# ECE 3317
# Applied Electromagnetic Waves

Prof. David R. Jackson

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# Notes 18

# Reflection and Transmission of Plane Waves

# General Plane Wave

Consider a plane wave traveling in an arbitrary direction $\underline{\hat{p}}$ in space.

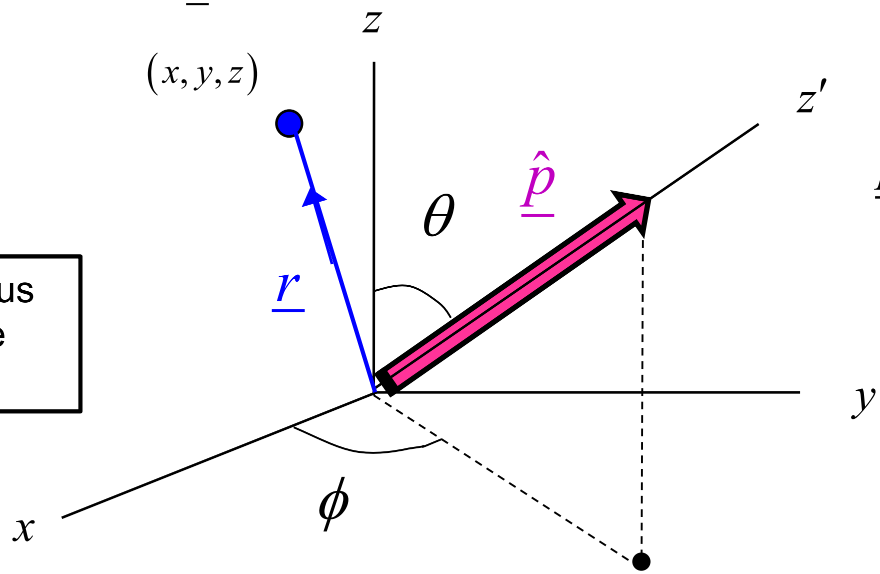
$\underline{r} = \underline{\hat{x}}x + \underline{\hat{y}}y + \underline{\hat{z}}z$

We call the axis of propagation $z'$.

Denote:

$$\psi = e^{-jkz'}$$

(This function $\psi$ tells us how the plane wave varies in space.)




$$\underline{\hat{p}} = \underline{\hat{x}}(\sin\theta\cos\phi) + \underline{\hat{y}}(\sin\theta\sin\phi) + \underline{\hat{z}}(\cos\theta)$$

**(direction of power flow)**

$$\begin{aligned} z' &= \underline{r} \cdot \underline{\hat{p}} \\ &= \left(\underline{\hat{x}}x + \underline{\hat{y}}y + \underline{\hat{z}}z\right) \cdot \underline{\hat{p}} \\ &= x\left(\underline{\hat{p}} \cdot \underline{\hat{x}}\right) + y\left(\underline{\hat{p}} \cdot \underline{\hat{y}}\right) + z\left(\underline{\hat{p}} \cdot \underline{\hat{z}}\right) \end{aligned}$$

so

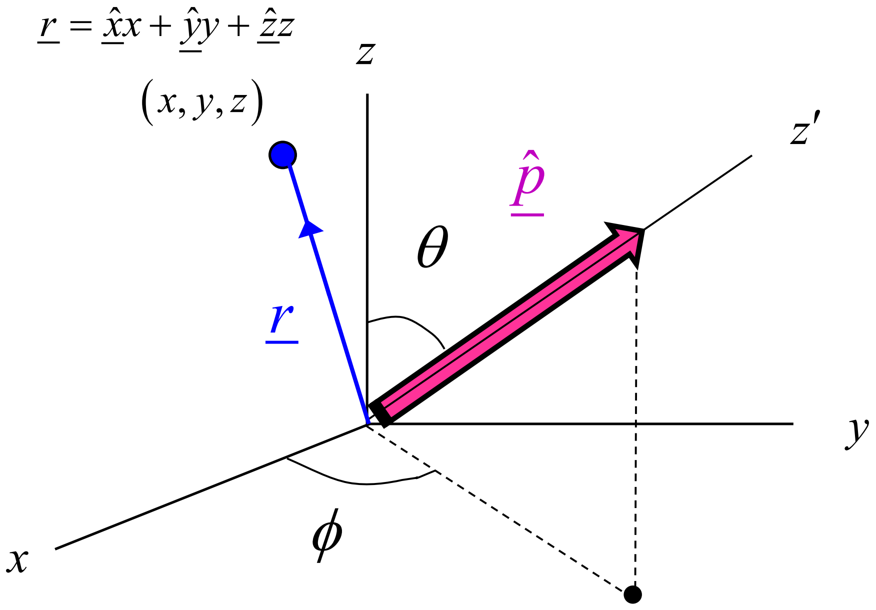
$$z' = x(\sin\theta\cos\phi) + y(\sin\theta\sin\phi) + z(\cos\theta)$$

# General Plane Wave (cont.)

Hence

$$\psi = e^{jkz'} = e^{-j\left(k_x x + k_y y + k_z z\right)}$$

$$k_x = k \sin\theta \cos\phi$$
$$k_y = k \sin\theta \sin\phi$$
$$k_z = k \cos\theta$$




Note:

$$k_x^2 + k_y^2 + k_z^2 = k^2 \sin^2\theta\left(\cos^2\phi + \sin^2\phi\right) + k^2\cos^2\theta$$

or

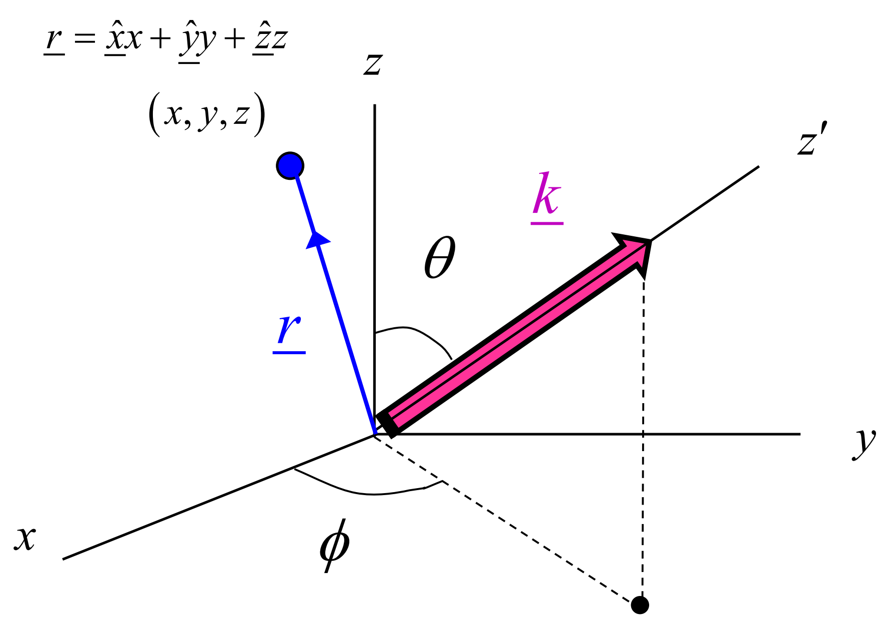
$$k_x^2 + k_y^2 + k_z^2 = k^2$$ (wavenumber equation)

# General Plane Wave (cont.)

We define the wavevector:

$$\underline{k} \equiv \underline{\hat{x}}\, k_x + \underline{\hat{y}}\, k_y + \underline{\hat{z}}\, k_z$$

$$k_x = k \sin\theta \cos\phi$$
$$k_y = k \sin\theta \sin\phi$$
$$k_z = k \cos\theta$$




$$\psi = e^{-j\left(k_x x + k_y y + k_z z\right)} = e^{-j\underline{k}\cdot\underline{r}}$$

**Recall:** $\underline{\hat{p}} = \underline{\hat{x}}\left(\sin\theta\cos\phi\right) + \underline{\hat{y}}\left(\sin\theta\sin\phi\right) + \underline{\hat{z}}\left(\cos\theta\right)$

$$\Rightarrow \quad \underline{k} = k\,\underline{\hat{p}}$$

The $\underline{k}$ vector tells us which direction the wave is traveling in.

