ECE 3317
Applied Electromagnetic Waves

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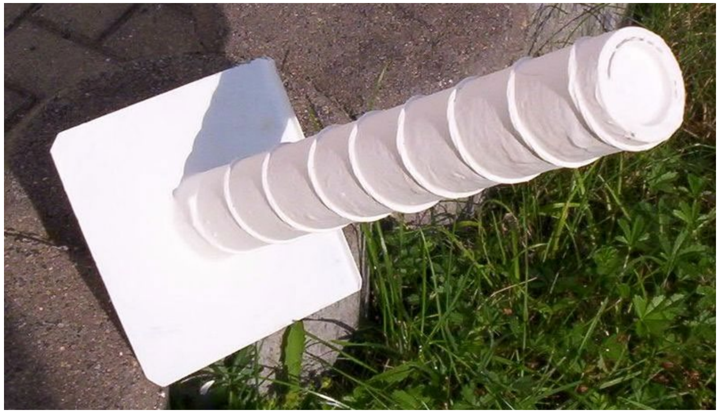
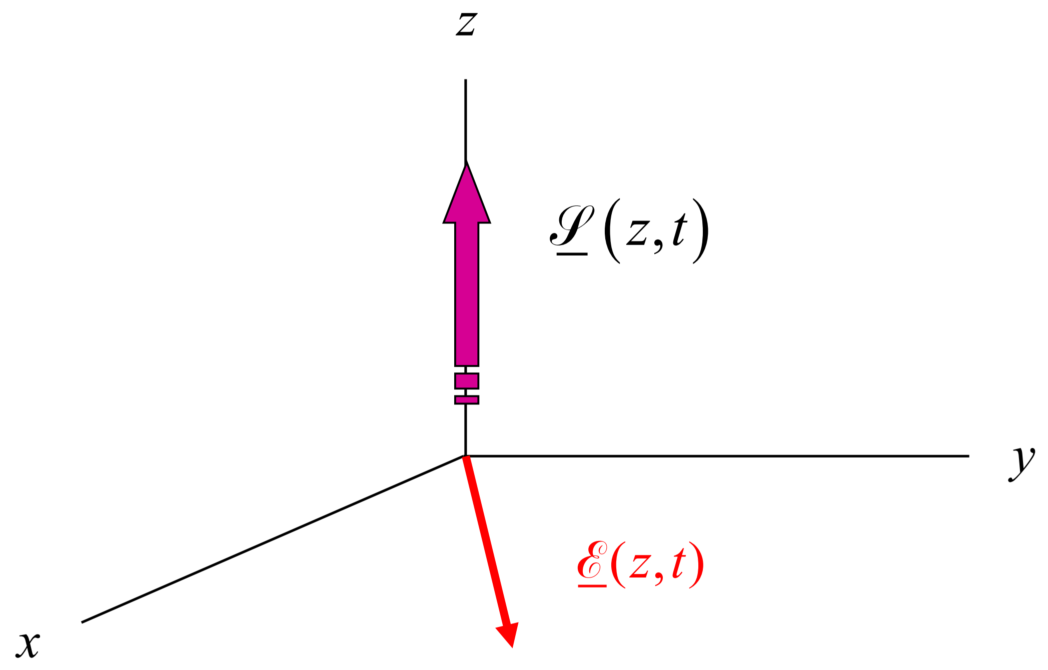
Notes 17

Polarization of Plane Waves



Polarization

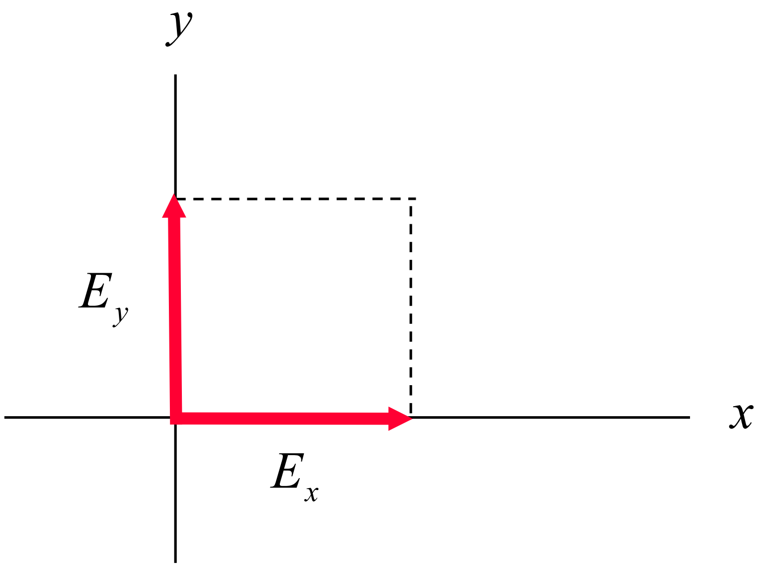
The polarization of a plane wave refers to the direction of the electric field vector in the time domain.



We assume here that the wave is traveling in the positive z direction.

Polarization (cont.)

Consider a plane wave with both x and y components



Phasor domain:

$$\underline{E}(z) = (\underline{\hat{x}}\, E_x + \underline{\hat{y}}\, E_y)\, e^{-jkz}$$

Assume:

$$E_x = a = \text{real number}$$

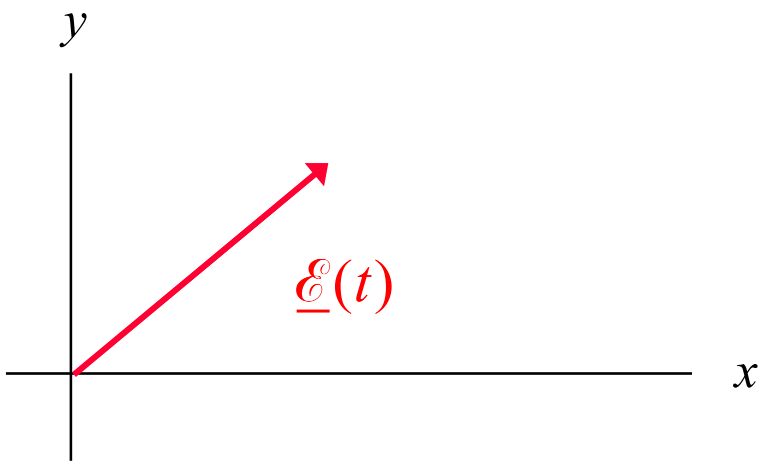
$$E_y = b e^{j\beta}$$

(In general, β = phase of E_y – phase of E_x)

Polarization (cont.)

Time Domain:

$z = 0$



$$\mathcal{E}_x = \mathrm{Re}\left(a e^{j\omega t}\right) = a\cos\left(\omega t\right)$$

$$\mathcal{E}_y = \mathrm{Re}\left(b e^{j\beta}\, e^{j\omega t}\right) = b\cos\left(\omega t + \beta\right)$$

Depending on b/a and β, three different cases arise:

- Linear polarization
- Circular polarization
- Elliptical polarization

Polarization (cont.)

Power Density:

$$\underline{S} = \frac{1}{2}\underline{E} \times \underline{H}^*$$

From Faraday's law:

$$H_y = \frac{E_x}{\eta} \; , \quad H_x = -\frac{E_y}{\eta}$$

Hence

$$\underline{S} = \frac{1}{2}\left(\underline{\hat{x}}\,E_x + \underline{\hat{y}}\,E_y\right) \times \left(\underline{\hat{x}}\left(-\frac{E_y}{\eta}\right) + \underline{\hat{y}}\left(\frac{E_x}{\eta}\right)\right)^*$$

Assume lossless medium (η is real):

$$\underline{S} = \frac{1}{2\eta}\underline{\hat{z}}\left(\left|E_x\right|^2 + \left|E_y\right|^2\right)$$

or

$$\underline{S} = \frac{1}{2\eta}\underline{\hat{z}}\left|\underline{E}\right|^2$$

Linear Polarization

$$\beta = 0 \quad \text{or} \quad \beta = \pi$$

At $z = 0$:

$$\begin{cases} \mathcal{E}_x = a\cos(\omega t) \\ \mathcal{E}_y = b\cos(\omega t + \beta) \end{cases}$$

Recall:

$E_x = a =$ real number

$E_y = be^{j\beta}$

$$\mathcal{E}_x = a\cos\omega t$$

$$\mathcal{E}_y = \pm b\cos\omega t$$

$+$sign: $\beta = 0$

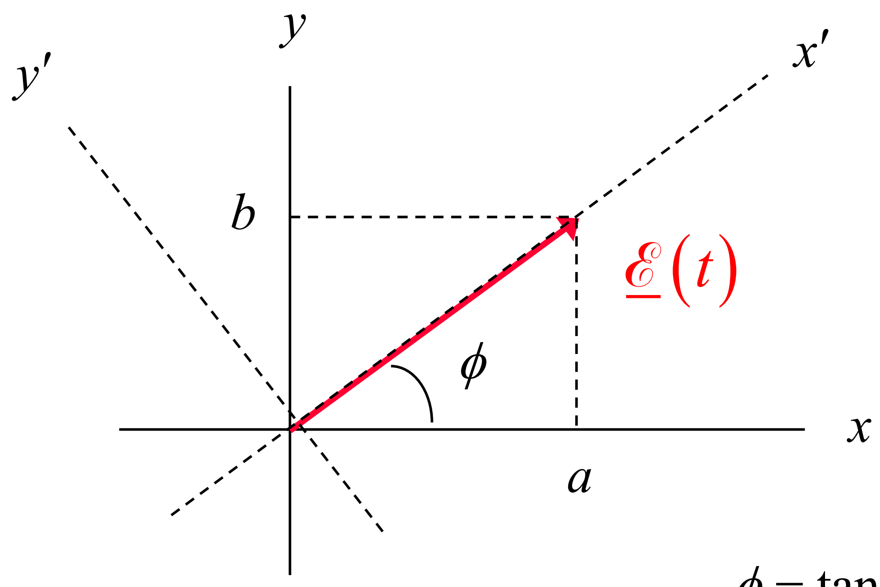
$-$sign: $\beta = \pi$

$$\underline{\mathcal{E}} = \left(\underline{\hat{x}}\,a \pm \underline{\hat{y}}\,b\right)\cos\omega t$$

Linear Polarization (cont.)

$$\underline{\mathcal{E}} = \left(\underline{\hat{x}}\, a \pm \underline{\hat{y}}\, b\right)\cos\omega t$$

This is simply a "tilted" plane wave.



$$\phi = \tan^{-1}\left(\pm\frac{b}{a}\right)$$

(shown for $\beta = 0$)

Circular Polarization

$$b = a \quad \text{and} \quad \beta = \pm\pi / 2$$

At $z = 0$:

$$\begin{cases} \mathcal{E}_x = a\cos(\omega t) \\ \mathcal{E}_y = b\cos(\omega t + \beta) \end{cases}$$

Recall:

