2\sigma}}$$

Example: copper

$$\mu = \mu_0 = 4\pi \times 10^{-7} \text{ [H/m]}$$

$$\sigma = 5.8 \times 10^{7} \text{ [S/m]}$$

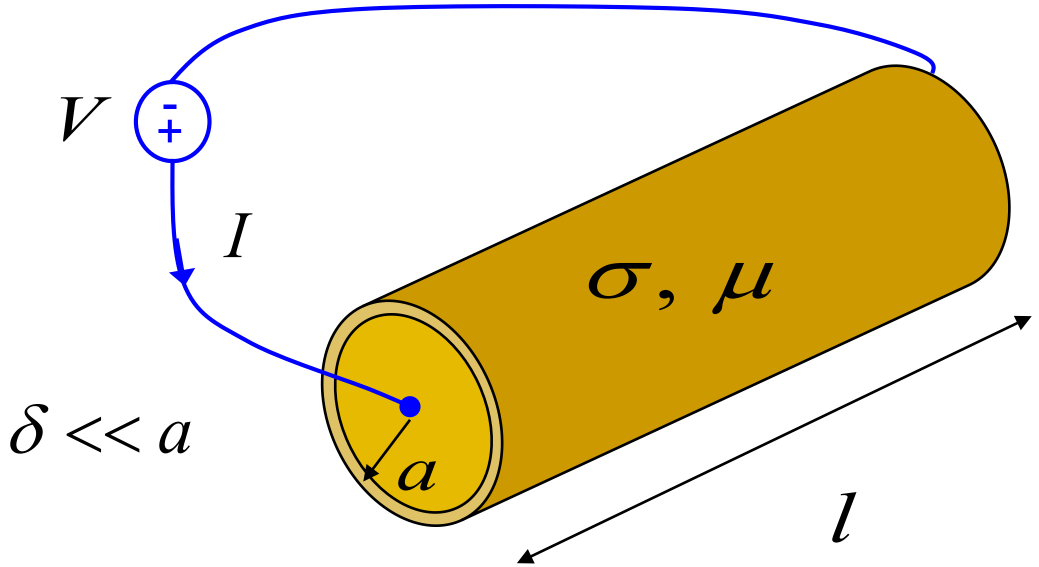
Frequency	R_s
1 [Hz]	2.6×10^{-7} [Ω]
10 [Hz]	8.3×10^{-7} [Ω]
100 [Hz]	2.6×10^{-6} [Ω]
1 [kHz]	8.3×10^{-6} [Ω]
10 [kHz]	2.6×10^{-5} [Ω]
100 [kHz]	8.3×10^{-5}[Ω]
1 [MHz]	2.6×10^{-4} [Ω]
10 [MHz]	8.3×10^{-4} [Ω]
100 [MHz]	0.0026 [Ω]
1 [GHz]	0.0083 [Ω]
10 [GHz]	0.026 [Ω]
100 [GHz]	0.083[Ω]

Impedance of Wire

Find the high-frequency impedance for a solid wire.

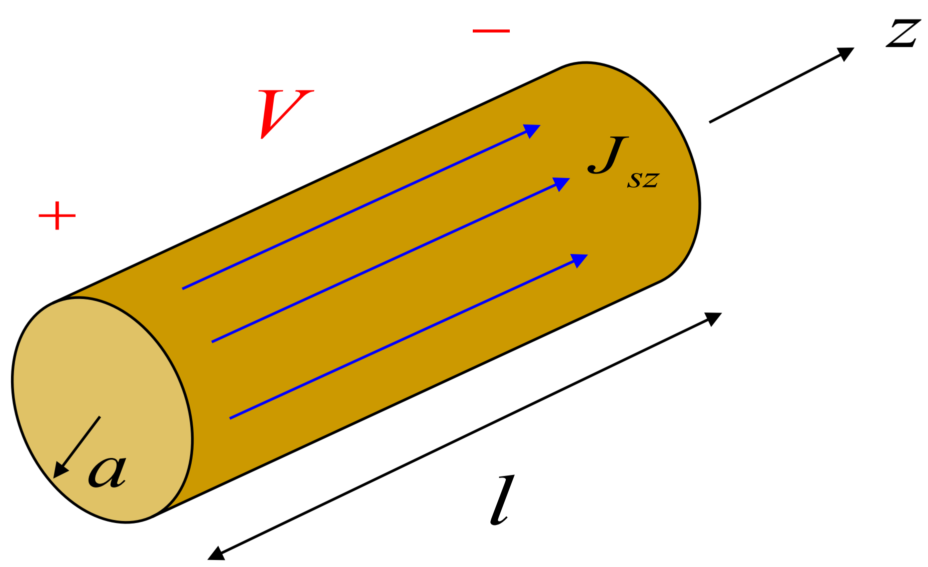
$Z =$ impedance seen by source

$$Z = \frac{V}{I}$$



Note: The current mainly flows on the outside surface of the wire!