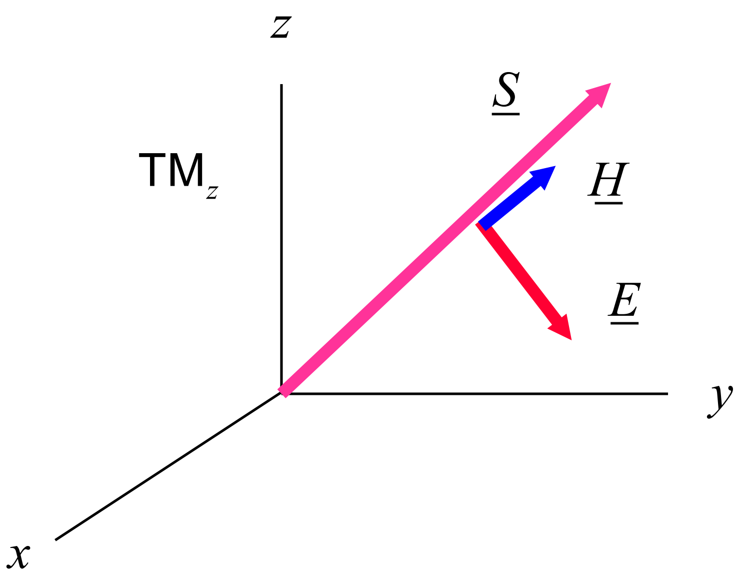
# $\mathrm{TM}_z$ and $\mathrm{TE}_z$ Plane Waves

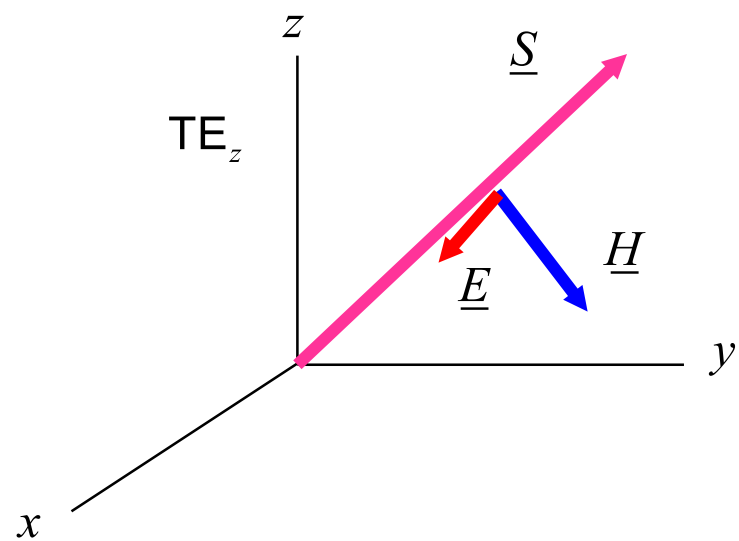
There are two fundamental polarizations possible.

- Transverse Magnetic ($\mathrm{TM}_z$) $H_z = 0$
- Transverse Electric ($\mathrm{TE}_z$) $E_z = 0$

$$\mathrm{TM}_z: \quad \underline{E} = \underline{\hat{\theta}}\, E_\theta \;,\; \underline{H} = \underline{\hat{\phi}}\, H_\phi$$

$$\mathrm{TE}_z: \quad \underline{E} = \underline{\hat{\phi}}\, E_\phi \;,\; \underline{H} = \underline{\hat{\theta}}\, H_\theta$$

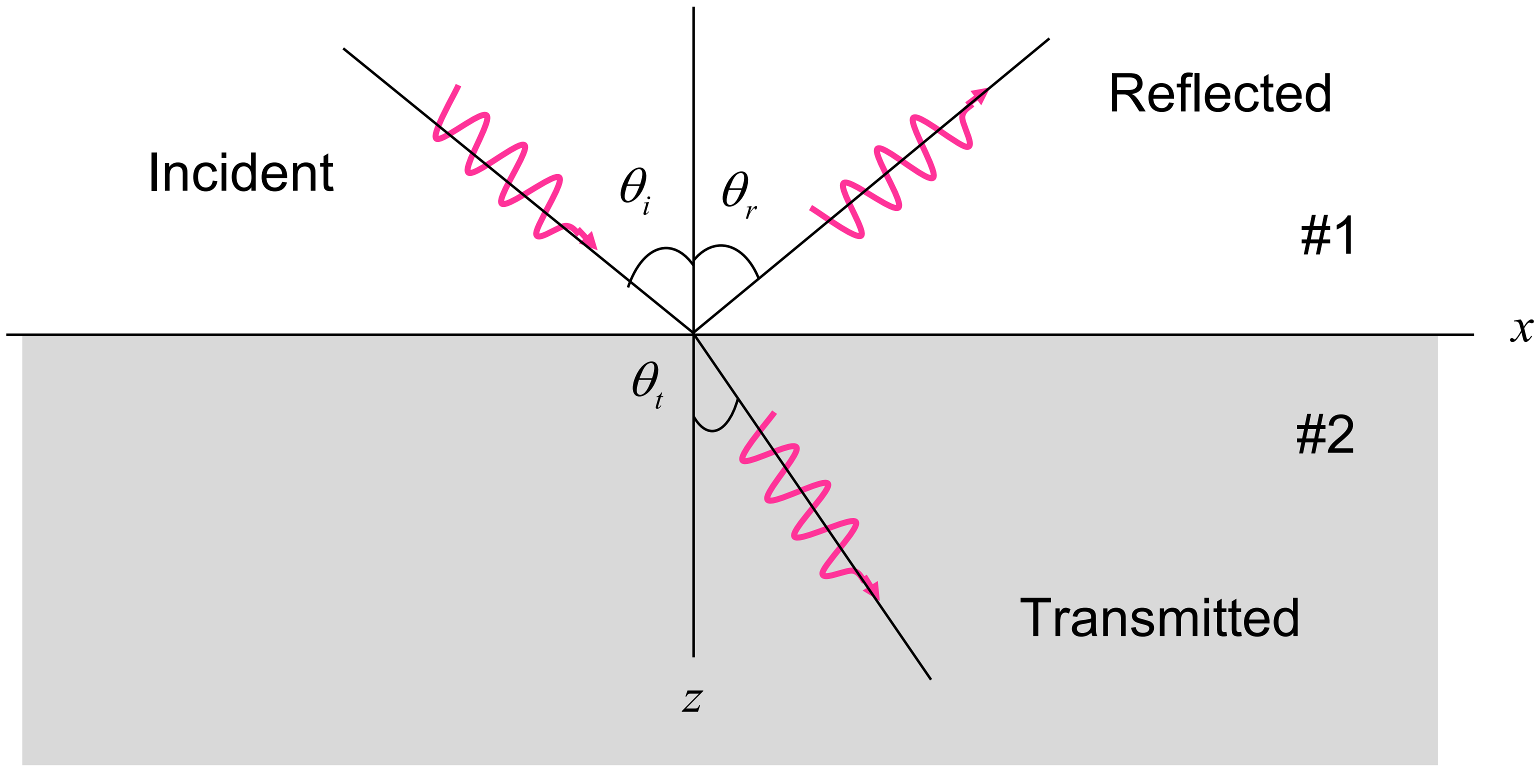



**Note:** The word "transverse" means "perpendicular to."

# Reflection and Transmission

As we will show, each type of plane wave ($TE_z$ and $TM_z$) reflects differently (and independently) from a material interface.