$E_x = a =$ real number

$E_y = b e^{j\beta}$

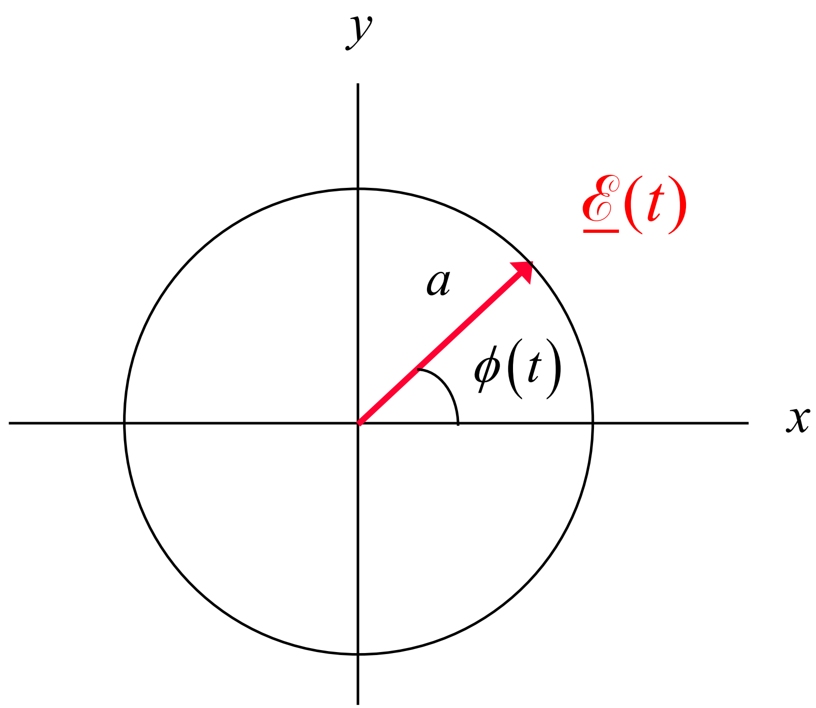
Note: The top sign is always for $\beta = +\pi/2$.

$$\mathcal{E}_x = a\cos\omega t$$

$$\mathcal{E}_y = a\cos(\omega t \pm \pi / 2) = \mp a\sin\omega t$$

$$\left|\underline{\mathcal{E}}\right|^2 = \mathcal{E}_x^2 + \mathcal{E}_y^2 = a^2\cos^2\omega t + a^2\sin^2\omega t = a^2$$

Circular Polarization (cont.)



$$\phi(t) = \tan^{-1}\left(\frac{\mathscr{E}_y}{\mathscr{E}_x}\right) = \tan^{-1}(\mp \tan \omega t) = \mp \tan^{-1}(\tan \omega t)$$

$$\Rightarrow \quad \phi(t) = \mp \omega t$$

Circular Polarization (cont.)

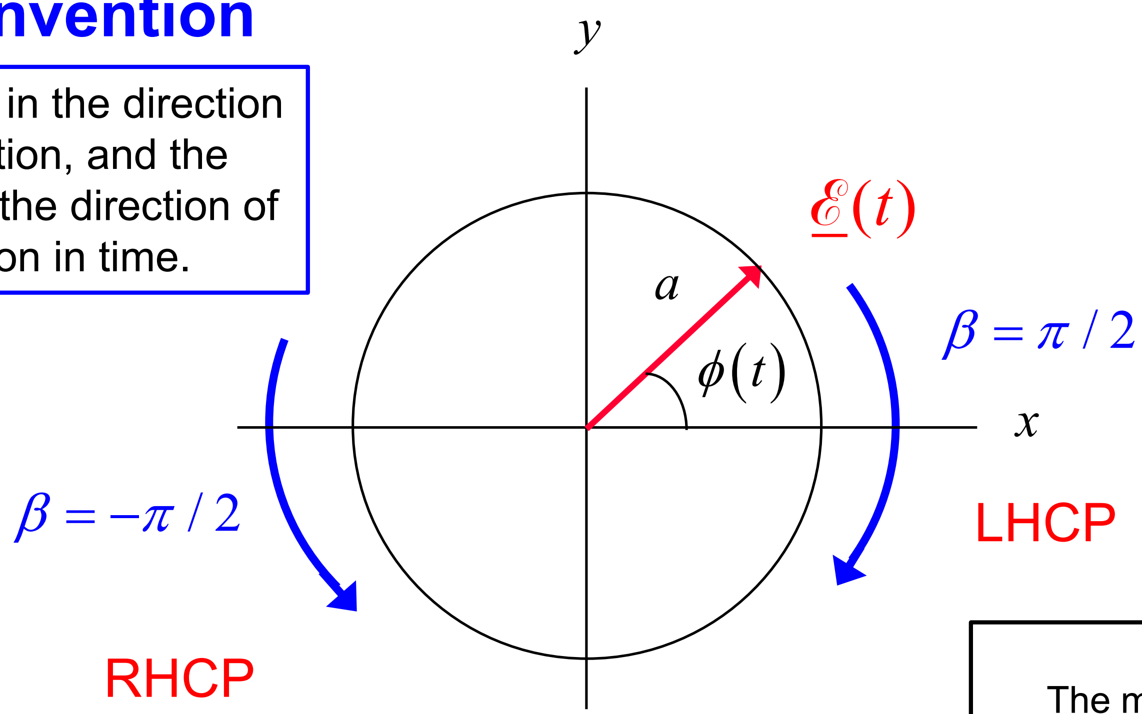
$$\beta = \pm \pi / 2$$

$$\mathscr{E}_x = a\cos(\omega t)$$

$$\mathscr{E}_y = b\cos(\omega t + \beta)$$

IEEE convention

Your thumb is in the direction of propagation, and the fingers are in the direction of the rotation in time.



Note:
The mechanical angular velocity is the same as the electrical radian frequency ω.

$$\phi(t) = \mp \omega t \;\Rightarrow\; \frac{d\phi(t)}{dt} = \mp \omega$$

Circular Polarization (cont.)

Rotation in space vs. rotation in time

Examine how the field varies in both space and time:

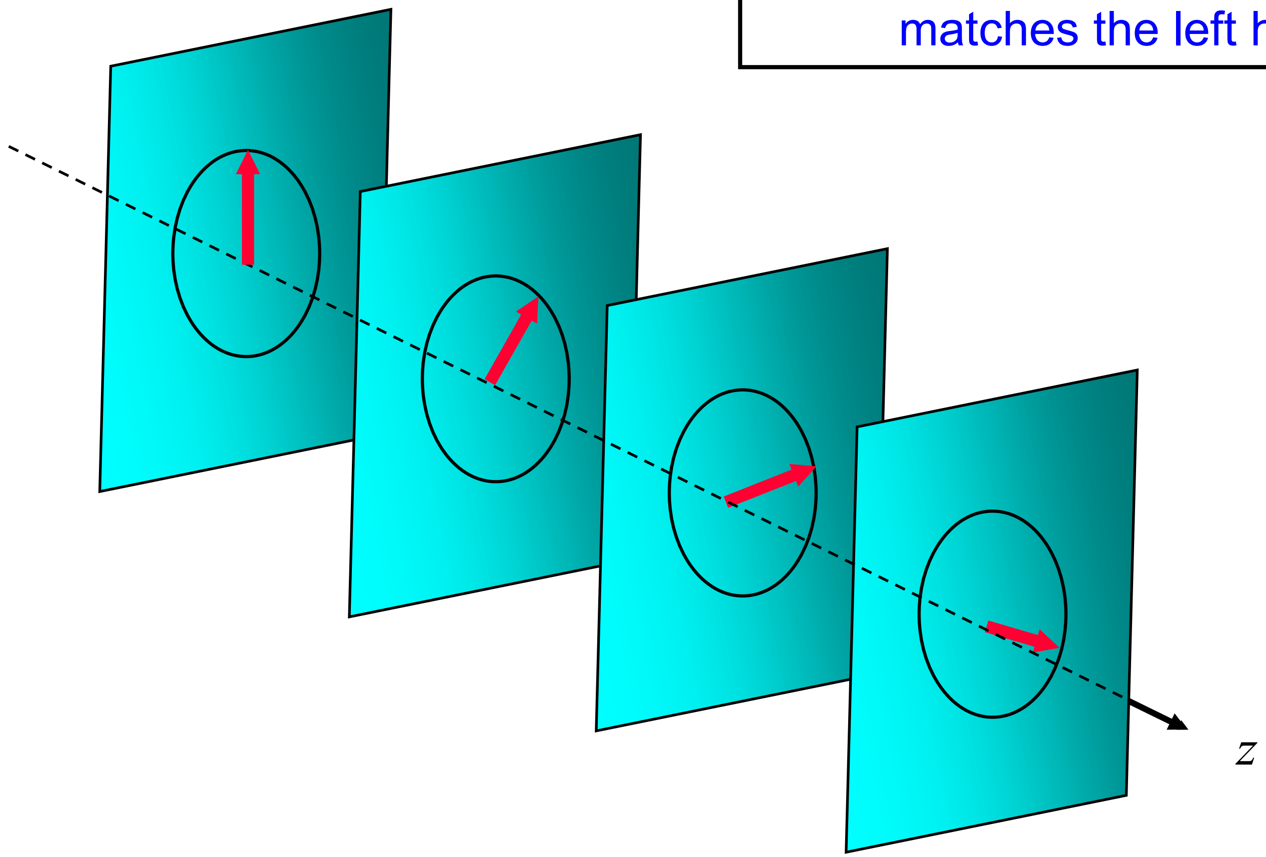
$$\underline{E}(z) = \left(\hat{\underline{x}}\,a + \hat{\underline{y}}\,be^{j\beta}\right)e^{-jkz}$$ Phasor domain

$$\underline{\mathscr{E}}(z,t) = \hat{\underline{x}}\,a\cos\left(\omega t - kz\right) + \hat{\underline{y}}\,b\cos\left(\omega t - kz + \beta\right)$$ Time domain

There is opposite rotation in space and time, due to the minus sign.

Circular Polarization (cont.)

A snapshot of the electric field vector, showing the vector at different points.



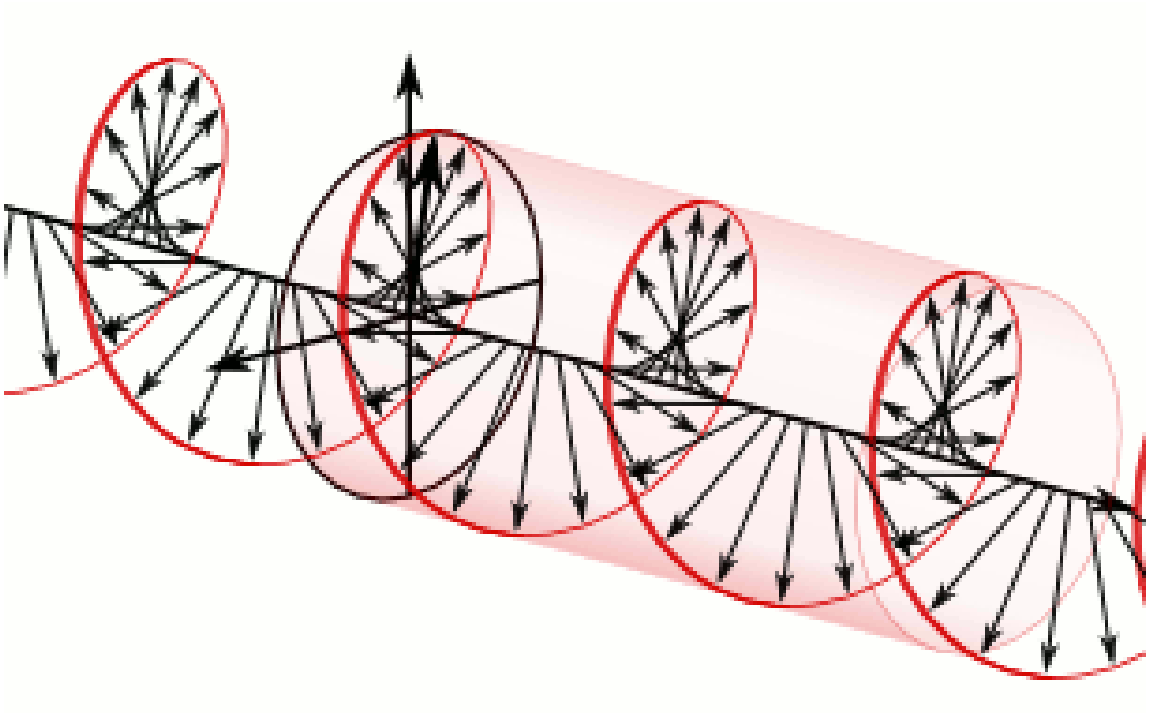
Notice that the rotation in space matches the left hand!