Impedance of Wire (cont.)



Surface-current model:

$$I = (2\pi a) J_{sz}$$

$$V = l\,E_{z0}$$

$$Z = \frac{V}{I} = \frac{E_{z0}\,l}{(2\pi a) J_{sz}} = \left(\frac{l}{2\pi a}\right)\left(\frac{E_{z0}}{J_{sz}}\right)$$

Z_s

$Z = R + j\,X =$ impedance

Hence

$$Z = Z_s \left(\frac{l}{2\pi a}\right) = R_s (1 + j)\left(\frac{l}{2\pi a}\right)$$

Therefore, we have $R = X = R_s \left(\frac{l}{2\pi a}\right)$

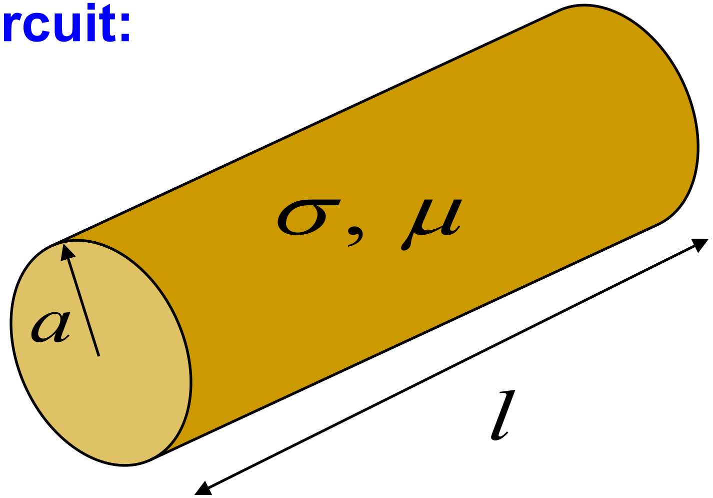
Reminder :

$$Z_s = R_s (1 + j)$$

$$R_s = \left(\frac{1}{\sigma\delta}\right) = \sqrt{\frac{\omega\mu}{2\sigma}}$$

Impedance of Wire (cont.)

Equivalent circuit:



$$R = X = R_s \left(\frac{l}{2\pi a} \right)$$

$$R_s = \left(\frac{1}{\sigma \delta} \right) = \sqrt{\frac{\omega \mu}{2\sigma}}$$

Impedance of Wire (cont.)

Example: copper wire

Assume:

$$\sigma = 5.8 \times 10^{7} \; [\text{S/m}]$$
$$l = 5.0 \; [\text{cm}]$$
$$f = 1.0 \; [\text{GHz}]$$

a	$R = X$
10 [μm]	6.57 [Ω]
0.1 [mm]	0.657 [Ω]
1 [mm]	0.0657 [Ω]
10 [mm]	0.00657 [Ω]

$$R = X = R_s \left(\frac{l}{2\pi a} \right)$$

$$R_s = \left(\frac{1}{\sigma \delta} \right) = \sqrt{\frac{\omega \mu}{2\sigma}}$$

$$R_s = 0.00825 \; [\Omega]$$

$$\delta = \sqrt{\frac{2}{\omega \mu \sigma}} = 2.090 \times 10^{-6} \; [\text{m}]$$

Impedance of Wire (cont.)

Compare with the same wire at DC:

$$R^{\mathrm{DC}} = \frac{l}{\sigma A} = \frac{l}{\sigma\left(\pi a^2\right)}$$

$$\sigma = 5.8\times 10^{7}\ [\mathrm{S/m}]$$
$$l = 5.0\ [\mathrm{cm}]$$

Derivation:

$$R^{\mathrm{DC}} = \frac{V}{I} = \frac{E_{z0} l}{A J_z} = \frac{E_{z0} l}{A\sigma E_{z0}} = \frac{l}{\sigma A} = \frac{l}{\sigma\left(\pi a^2\right)}$$

DC

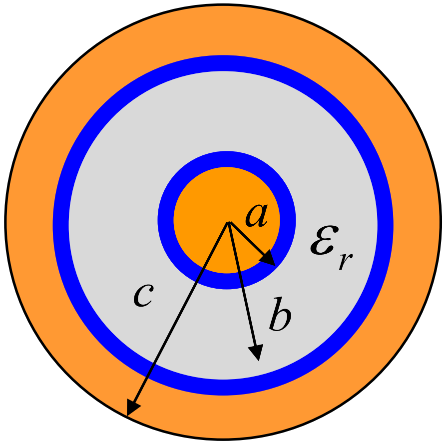
a	R
10 [μm]	2.7 [Ω]
0.1 [mm]	0.027 [Ω]
1 [mm]	2.7×10^{-4} [Ω]
10 [mm]	2.7×10^{-6} [Ω]

1.0 GHz

a	$R = X$
10 [μm]	6.57 [Ω]
0.1 [mm]	0.657 [Ω]
1 [mm]	0.0657 [Ω]
10 [mm]	0.00657 [Ω]

Coax

We use the surface resistance concept to calculate the resistance per unit length R of coax.