**Note:** We assume that the Poynting vector of the incident plane wave lies in the $xz$ plane ($\phi = 0$). This is called the plane of incidence. There is then no variation of the fields in the $y$ direction ($k_y = 0$).

# Boundary Conditions

Here we review the boundary conditions at an interface (from ECE 3318).

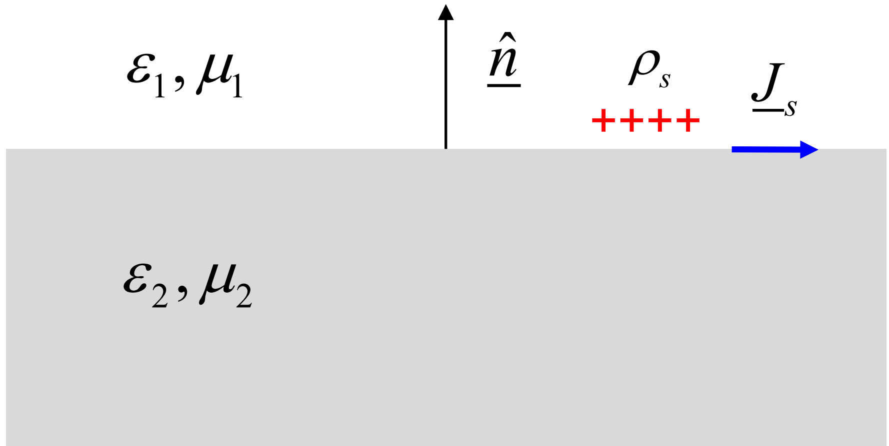
General BCs:

$$\underline{\hat{n}} \cdot \left(\underline{D}_1 - \underline{D}_2\right) = \rho_s$$

$$\underline{\hat{n}} \times \left(\underline{E}_1 - \underline{E}_2\right) = \underline{0}$$

$$\underline{\hat{n}} \cdot \left(\underline{B}_1 - \underline{B}_2\right) = 0$$

$$\underline{\hat{n}} \times \left(\underline{H}_1 - \underline{H}_2\right) = \underline{J}_s$$




**Note:** The unit normal points towards region 1.

No sources on interface:

$$\underline{\hat{n}} \times \underline{E}_1 = \underline{\hat{n}} \times \underline{E}_2$$

$$\underline{\hat{n}} \times \underline{H}_1 = \underline{\hat{n}} \times \underline{H}_2$$

$$\underline{\hat{n}} \cdot \underline{D}_1 = \underline{\hat{n}} \cdot \underline{D}_2$$

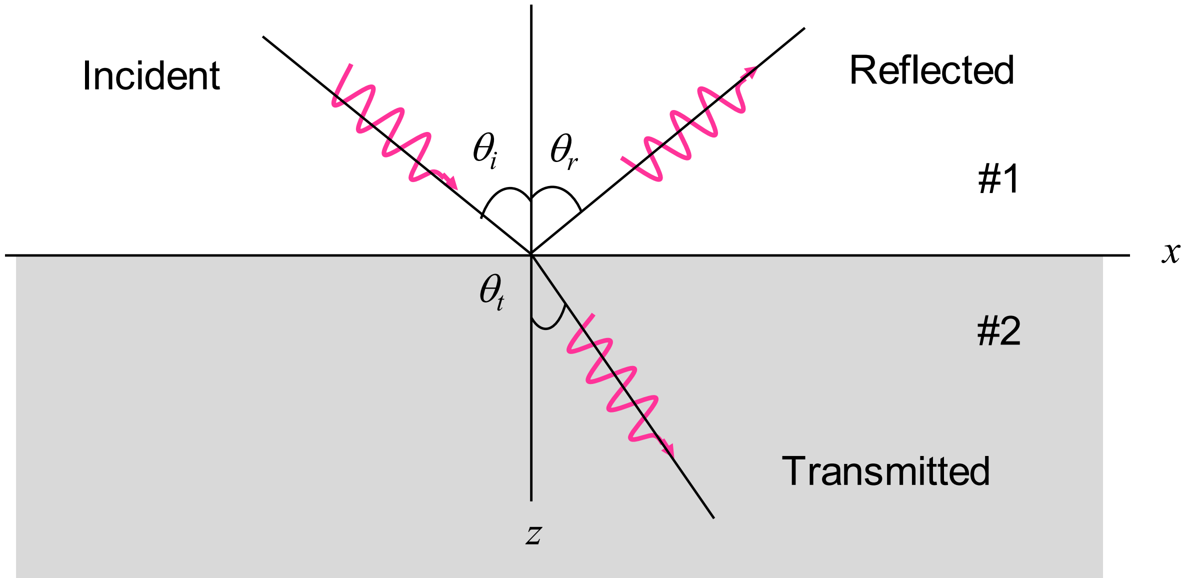
$$\underline{\hat{n}} \cdot \underline{B}_1 = \underline{\hat{n}} \cdot \underline{B}_2$$

⇨ The tangential electric and magnetic fields are continuous.

(The normal electric and magnetic flux densities are also continuous.)

# Reflection at Interface

Assume that the Poynting vector of the incident plane wave lies in the $xz$ plane ($\phi = 0$). This is called the plane of incidence.




We first consider the $(x,z)$ variation of the fields.
(We will worry about the polarization later.)

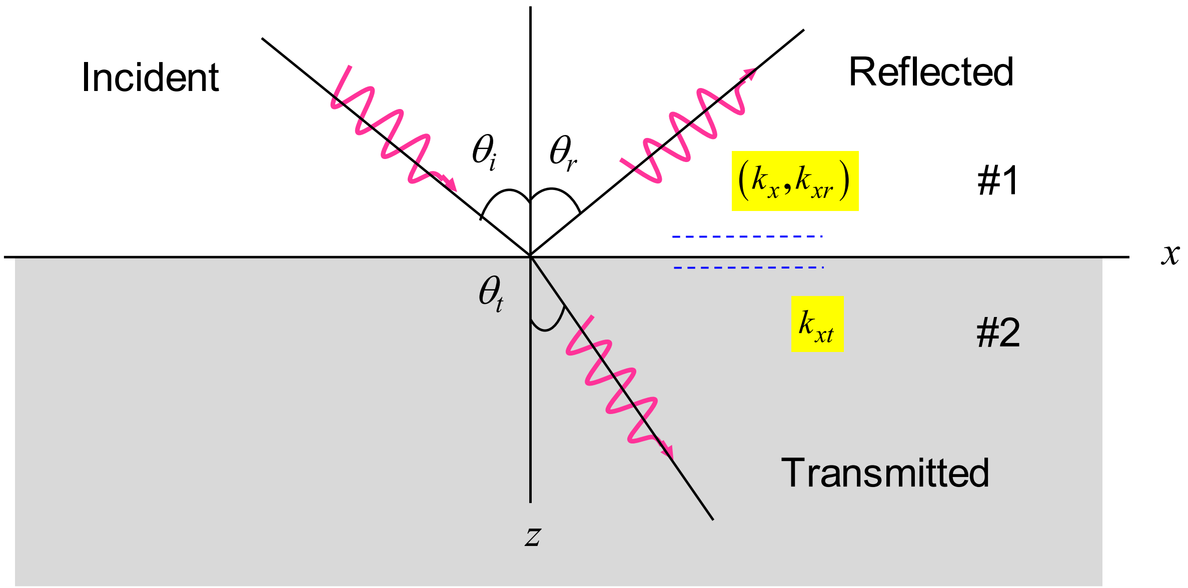
$$\underline{E}^i = \underline{E}_{0i}\, e^{-jk_{xi}x - jk_{zi}z} \qquad \underline{E}^r = \underline{E}_{0r}\, e^{-jk_{xr}x + jk_{zr}z} \qquad \underline{E}^t = \underline{E}_{0t}\, e^{-jk_{xt}x - jk_{zt}z}$$