RHCP

Circular Polarization (cont.)

Animation of LHCP wave

(Use pptx version in full-screen mode to see motion.)

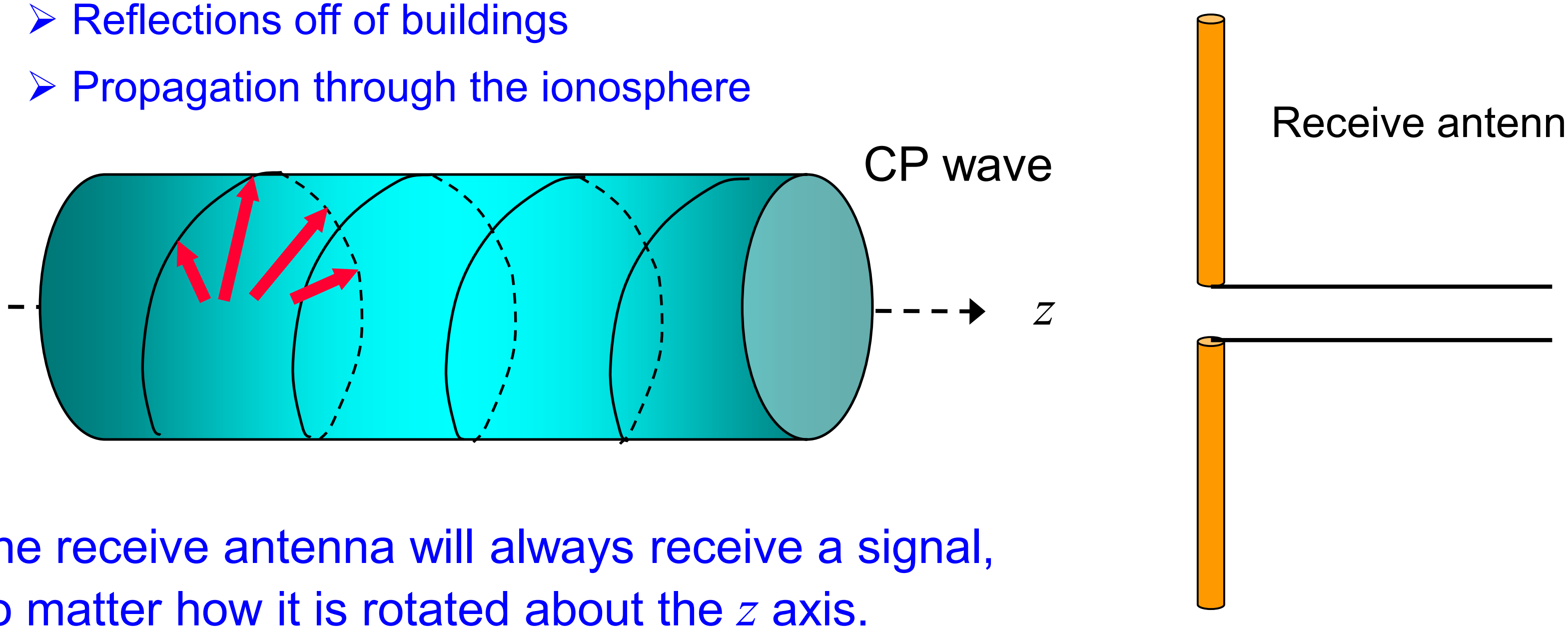


http://en.wikipedia.org/wiki/Circular_polarization

Circular Polarization (cont.)

Circular polarization is often used in wireless communications to avoid problems with signal loss due to polarization mismatch.

- Misalignment of transmit and receive antennas
- Reflections off of buildings
- Propagation through the ionosphere



The receive antenna will always receive a signal, no matter how it is rotated about the z axis.

However, for the same incident power density, an optimum <u>linearly-polarized wave</u> will give the maximum output signal from this linearly-polarized receive antenna (3 dB higher than from an incident CP wave). The linear receive antenna "throws away" half of the incident signal.

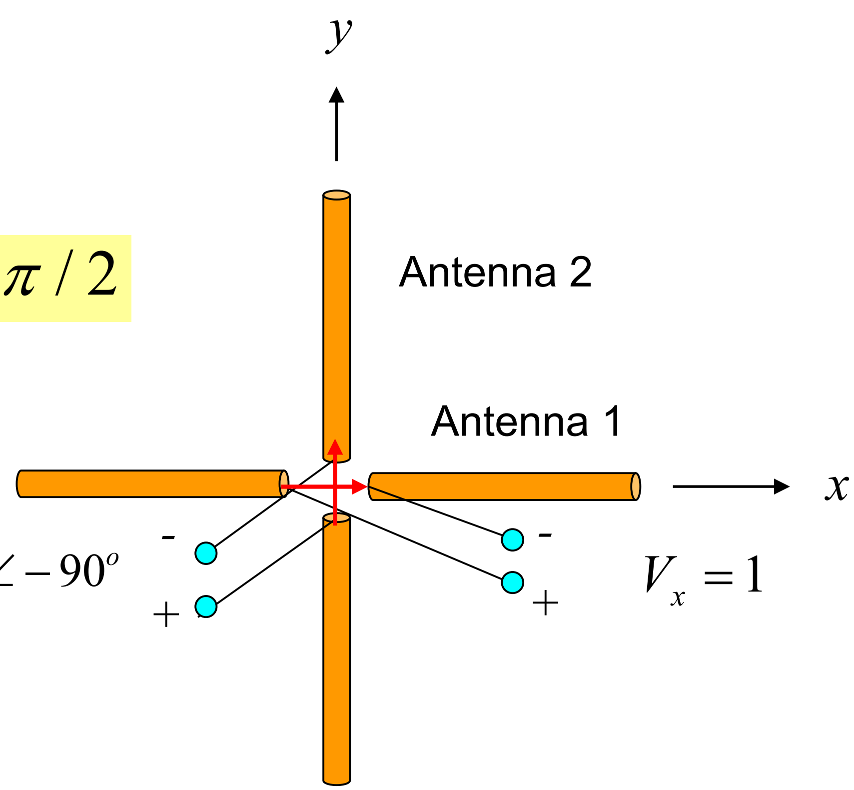
Circular Polarization (cont.)

Two ways in which circular polarization can be obtained:

Method 1)

Use two identical antennas rotated by 90°, and fed 90° out of phase.

$$\beta = -\pi / 2$$



$$V_y = -j = 1\angle -90^o \qquad V_x = 1$$

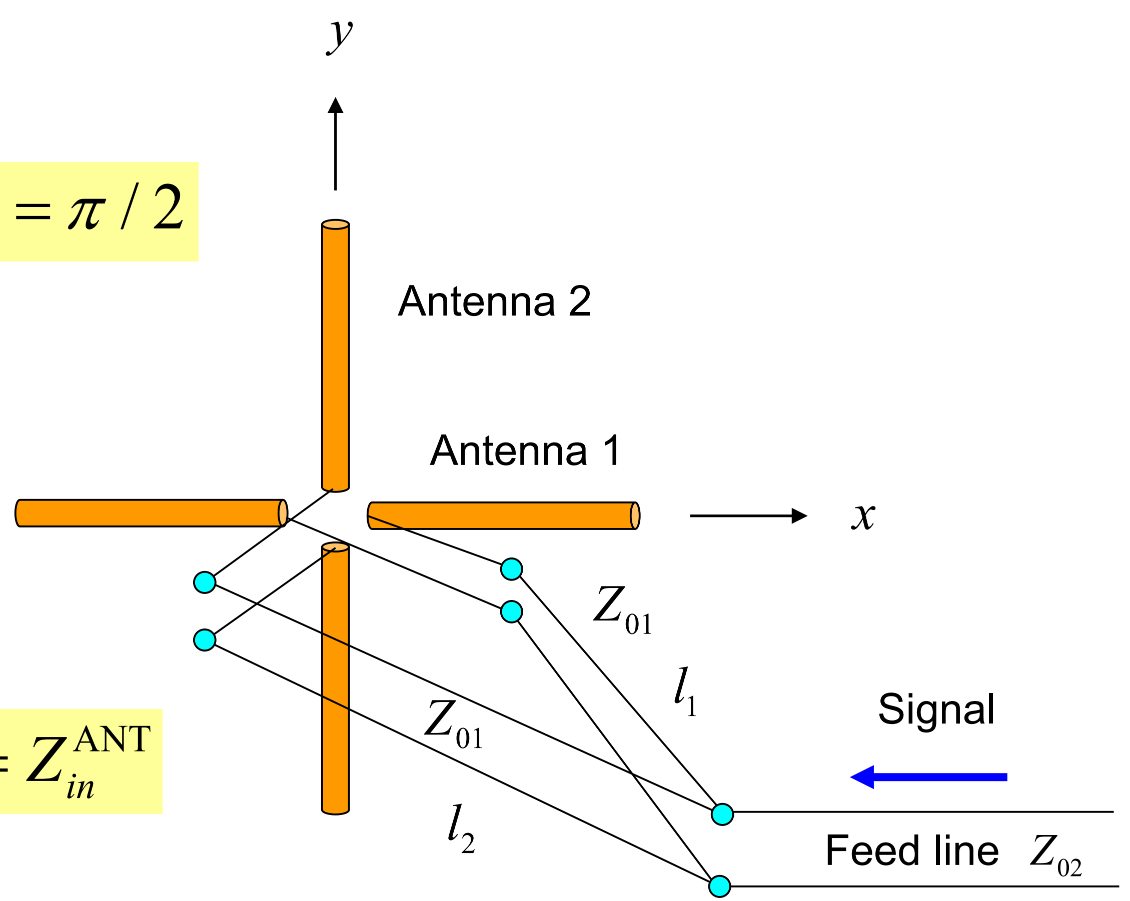
This antenna will radiate a RHCP signal in the positive z direction, and LHCP in the negative z direction.

Circular Polarization (cont.)