For a length l :

$$R_{\text{inner}} = R_{sa}\left(\frac{l}{2\pi a}\right)$$

$$R_{\text{outer}} = R_{sb}\left(\frac{l}{2\pi b}\right)$$

$$R_{\text{total}} = \left(\frac{R_{sa}}{2\pi a} + \frac{R_{sb}}{2\pi b}\right) l$$

Resistance per unit length:

$$R = \left(\frac{R_{sa}}{2\pi a} + \frac{R_{sb}}{2\pi b}\right) \quad [\Omega/\text{m}]$$

$$R_{sa} = \frac{1}{\sigma_a \delta_a} = \sqrt{\frac{\omega \mu_a}{2\sigma_a}} \quad , \quad R_{sb} = \frac{1}{\sigma_b \delta_b} = \sqrt{\frac{\omega \mu_b}{2\sigma_b}}$$

Coax (cont.)

Coax Formulas
(*R*, *L*, *G*, *C*)

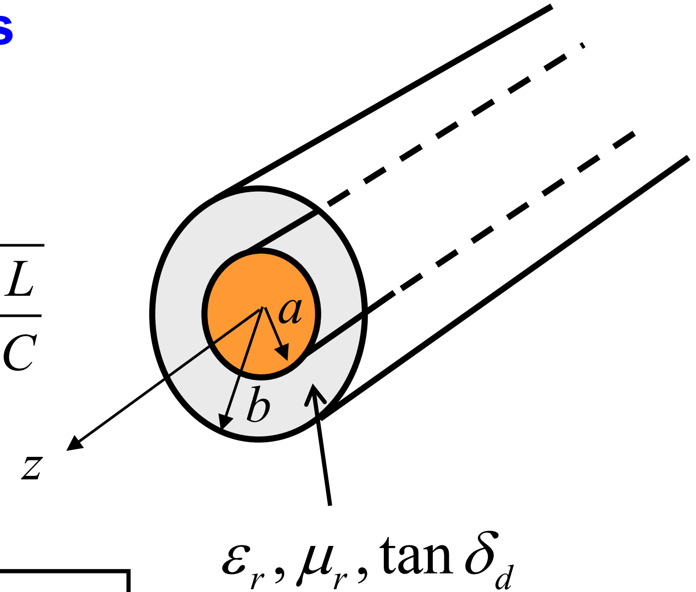
$$C = \frac{2\pi\varepsilon_0\varepsilon_r}{\ln\left(\frac{b}{a}\right)} \quad [\text{F/m}]$$

$$L = \frac{1}{2\pi}\ln\left(\frac{b}{a}\right)\mu_0\mu_r \quad [\text{H/m}]$$

$$G = (\omega C)\tan\delta_d \quad [\text{S/m}]$$

$$R = \left(\frac{R_{sa}}{2\pi a} + \frac{R_{sb}}{2\pi b}\right) \quad [\Omega/\text{m}]$$

$$Z_0 = \sqrt{\frac{L}{C}}$$



Note : $\tan\delta_d = \dfrac{\sigma_d}{\omega\varepsilon_0\varepsilon_r}$

Recall:

$$\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)}$$

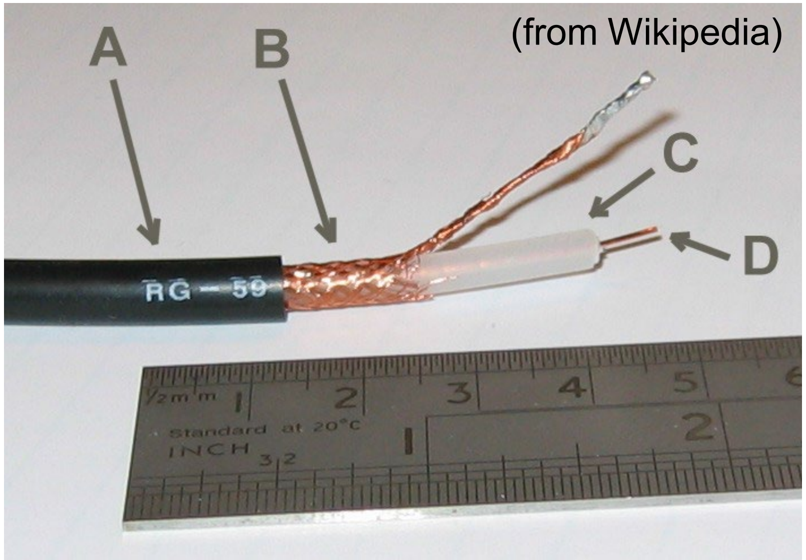
α = attenuation constant (np / m)

Attenuation (dB / m) = 8.686α

Coax (cont.)