**Note:**
The plus sign for the exponent term in the reflected wave is chosen because this wave is traveling in the negative $z$ direction.

# Reflection at Interface (cont.)

Phase matching condition:

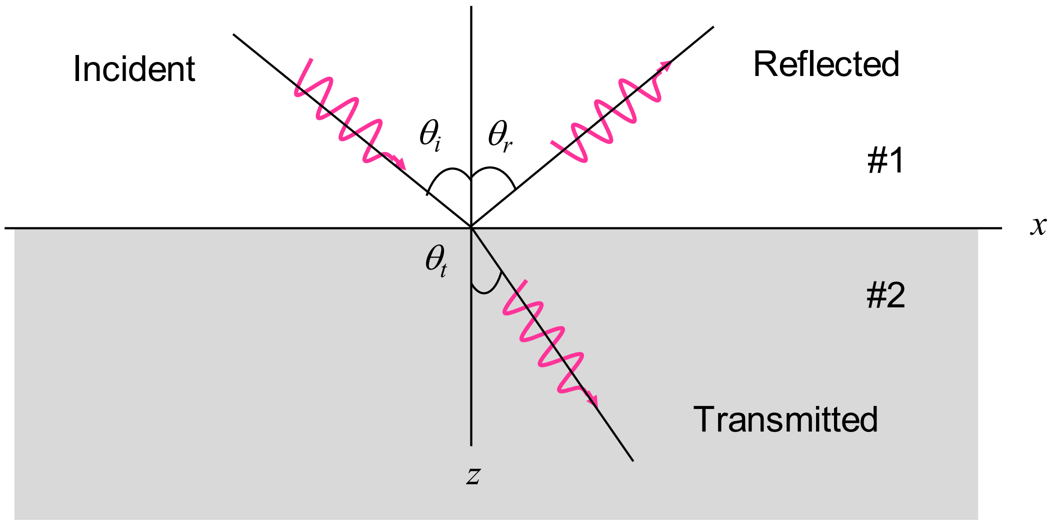
$$k_{xi} = k_{xr} = k_{xt}$$




This follows from the fact that the fields must match everywhere at the interface ($z = 0$).

$$\underline{E}^i(x,0) = \underline{E}_{0i}\, e^{-jk_{xi}x} \qquad \underline{E}^r(x,0) = \underline{E}_{0r}\, e^{-jk_{xr}x} \qquad \underline{E}^t(x,0) = \underline{E}_{0t}\, e^{-jk_{xt}x}$$

# Law of Reflection




$$k_{xi} = k_{xr}$$

(This comes from the first part of the phase matching condition.)

Incident and reflected waves:

$$k_{xi} = k_1 \sin\theta_i \qquad k_{yi} = 0 \qquad k_{zi} = k_1 \cos\theta_i$$

$$k_{xr} = k_1 \sin\theta_r \qquad k_{yr} = 0 \qquad k_{zr} = k_1 \cos\theta_r$$

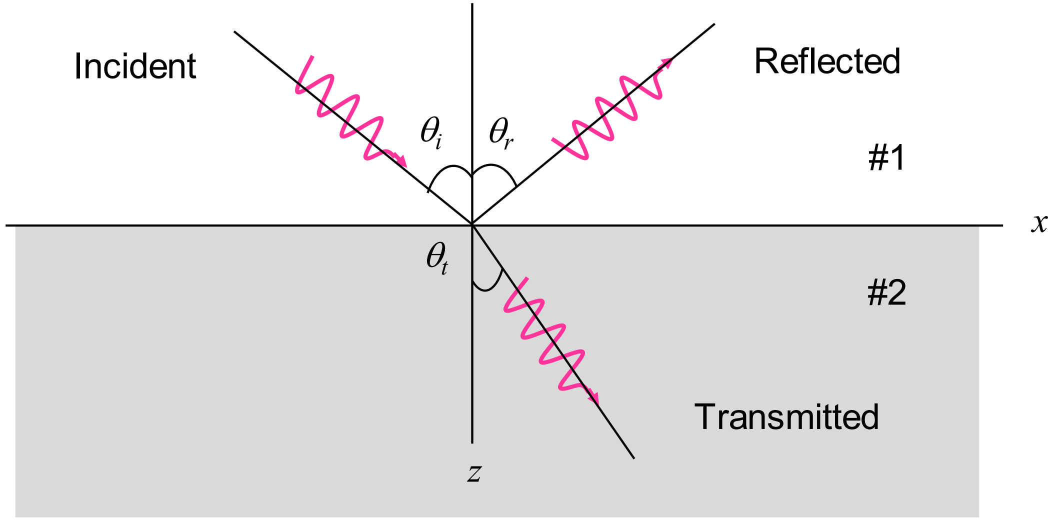
$$k_1 \sin\theta_i = k_1 \sin\theta_r \Rightarrow \theta_i = \theta_r$$

Law of reflection

# Snell's Law

$$k_{xi} = k_{xt}$$

(This comes from the second part of the phase matching condition.)




Incident and transmitted waves:

$$k_{xi} = k_1 \sin\theta_i \qquad k_{yi} = 0 \qquad k_{zi} = k_1 \cos\theta_i$$

$$k_{xt} = k_2 \sin\theta_t \qquad k_{yt} = 0 \qquad k_{zt} = k_2 \cos\theta_t$$

We define the index of refraction:

$$n_i = k_i / k_0 = \omega\sqrt{\mu_i\varepsilon_i} \,/\, \omega\sqrt{\mu_0\varepsilon_0} = \sqrt{\varepsilon_{ri}\mu_{ri}}$$