Realization of method 1 using a 90° delay line

$$\beta l_2 - \beta l_1 = \pi / 2$$

$$Z_{01} = Z_{in}^{\mathrm{ANT}}$$



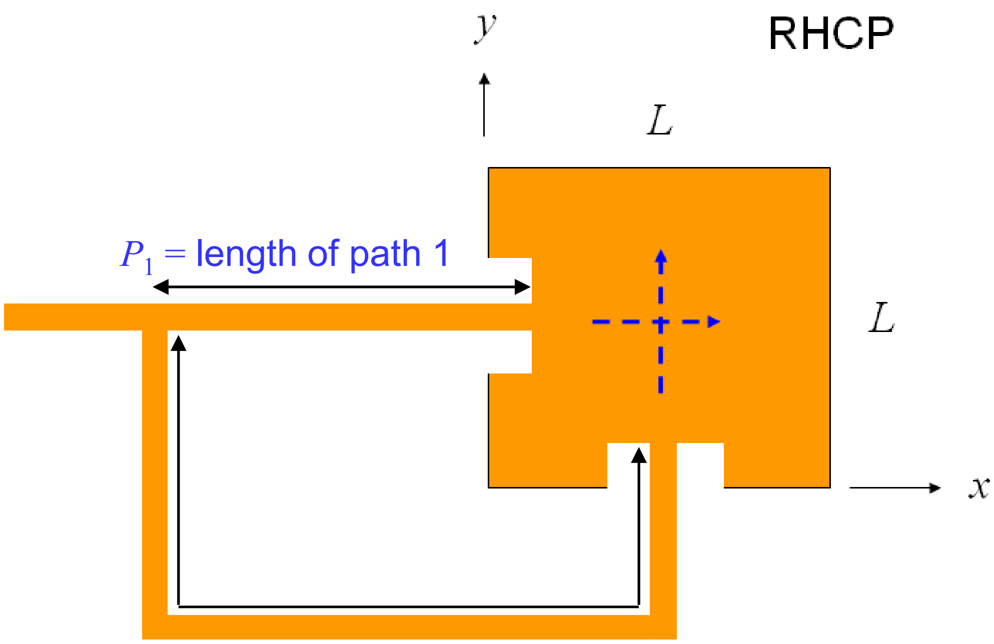
Power splitter : $Z_{02} = Z_{01} / 2$

Circular Polarization (cont.)

An array of CP antennas

Circular Polarization (cont.)

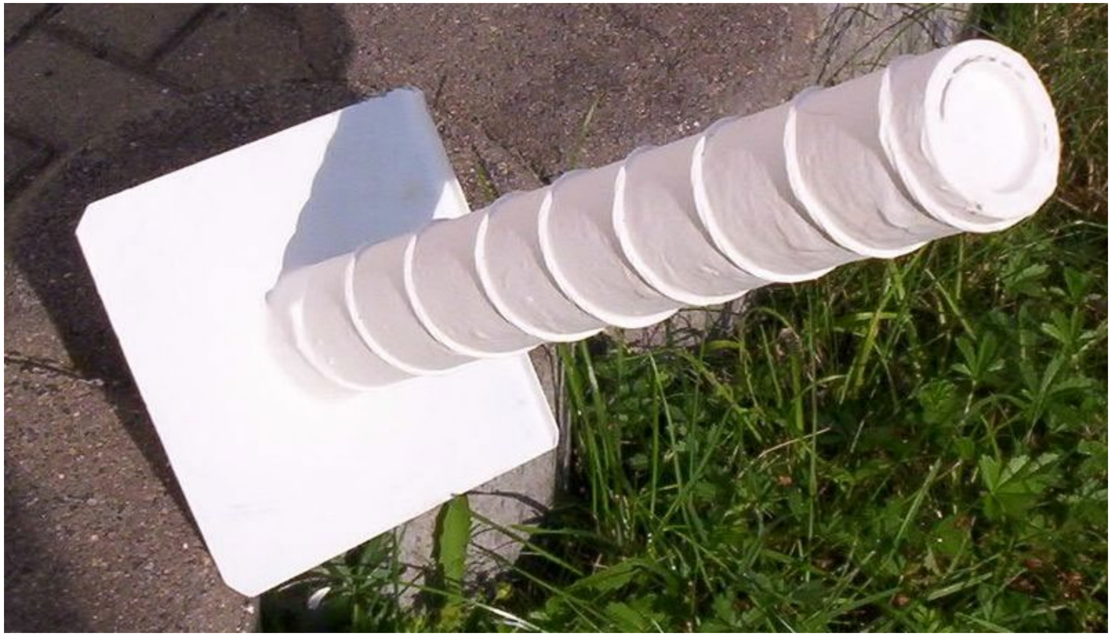
The two antennas can realized by using two different modes of a single microstrip or dielectric resonator antenna.



$$P_2 = P_1 + \lambda_g / 4$$

Circular Polarization (cont.)

Method 2) Use an antenna that inherently radiates circular polarization.

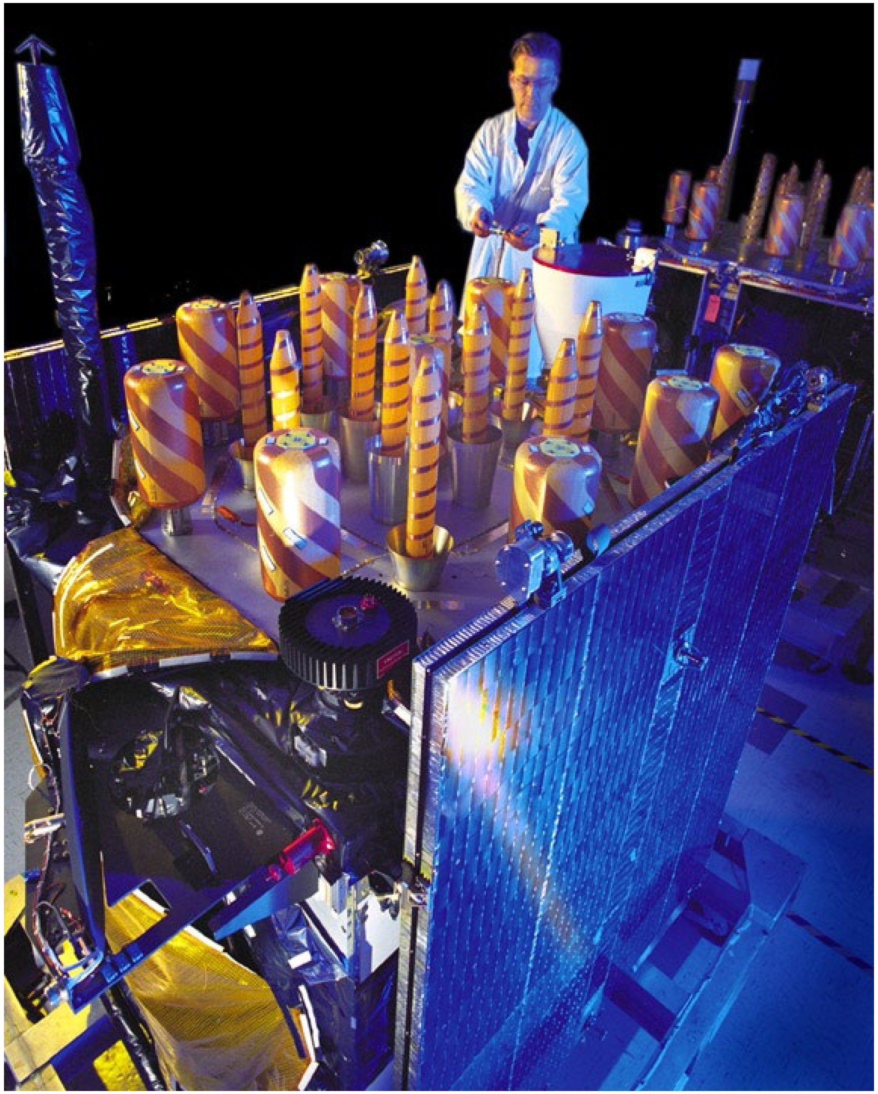


Helical antenna for WLAN communication at 2.4 GHz

http://en.wikipedia.org/wiki/Helical_antenna

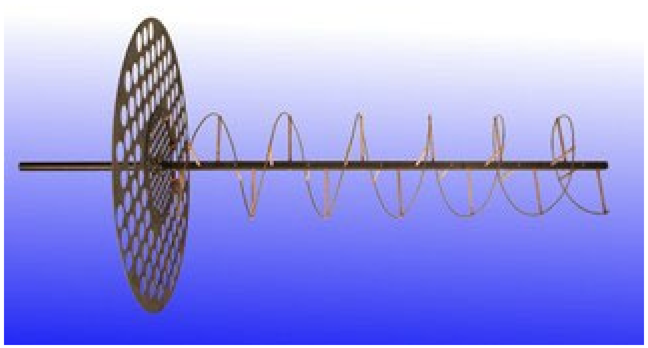
Circular Polarization (cont.)

Helical antennas on a GPS satellite



Circular Polarization (cont.)

Other Helical antennas

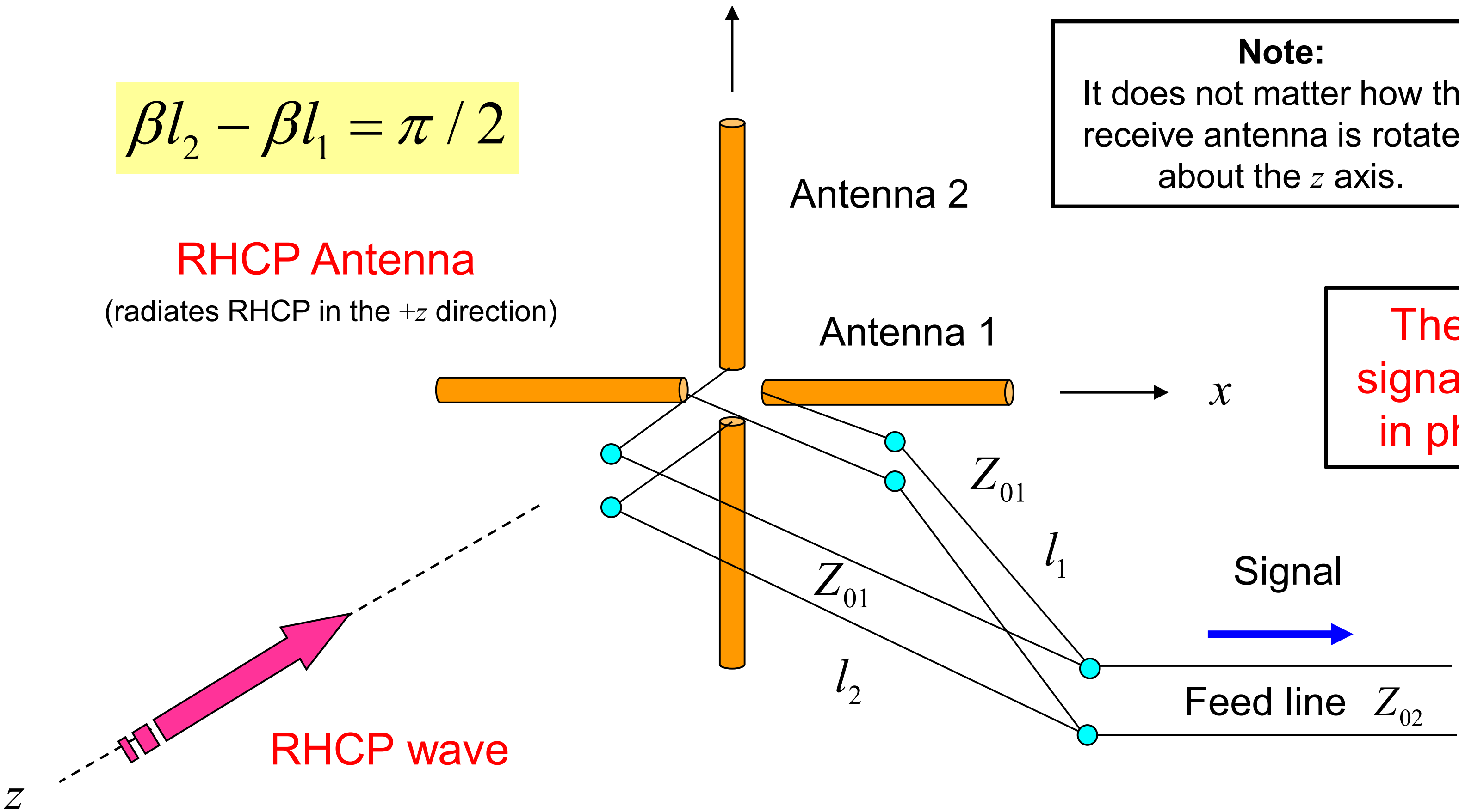


Circular Polarization (cont.)