Approximate attenuation in dB/m for RG59 coax:

Frequency	dB/m
1 [MHz]	0.01
10 [MHz]	0.03
100 [MHz]	0.11
1 [GHz]	0.40
5 [GHz]	1.0
10 [GHz]	1.5
20 [GHz]	2.3



$$Z_0 = 75 \; [\Omega]$$

$$a = 0.292 \; [\text{mm}]$$

$$b = 1.85 \; [\text{mm}]$$

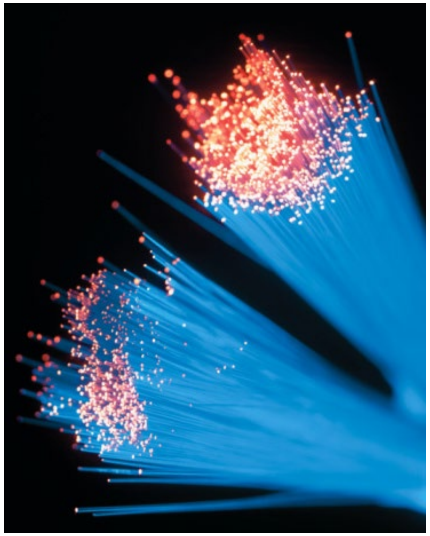
$$\varepsilon_r = 2.25$$

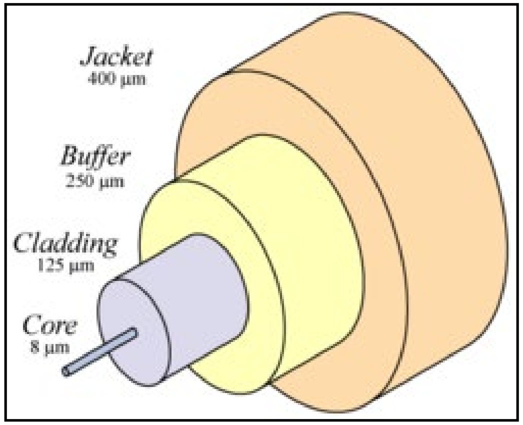
Coax (cont.)

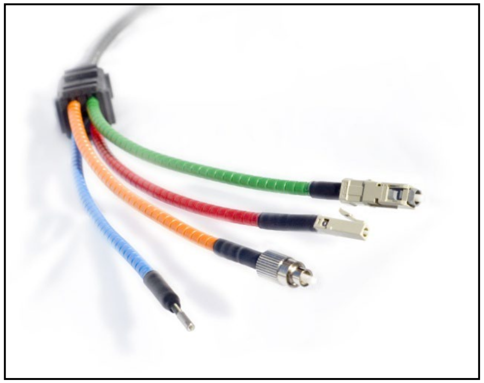
Fiber-optic guides give a much lower attenuation than coax:

RG59 Coax

Frequency	dB/m
1 [MHz]	0.01
10 [MHz]	0.03
100 [MHz]	0.11
1 [GHz]	0.40
5 [GHz]	1.0
10 [GHz]	1.5
20 [GHz]	2.3







Fiber optic cable

Typical single-mode fiber optic cable: 0.3 dB/km
Typical multimode fiber optic cable: 3 dB/km

Appendix

Summary of Formulas

(Good Conductors)

$$E_x(z) = E_{x0}\, e^{-jkz} = E_{x0}\, e^{-jk'z} e^{-k''z} = E_{x0}\, e^{-j(z/\delta)}\, e^{-(z/\delta)}$$

$$k' \approx k'' \approx \sqrt{\frac{\omega\mu\sigma}{2}}$$

$$R = X = R_s \left(\frac{l}{2\pi a} \right) \; [\Omega]$$ (round wire of length l)

$$\delta = d_p = \sqrt{\frac{2}{\omega\mu\sigma}}$$

$$R = \left(\frac{R_{sa}}{2\pi a} + \frac{R_{sb}}{2\pi b} \right) \; [\Omega/\mathrm{m}]$$ (coax)

$$k' \approx k'' \approx \frac{1}{\delta}$$

$$R_{sa} = \frac{1}{\sigma_a \delta_a} = \sqrt{\frac{\omega\mu_a}{2\sigma_a}} \quad , \quad R_{sb} = \frac{1}{\sigma_b \delta_b} = \sqrt{\frac{\omega\mu_b}{2\sigma_b}}$$

$$Z_s = R_s(1 + j)$$

$$R_s = \left(\frac{1}{\sigma\delta} \right) = \sqrt{\frac{\omega\mu}{2\sigma}}$$