$$i = 1, 2$$

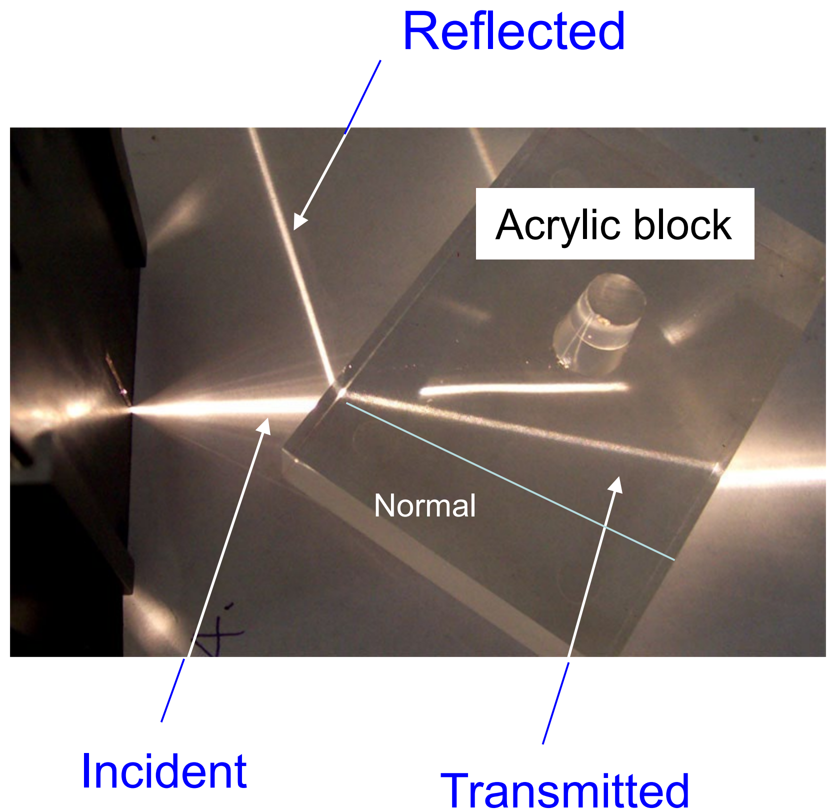
$$k_1 \sin\theta_i = k_2 \sin\theta_t \quad \Rightarrow \quad n_1 \sin\theta_i = n_2 \sin\theta_t$$

Snell's law

**Note:** The wave is bent <u>towards</u> the normal ($z$ axis) when entering a more optically "dense" region (a higher index of refraction).

# Snell’s Law (cont.)

The bending of light (or EM waves in general) is called refraction.

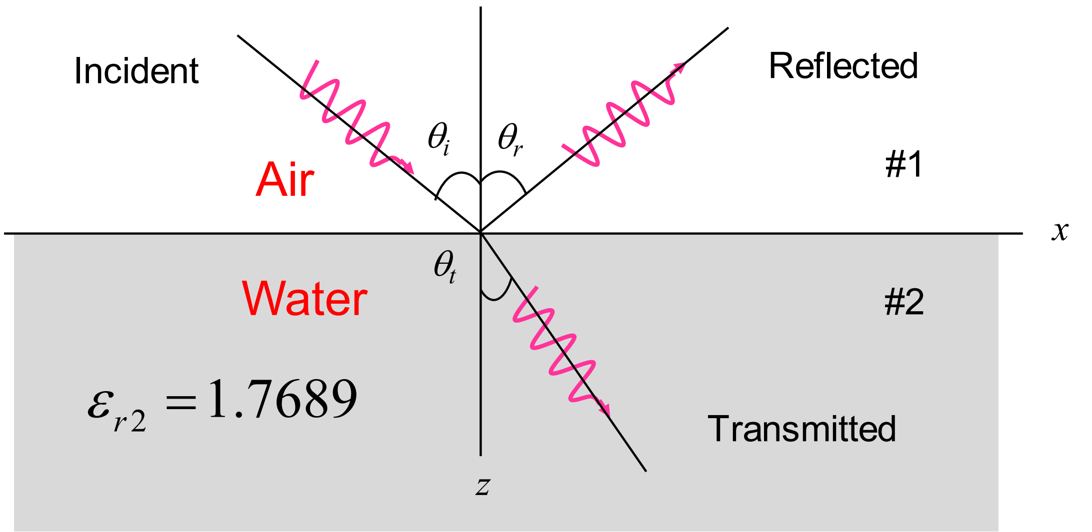



http://en.wikipedia.org/wiki/Refraction

# Example

Given: $\theta_i = 45^{\circ}$

Find the transmitted angle.




$$n_1 \sin\theta_i = n_2 \sin\theta_t$$

$$\sqrt{1}\sin\theta_i = \sqrt{1.7689}\sin\theta_t$$

$$\Rightarrow \quad \theta_t = 32.1^{\circ}$$

Note that in going from a less dense to a more dense medium, the wavevector is bent towards the normal.

**Note**:
If the wave is incident from the water region at an incident angle of 32.1°, the wave will exit into the air region at an angle of 45°.

**Note:**
At microwave frequencies and below, the relative permittivity of pure water is about 81. At optical frequencies it is about 1.7689.

# Critical Angle

The wave is incident from a more dense region onto a less dense region.

$n_1 > n_2$

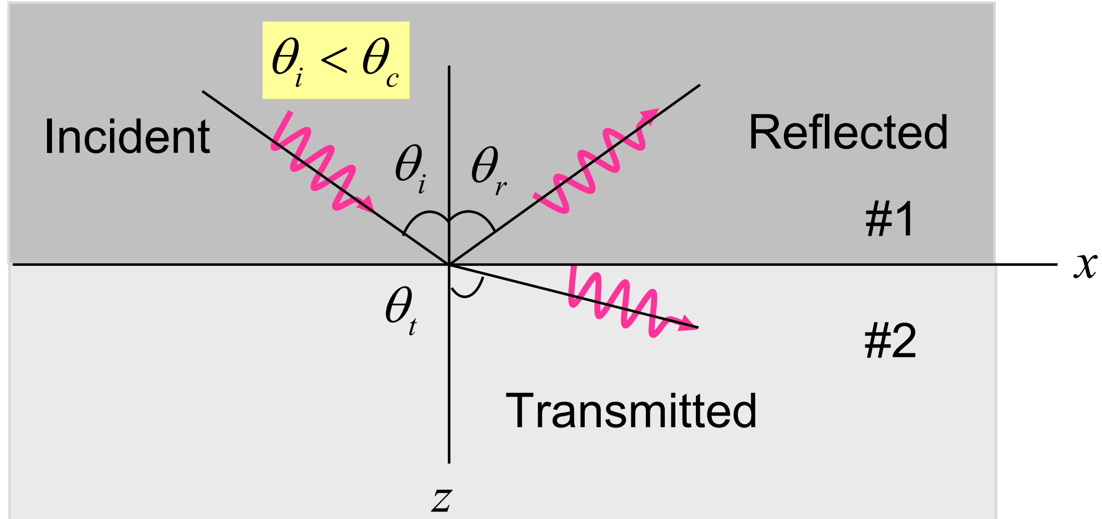
$$n_1 \sin\theta_i = n_2 \sin\theta_t$$

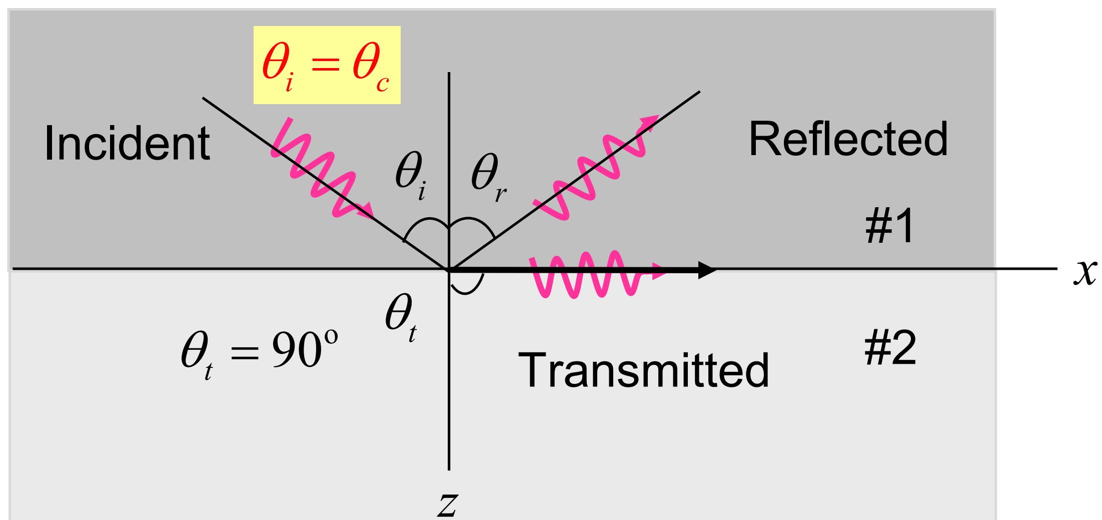
$$\Rightarrow \quad \sin\theta_t = \frac{n_1}{n_2}\sin\theta_i$$