An antenna that radiates circular polarization will also <u>receive</u> circular polarization of the <u>same</u> handedness, and be blind to the opposite handedness.
(The proof is omitted.)

$$\beta l_2 - \beta l_1 = \pi / 2$$

Note:
It does not matter how the receive antenna is rotated about the z axis.



The two signals add in phase.

Circular Polarization (cont.)

Summary of Possible Polarization Scenarios

1) Transmit antenna is LP, receive antenna is LP

- Simple, works good if both antennas are aligned.
- The received signal is less if there is a misalignment.

2) Transmit antenna is CP, receive antenna is LP

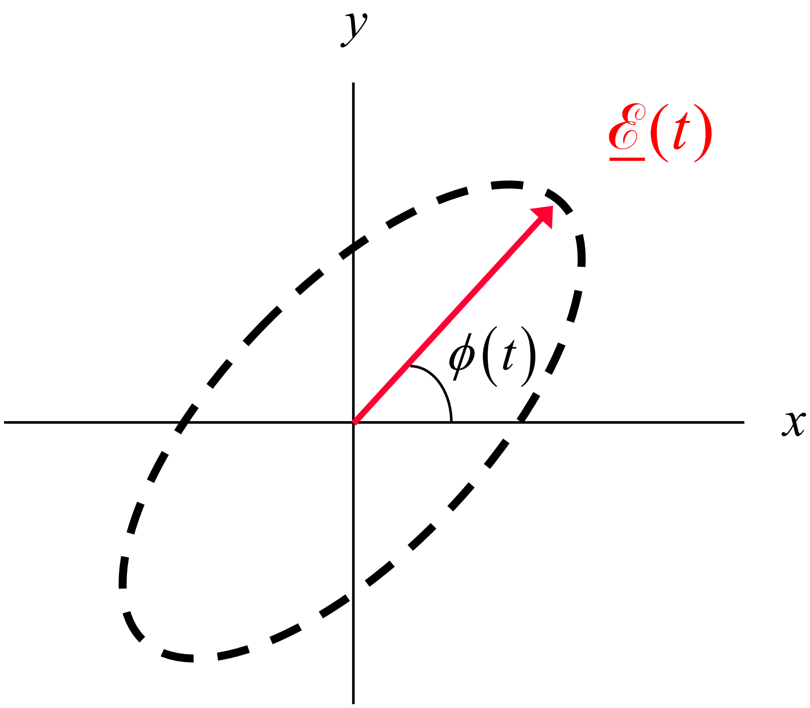
- Signal can be received no matter what the alignment is.
- The received signal is 3 dB less then for two aligned LP antennas.

3) Transmit antenna is CP, receive antenna is CP (of the same handedness)

- Signal can be received no matter what the alignment is.
- There is never a loss of signal, no matter what the alignment is.
- The system is now more complicated.

Elliptic Polarization

Includes all other cases that are not linear or circular



The tip of the electric field vector stays on an ellipse.

(This is proved in Appendix A.)

Elliptic Polarization (cont.)

Rotation Property

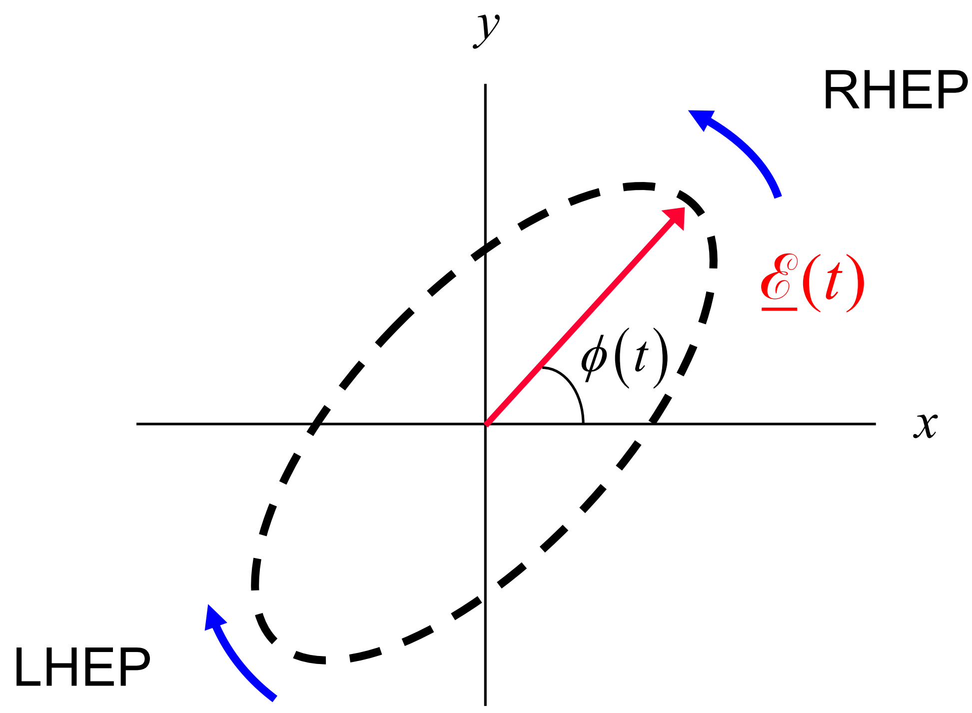
$$\mathcal{E}_x = a\cos(\omega t)$$

$$\mathcal{E}_y = b\cos(\omega t + \beta)$$

Recall:

$E_x = a =$ real number

$E_y = b e^{j\beta}$



$0 < \beta < \pi$	LHEP
$-\pi < \beta < 0$	RHEP

(This is proved in Appendix B.)

Rotation Rule

Here we give a simple graphical method for determining the type of polarization (left-handed or right handed).

Rotation Rule (cont.)

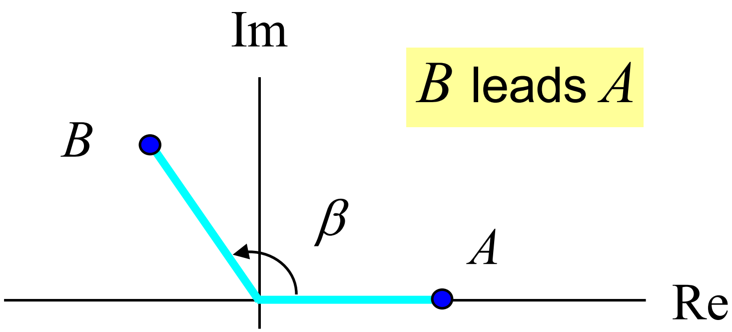
First, we review the concept of leading and lagging sinusoidal waves.

Two phasors: A and B

$$A = ae^{j0}$$
$$B = be^{j\beta}$$

B leads A

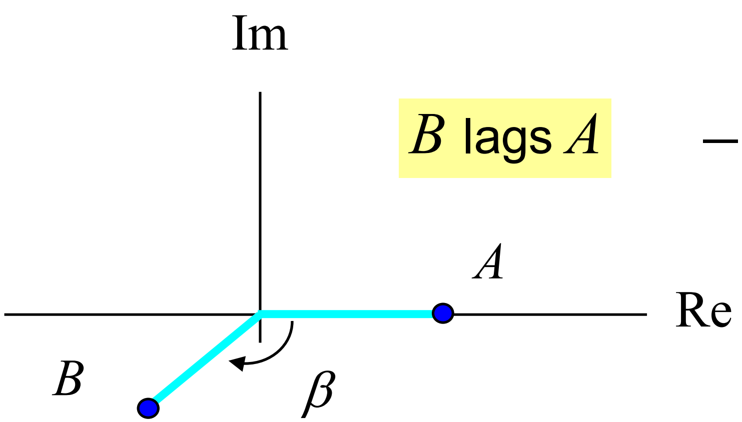
$$0 < \beta < \pi$$



Note:
We can always assume that the phasor A is on the real axis (zero degrees phase) without loss of generality, since it is only the phase difference between the two phasors that is important.

B lags A

$$-\pi < \beta < 0$$

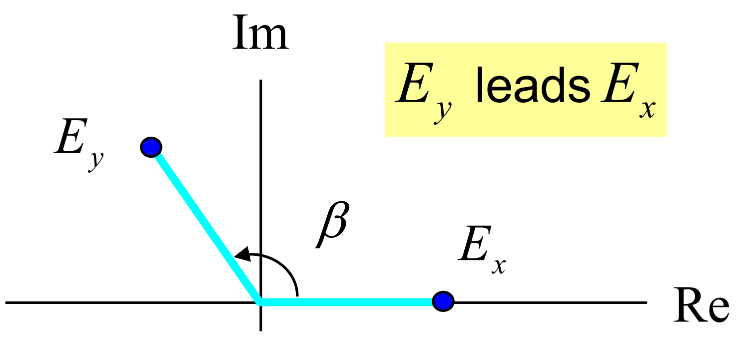


Note:
A lagging sinusoidal wave will appear shifted to the right (later time) on an oscilloscope trace.

Rotation Rule (cont.)

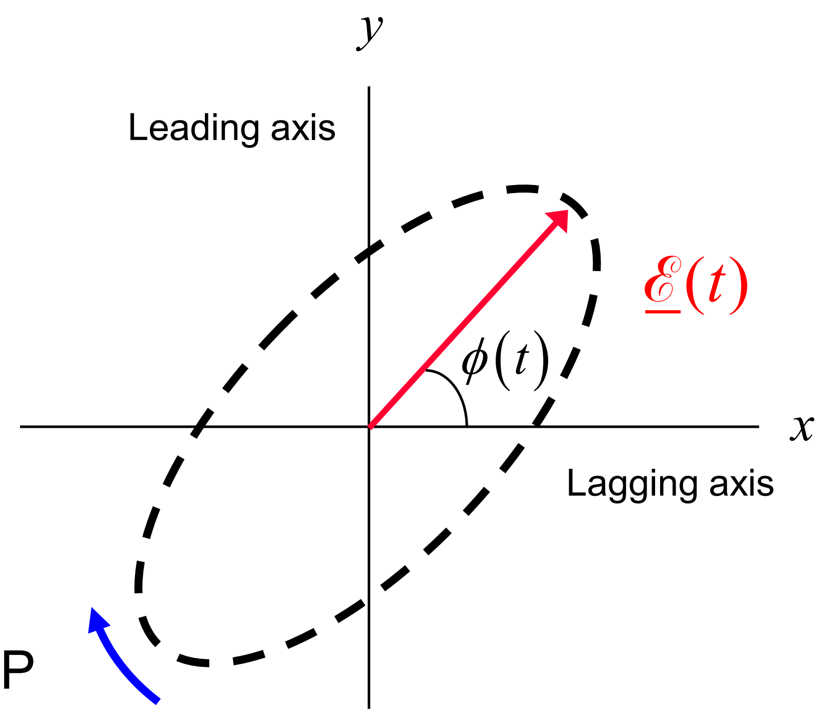
Now consider the case of a plane wave.

$(a) \quad 0 < \beta < \pi$ LHEP



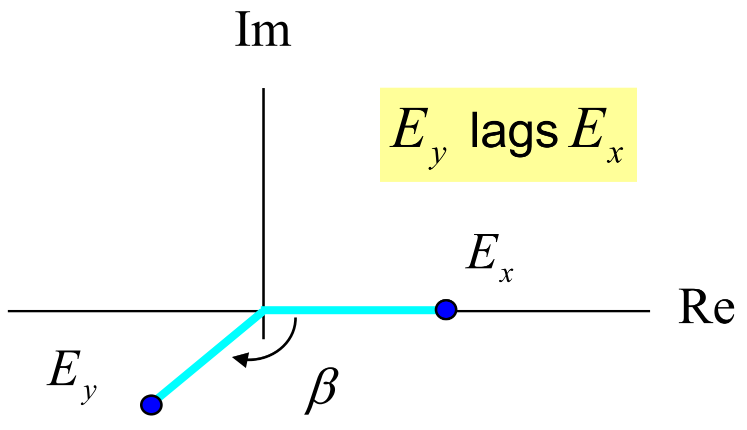
<u>Observation:</u>

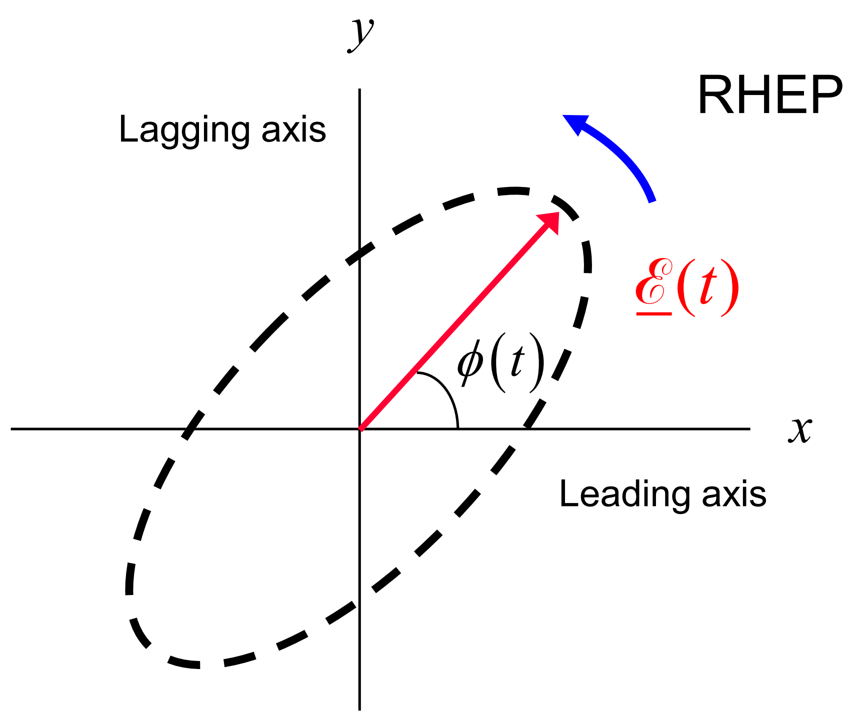
The electric field vector rotates in time from the leading axis to the lagging axis.



Rotation Rule (cont.)

$(b) \quad -\pi < \beta < 0$ RHEP





Observation:

The electric field vector rotates in time from the leading axis to the lagging axis.

Rotation Rule (cont.)

The rule works in both cases, so we can call it a general rule:

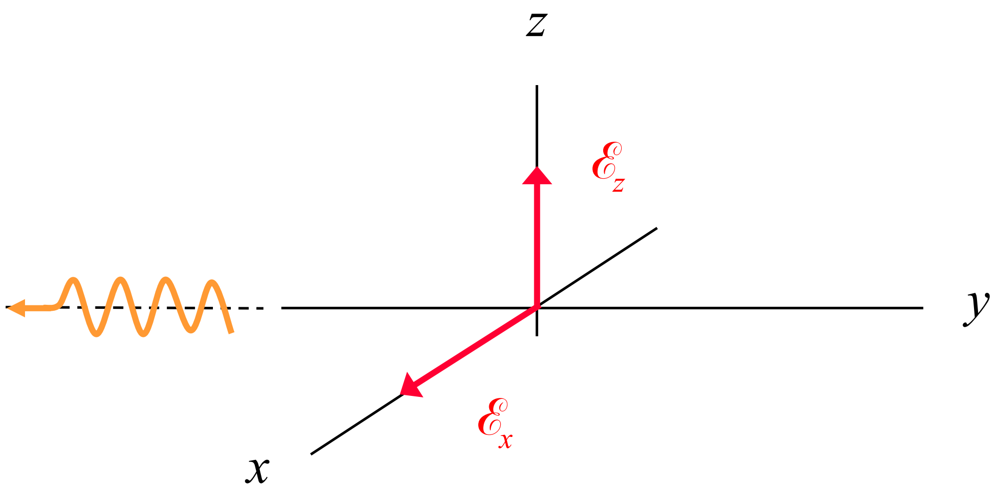
Rotation Rule:

In time, the electric field vector rotates from the leading axis to the lagging axis.

Rotation Rule (cont.)

Example

$$\underline{E} = \left[\underline{\hat{z}}(1+j) + \underline{\hat{x}}(2-j) \right] e^{jky}$$

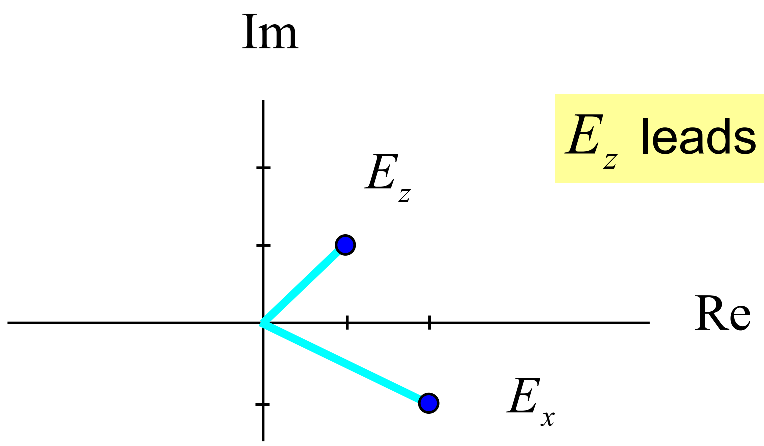


What is this wave's polarization?

Rotation Rule (cont.)

Example (cont.)

$$\underline{E} = \left[\underline{\hat{z}}\,(1+j) + \underline{\hat{x}}\,(2-j) \right] e^{jky}$$



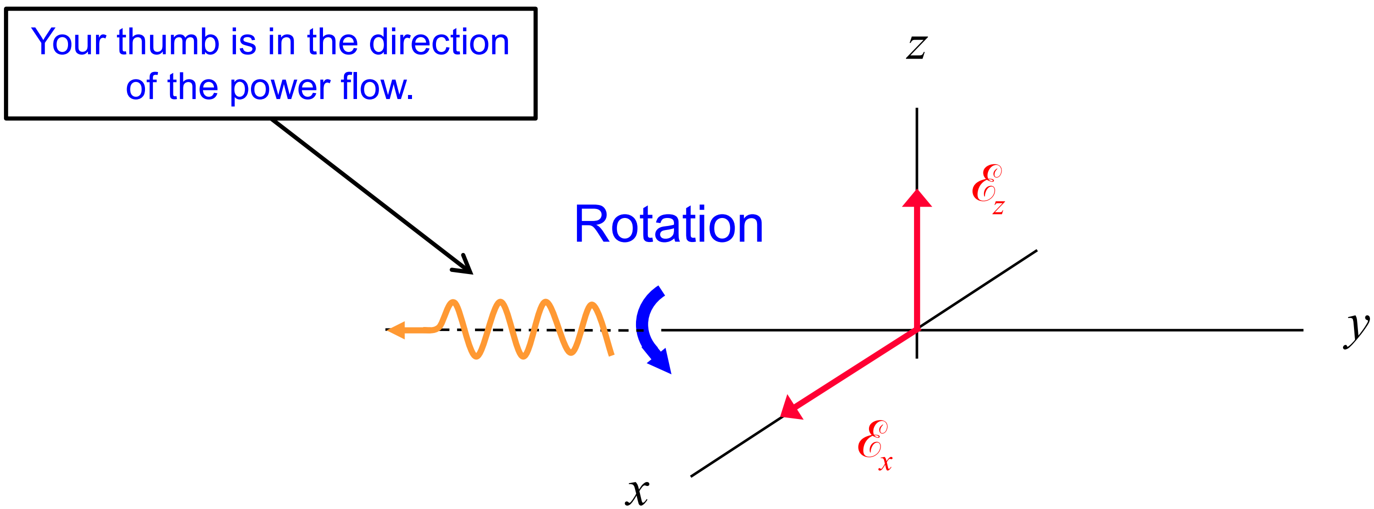
E_z leads E_x

Therefore, in <u>time</u> the wave rotates from the z axis to the x axis.

Rotation Rule (cont.)

Example (cont.)

$$\underline{E} = \left[\hat{\underline{z}}(1+j) + \hat{\underline{x}}(2-j)\right]e^{jky}$$



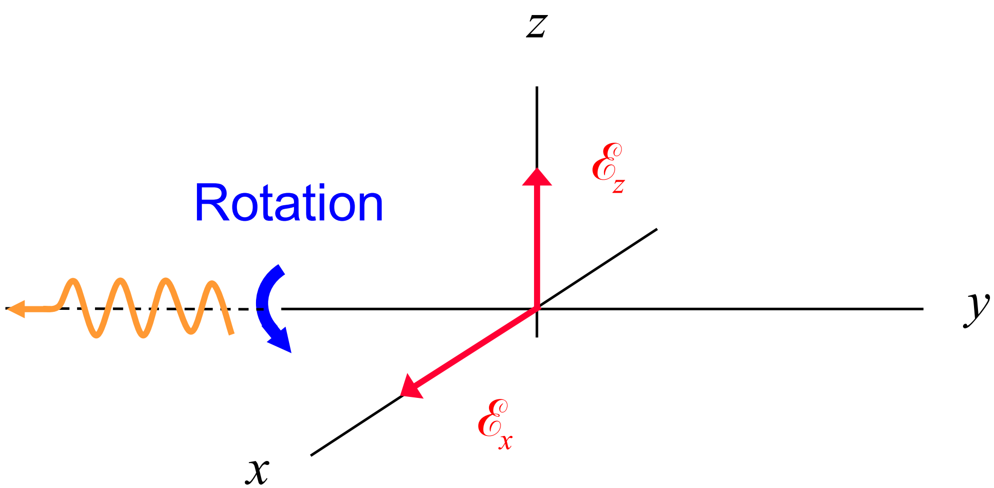
⇒ LHEP or LHCP

Note: $|E_x| \neq |E_z|$ and $\beta \neq \pm\dfrac{\pi}{2}$ (so this is not LHCP)

Rotation Rule (cont.)