At the critical angle: $\theta_t = 90^\circ$

$$\sin\left(90^\circ\right) = \frac{n_1}{n_2}\sin\theta_c$$

$$\theta_c = \sin^{-1}\left(\frac{n_2}{n_1}\right)$$



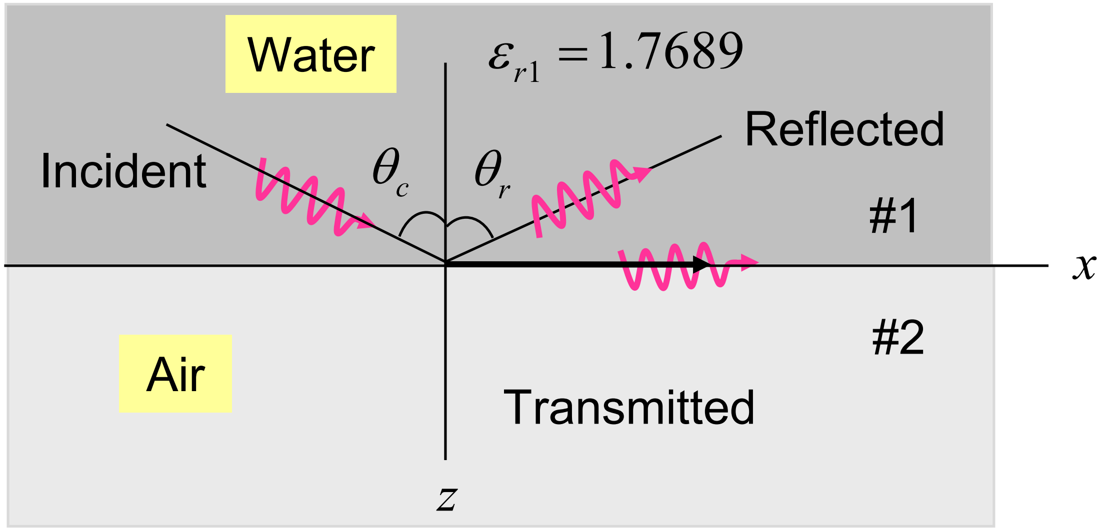




$$\theta_i = \theta_c: \quad k_{xi} = k_{xr} = k_{xt} = k_2 \sin\theta_t = k_2$$

# Critical Angle (cont.)

**Example**

Find the critical angle.




$$\theta_c = \sin^{-1}\left(\frac{n_2}{n_1}\right)$$

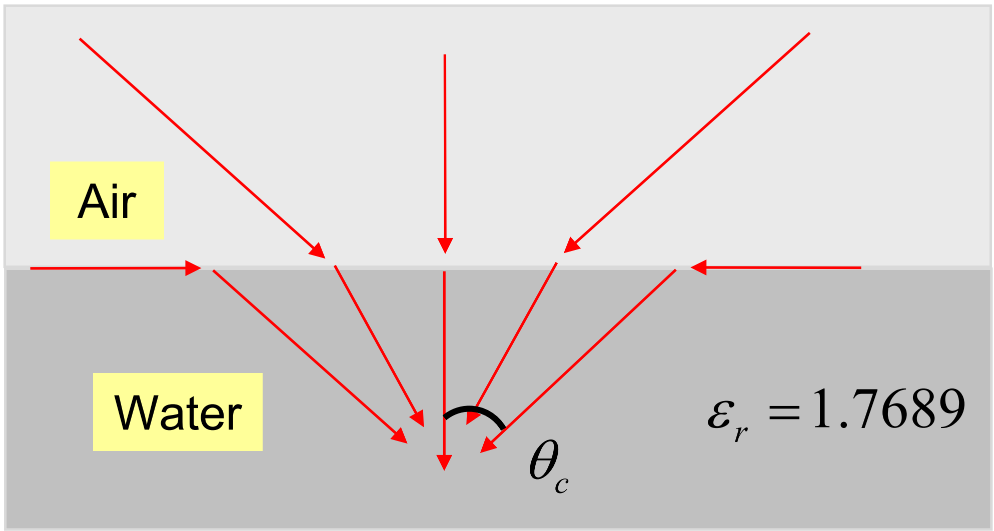
$$= \sin^{-1}\left(\frac{\sqrt{1}}{\sqrt{1.7689}}\right)$$

$$\theta_c = 48.75^{\circ}$$

# Critical Angle (cont.)

Example: "fish-eye" effect

$$\theta_c = 48.75^{\circ}$$




The critical angle explains the "fish eye" effect that you can observe in a swimming pool.

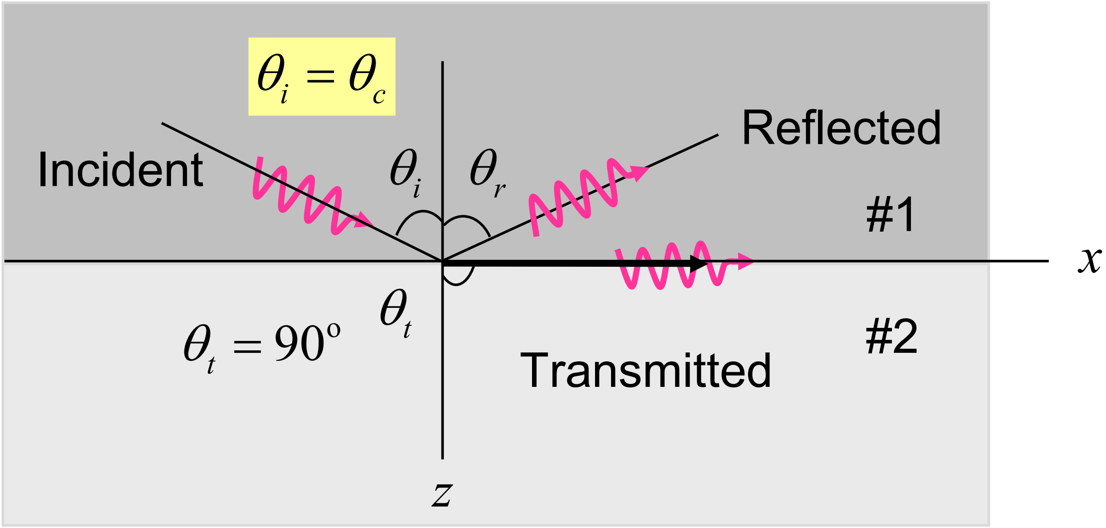
A fish can see everything above the water by only looking no farther than $49^{o}$ from the vertical!

# Critical Angle (cont.)

At the critical angle:

$$k_{xi} = k_{xr} = k_{xt} = k_2 \sin\theta_t = k_2$$

$$k_{zt} = \sqrt{k_2^2 - k_{xt}^2}$$
$$= \sqrt{k_2^2 - k_2^2}$$
$$= 0$$




There is no vertical variation of the field in the less-dense (transmitted) region.
There are still fields in the less dense region, however.

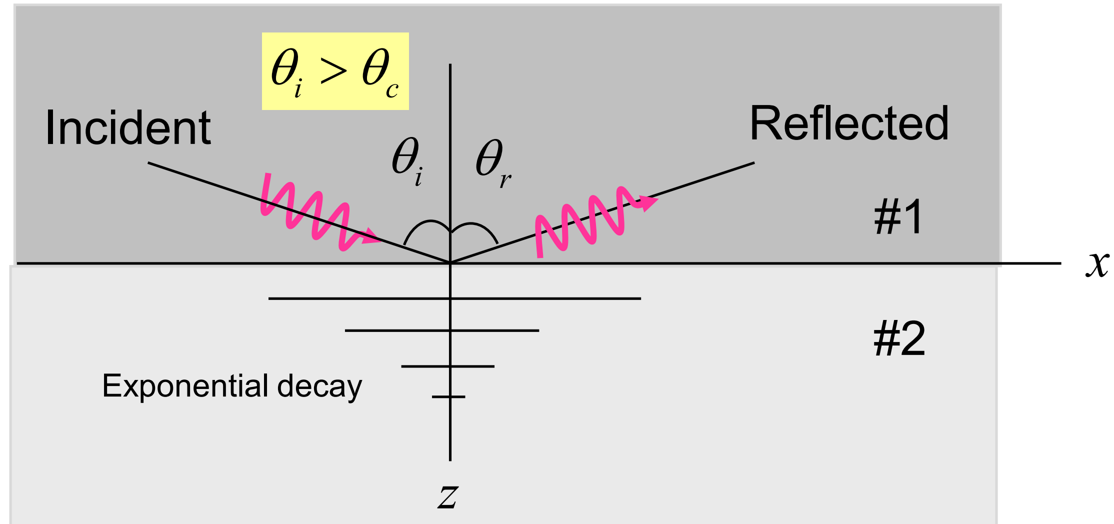
# Critical Angle (cont.)

Beyond the critical angle:

$$\theta_i > \theta_c$$

$$\Rightarrow \quad k_{xi} > k_2$$

$$\left(k_{xi} = k_1 \sin\theta_i > k_1 \sin\theta_c = k_2 \sin 90^o = k_2\right)$$




$$\begin{aligned} k_{zt} &= \sqrt{k_2^2 - k_{xt}^2} \\ &= \sqrt{k_2^2 - k_{xi}^2} \\ &= -j\sqrt{k_{xi}^2 - k_2^2} \\ &= -j\sqrt{k_1^2 \sin^2\theta_i - k_2^2} \\ &= -jk_0\sqrt{n_1^2 \sin^2\theta_i - n_2^2} \\ &= -j\alpha_{zt} \end{aligned}$$

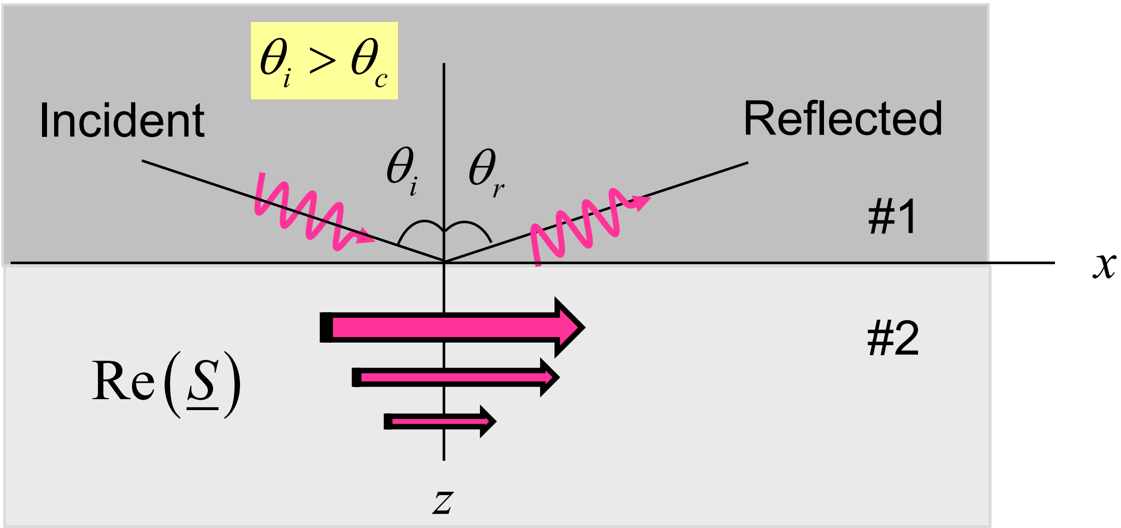
There is an exponential decay of the field in the vertical direction in the less-dense region.

$$\psi_t(x,z) \equiv e^{-j(k_{xt}x + k_{zt}z)} = e^{-j(k_{xt}x)}\, e^{-\alpha_{zt}z}$$

$$\alpha_{zt} = k_0\sqrt{n_1^2 \sin^2\theta_i - n_2^2}$$

# Critical Angle (cont.)

Beyond the critical angle:




The power flows completely <u>horizontally</u> in the lower region.

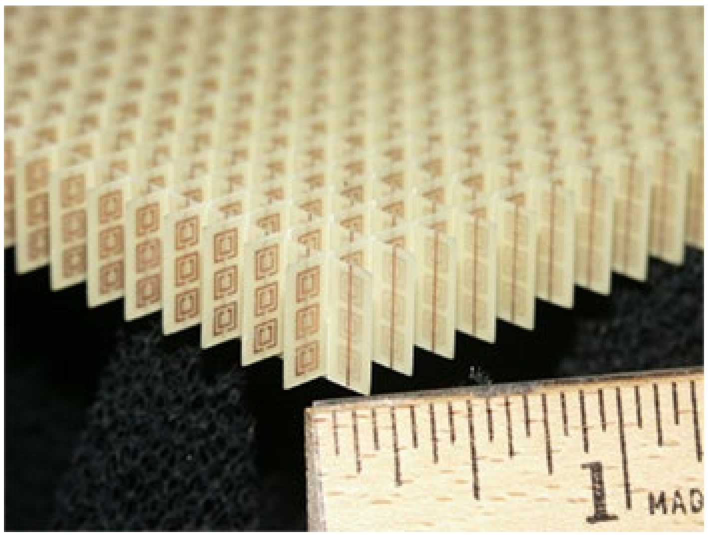
No power crosses the boundary and enters into the less dense region.

This must be true from conservation of energy, since the field decays exponentially in the lossless region 2.

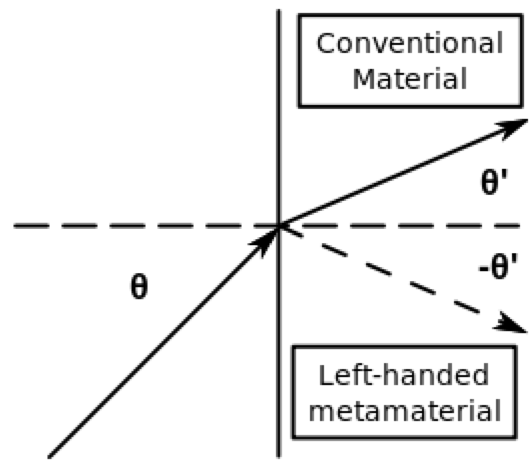
# Exotic Materials

Artificial “metamaterials” that have been designed that have exotic permittivity and/or permeability performance.

https://en.wikipedia.org/wiki/Metamaterial




Negative index metamaterial array configuration, which was constructed of copper split-ring resonators and wires mounted on interlocking sheets of fiberglass circuit board. The total array consists of 3 by 20×20 unit cells with overall dimensions of 10×100×100 mm.