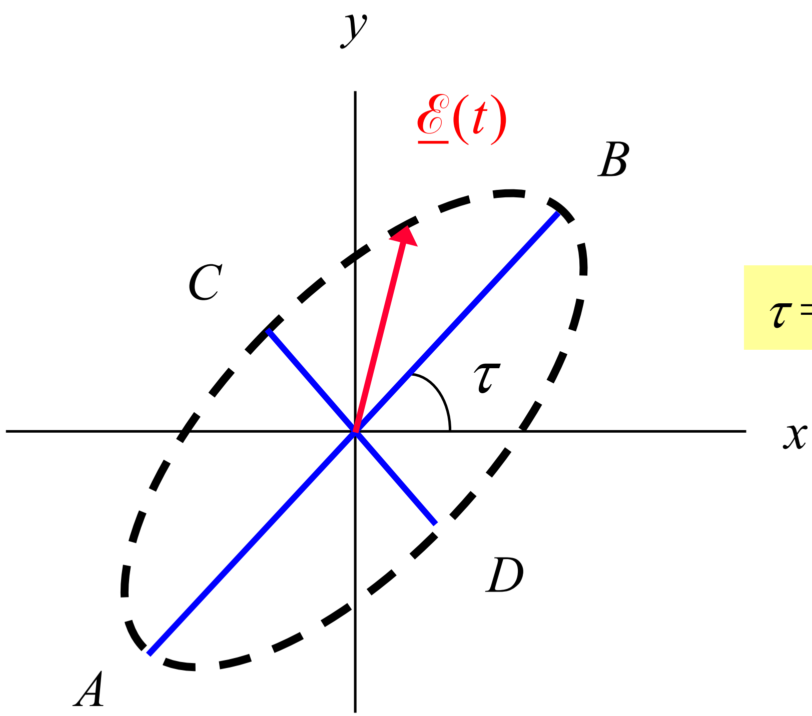
Example (cont.)

$$\underline{E} = \left[\underline{\hat{z}} (1 + j) + \underline{\hat{x}} (2 - j) \right] e^{jky}$$



Conclusion: LHEP

Axial Ratio (AR) and Tilt Angle (τ)



$\tau =$ tilt angle

$$\text{AR} = \frac{\text{major axis}}{\text{minor axis}} = \frac{\text{AB}}{\text{CD}} > 1$$

Note: In dB we have $\text{AR}_{\text{dB}} = 20\log_{10}\left(\text{AR}\right)$

Axial Ratio (AR) and Tilt Angle (τ) Formulas

These formulas assume that the wave has E_x and E_y components, and the power is flowing in the $+z$ direction.

$$\gamma \equiv \tan^{-1}\left(\frac{b}{a}\right)$$

$$0 \le \gamma \le 90^{\circ}$$

Tilt Angle

$$\tan 2\tau = \tan 2\gamma \cos\beta$$

Note:
The tilt angle τ is ambiguous by the addition of $\pm 90^{\circ}$.

Axial Ratio and Handedness

$$\mathrm{AR} = \left|\cot\xi\right|$$

$$\xi > 0: \quad \text{LHEP}$$

$$\xi < 0: \quad \text{RHEP}$$

where

$$\sin 2\xi = \sin 2\gamma \sin\beta$$

$$-45^{\circ} \le \xi \le +45^{\circ}$$

Note on Tilt Angle

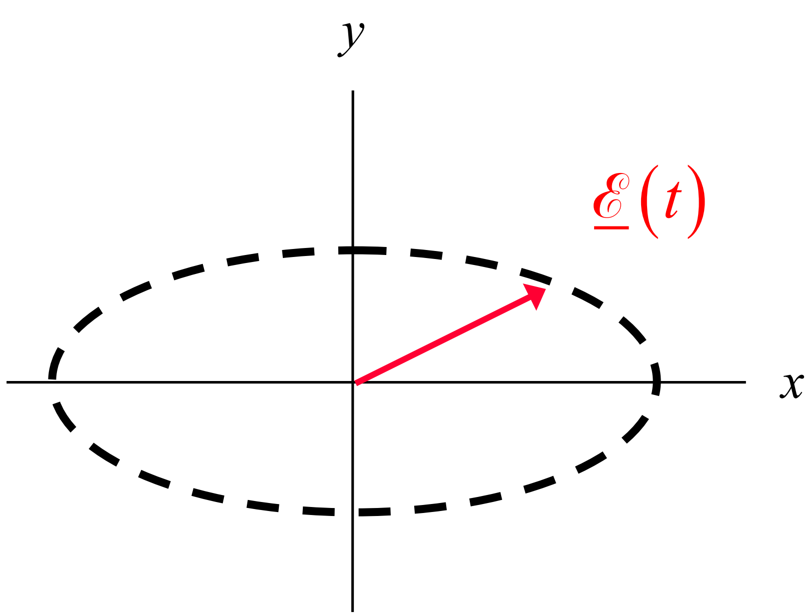
The title angle τ is zero or 90° if:

$$\beta = \pm\pi / 2$$

Tilt Angle: $\tan 2\tau = \tan 2\gamma \cos \beta$

$\Rightarrow 2\tau = 0$ or 180°

$\Rightarrow \tau = 0$ or 90°

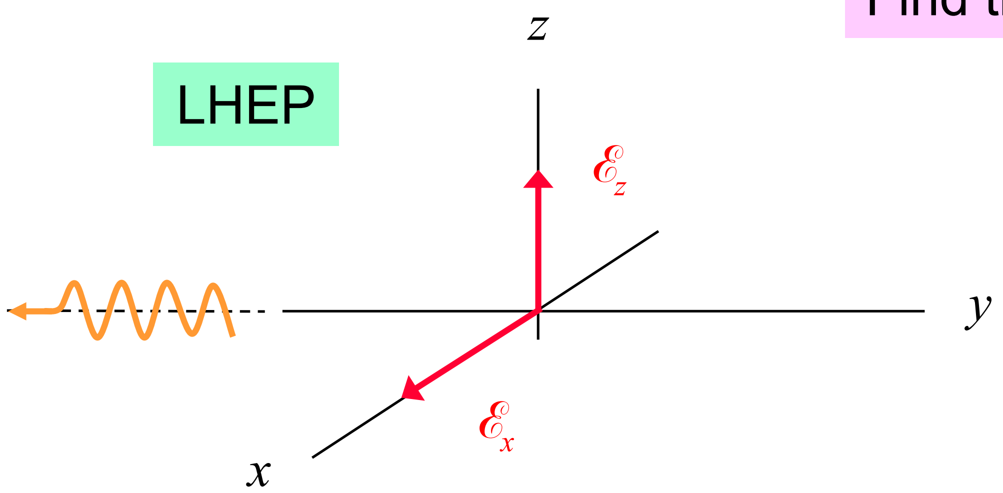


Example

$$\underline{E} = \left[\underline{\hat{z}}(1+j) + \underline{\hat{x}}(2-j)\right] e^{jky}$$

Find the axial ratio and tilt angle.

LHEP



Note:
In order to use the formulas for tilt angle and axial ratio, we need to relabel to coordinate system so that the wave has E_x and E_y components, and the power is flowing in the $+z$ direction.

Re-label the coordinate system:

$$x \to x$$

$$z \to y$$

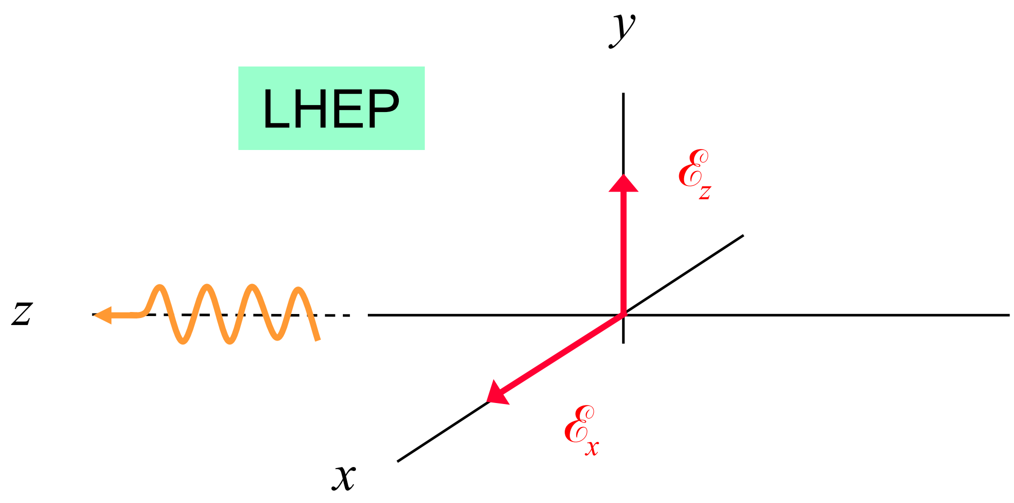
$$y \to -z$$

Note: The new coordinate system needs to be a valid right-handed coordinate system:

$$\underline{\hat{x}} \times \underline{\hat{y}} = \underline{\hat{z}}$$

Example (cont.)

$$\underline{E} = \left[\underline{\hat{y}}(1+j) + \underline{\hat{x}}(2-j) \right] e^{-jkz}$$



Normalize:

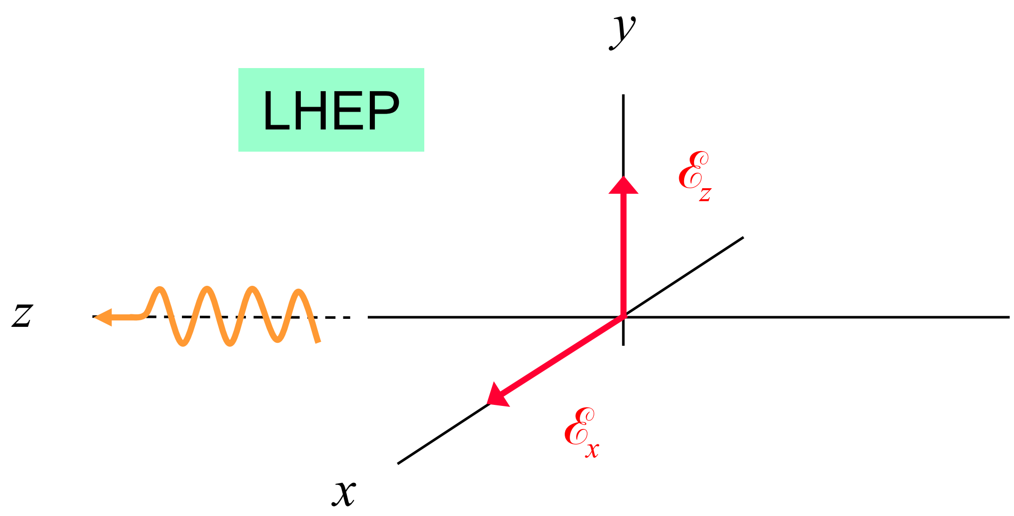
$$\underline{E} = \left[\underline{\hat{x}}(1) + \underline{\hat{y}}\left(\frac{1+j}{2-j} \right) \right] e^{-jkz}$$

or

$$\underline{E} = \left[\underline{\hat{x}}(1) + \underline{\hat{y}}\left(0.6324\, e^{j1.249} \right) \right] e^{-jkz}$$

Example (cont.)

$$\underline{E} = \left[\underline{\hat{x}}(1) + \underline{\hat{y}}\left(0.6324\, e^{j1.249}\right) \right] e^{-jkz}$$



Hence

$a = 1$

$b = 0.6324$

$\beta = 1.249\,[\text{rad}] = 71.565^{\circ}$

$$\beta = \angle E_y - \angle E_x = 71.565^{\circ}$$

$$\gamma = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left(0.632\right) = 0.564\ [\text{rad}]$$

Example (cont.)

$$\tan 2\tau = \tan 2\gamma \cos \beta$$

$$\mathrm{AR} = |\cot \xi|$$

$$\xi > 0: \quad \mathrm{LHEP}$$

$$\xi < 0: \quad \mathrm{RHEP}$$

where

$$\sin 2\xi = \sin 2\gamma \sin \beta$$

$$-45^{\circ} \le \xi \le +45^{\circ}$$

$$\gamma = 0.564 \ [\mathrm{rad}]$$

$$\beta = 71.565^{\circ}$$

Results:

$$\tau = 16.845^{\circ}$$

$$\xi = 29.499^{\circ}$$

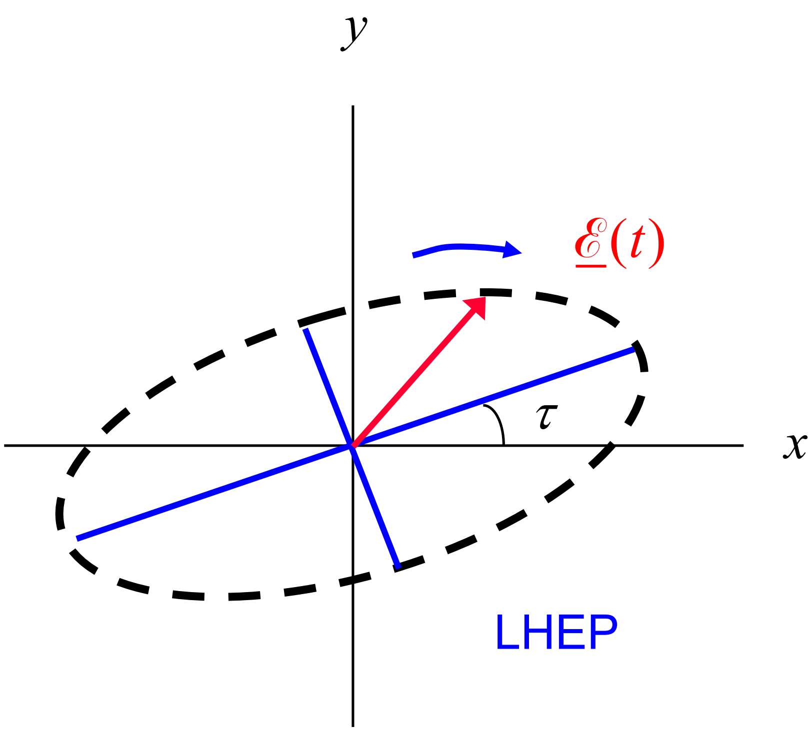
$$\mathrm{AR} = 1.768$$

LHEP

Example (cont.)

$$\mathrm{AR} = 1.768$$

$$\tau = 16.845^{\circ}$$



Note: We are not sure which choice is correct:

$$\tau = 16.845^{\circ} \checkmark$$

$$\tau = 16.845^{\circ} + 90^{\circ} \times$$

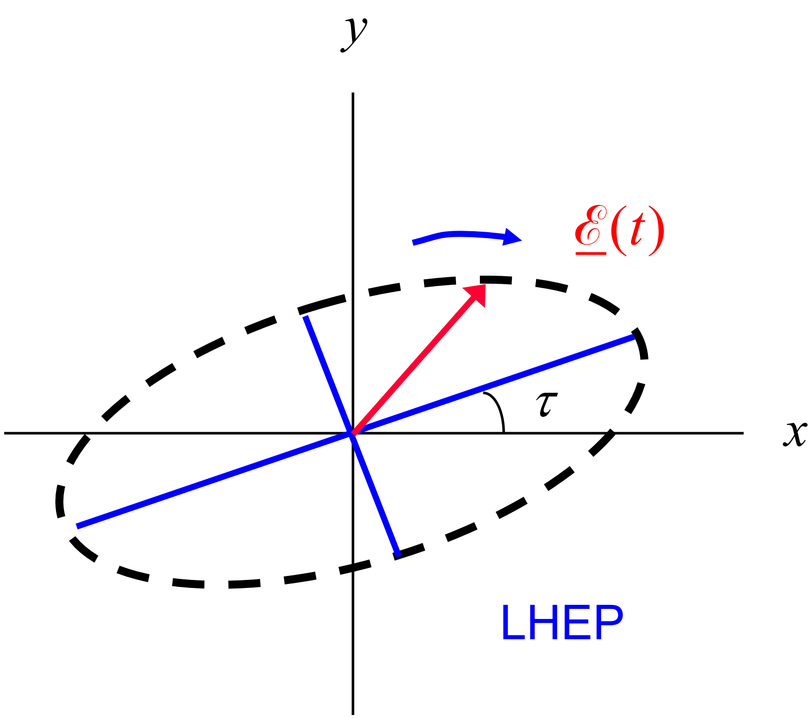
We can make a quick time-domain sketch to be sure.

Example (cont.)

Summary

$$\mathcal{E}_x = a\cos(\omega t)$$

$$\mathcal{E}_y = b\cos(\omega t + \beta)$$



Given:

$a = 1$

$b = 0.6324$

$\beta = 1.249[\text{rad}] = 71.565^{\circ}$

Results:

LHEP

$\text{AR} = 1.768$

$\tau = 16.845^{\circ}$

Appendix A

Here we give a proof that the tip of the electric field vector must stay on an ellipse.

$$\mathscr{E}_x = a\cos\omega t \qquad \mathscr{E}_y = b\cos(\omega t + \beta)$$

$$= b\cos\omega t\cos\beta - b\sin\omega t\sin\beta$$

so

$$\mathscr{E}_y = b\cos\beta\left[\frac{\mathscr{E}_x}{a}\right] - b\sin\beta\left[\sqrt{1-\left(\frac{\mathscr{E}_x}{a}\right)^2}\right] \qquad \left(\sin\omega t = \sqrt{1-\cos^2(\omega t)}\right)$$

or

$$\mathscr{E}_y - \mathscr{E}_x\left[\frac{b}{a}\cos\beta\right] = -\sin\beta\sqrt{b^2 - \left(\frac{b}{a}\right)^2\mathscr{E}_x^{\,2}}$$

Squaring both sides, we have

$$\mathscr{E}_y^{\,2} + \mathscr{E}_x^{\,2}\left[\frac{b}{a}\cos\beta\right]^2 - 2\mathscr{E}_x\mathscr{E}_y\left(\frac{b}{a}\cos\beta\right) = \sin^2\beta\left[b^2 - \left(\frac{b}{a}\right)^2\mathscr{E}_x^{\,2}\right]$$

Appendix A (cont.)

$$\mathcal{E}_y^{\,2} + \mathcal{E}_x^{\,2}\left[\frac{b}{a}\cos\beta\right]^2 - 2\mathcal{E}_x\mathcal{E}_y\left(\frac{b}{a}\cos\beta\right) = \sin^2\beta\left[b^2 - \left(\frac{b}{a}\right)^2\mathcal{E}_x^{\,2}\right]$$

Collecting terms, we have

$$\mathcal{E}_x^{\,2}\left[\left(\frac{b}{a}\right)^2\left(\cos^2\beta + \sin^2\beta\right)\right] + \mathcal{E}_y^{\,2} - 2\mathcal{E}_x\mathcal{E}_y\left(\frac{b}{a}\cos\beta\right) = b^2\sin^2\beta$$

or

$$\mathcal{E}_x^{\,2}\left(\frac{b}{a}\right)^2 + \mathcal{E}_x\mathcal{E}_y\left[-2\frac{b}{a}\cos\beta\right] + \mathcal{E}_y^{\,2} = b^2\sin^2\beta$$

This is in the form of a quadratic expression:

$$A\,\mathcal{E}_x^{\,2} + B\,\mathcal{E}_x\mathcal{E}_y + C\mathcal{E}_y^{\,2} = D$$

Appendix A (cont.)

Discriminant:

$$\Delta = B^2 - 4AC$$ (determines the type of curve)

$$= 4\left(\frac{b}{a}\right)^2 \cos^2\beta - 4\left(\frac{b}{a}\right)^2$$

$$= 4\left(\frac{b}{a}\right)^2 \left[\cos^2\beta - 1\right]$$

so

$$\Delta = -4\left(\frac{b}{a}\right)^2 \sin^2\beta < 0$$

Hence, this is an ellipse.

(This follows from analytic geometry.)

Appendix B

Here we give a proof of the rotation property.

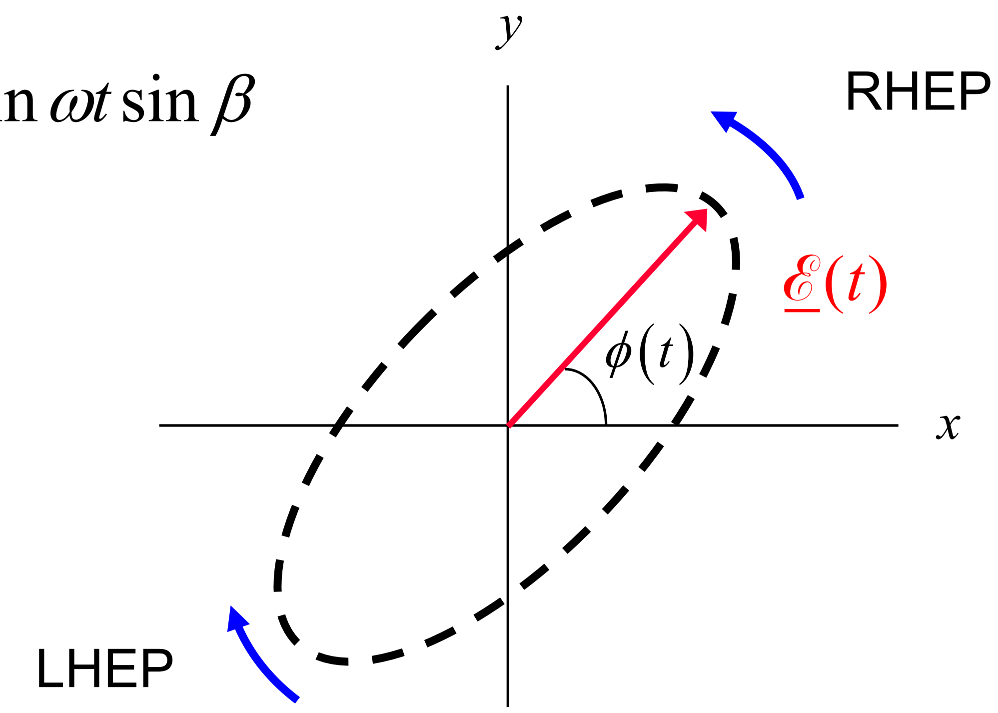
Rotation property:

$0 < \beta < \pi$	LHEP
$-\pi < \beta < 0$	RHEP

$$\mathcal{E}_x = a\cos\omega t$$

$$\mathcal{E}_y = b\cos(\omega t + \beta)$$

$$= b\cos\omega t\cos\beta - b\sin\omega t\sin\beta$$



Appendix B (cont.)

$$\tan\phi = \frac{\mathcal{E}_y}{\mathcal{E}_x} = \frac{b}{a}\left[\cos\beta - \tan\omega t \sin\beta\right]$$

Take the derivative:

$$\sec^2\phi \frac{d\phi}{dt} = \left(\frac{b}{a}\right)\left[-\sec^2(\omega t)(\omega)\sin\beta\right]$$

Hence

$$\frac{d\phi}{dt} = -\sin\beta\left[\left(\frac{b}{a}\right)\cos^2\phi\sec^2(\omega t)(\omega)\right]$$

The term in square brackets is always positive.

(a) $\quad 0 < \beta < \pi \quad \frac{d\phi}{dt} < 0$ LHEP

(b) $\quad -\pi < \beta < 0 \quad \frac{d\phi}{dt} > 0$ RHEP (proof complete)