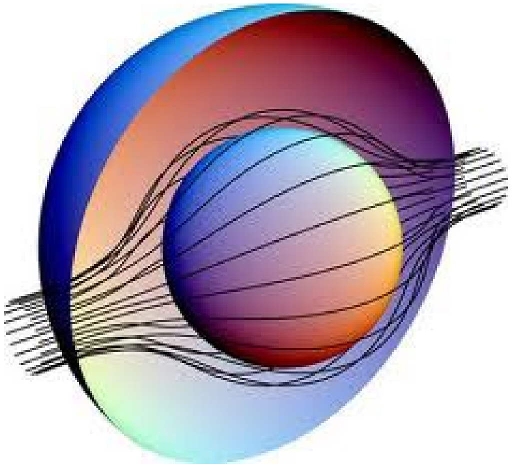


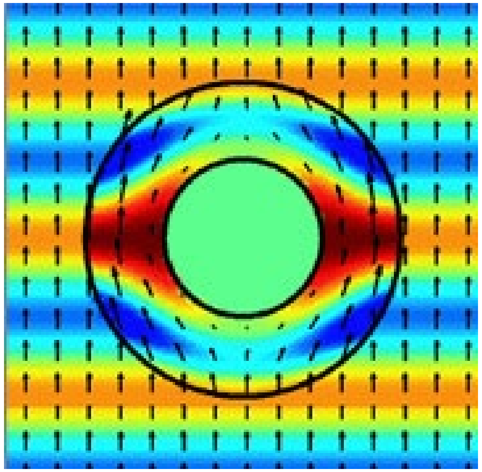

$$\varepsilon_r < 0$$

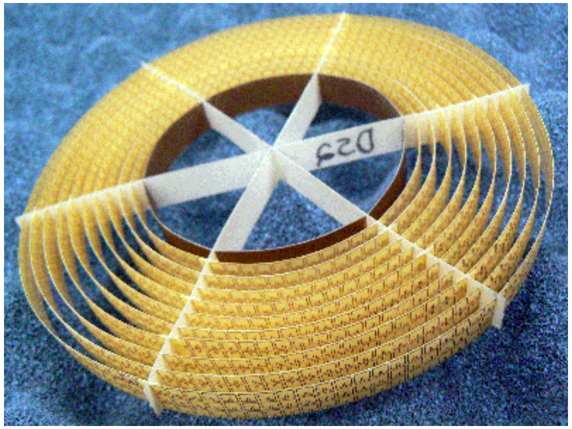
$$\mu_r < 0$$

(over a certain bandwidth of operation)

# Exotic Materials (cont.)







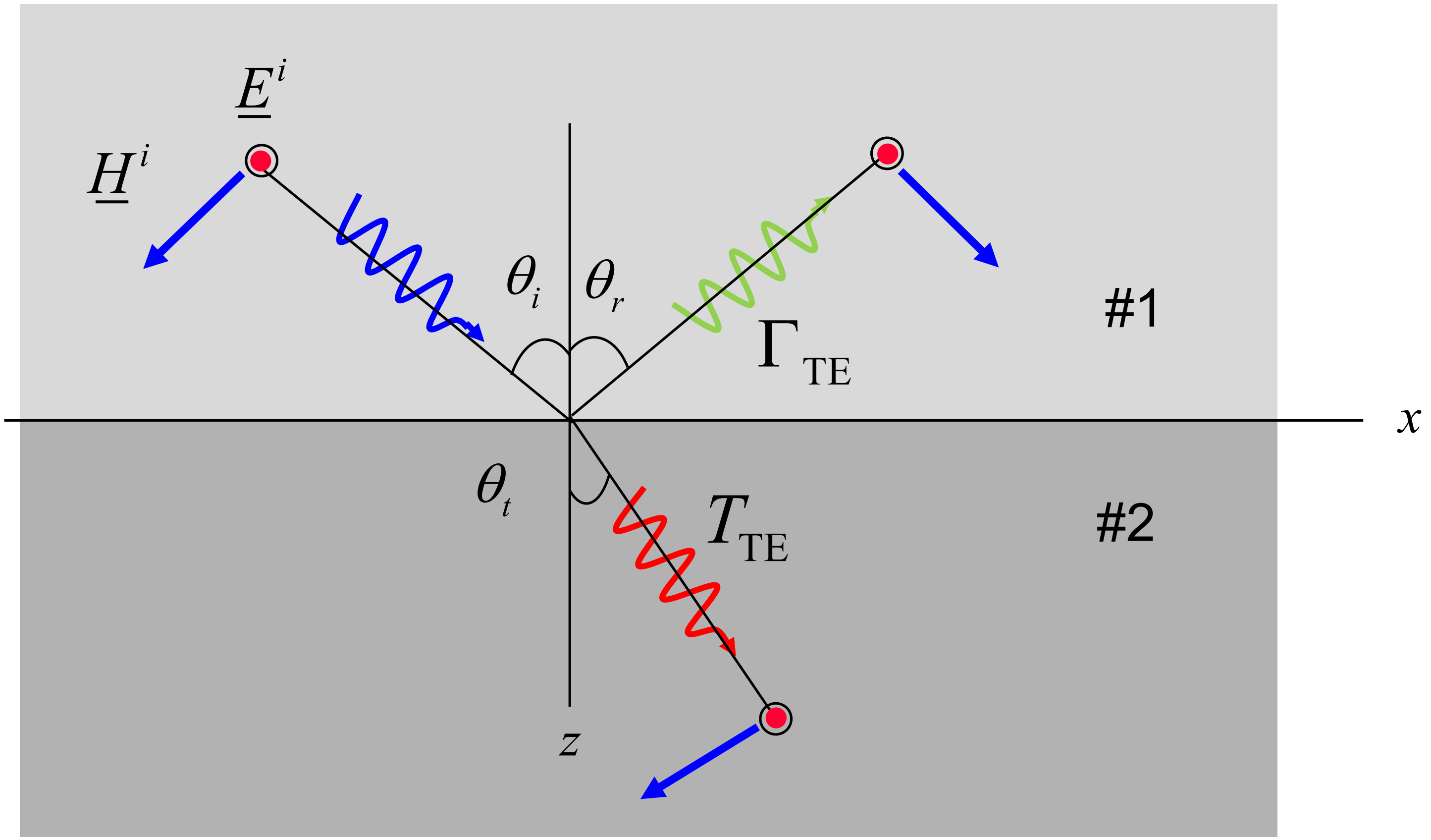
The Duke cloaking device masks an object at one microwave frequency.

Image courtesy Dr. David R. Smith.

Cloaking of objects is one area of research in metamaterials.

https://en.wikipedia.org/wiki/Metamaterial_cloaking

# $TE_z$ Reflection




Note that the electric field vector is in the $y$ direction.

(The wave is polarized perpendicular to the plane of incidence.)

# $TE_z$ Reflection (cont.)

**Incident Wave**

$$\underline{E}^i = \hat{y}\, E_0\, e^{-jk_{xi}x - jk_{zi}z}$$

**Reflected Wave**

$$\underline{E}^r = \hat{y}\Gamma_{\mathrm{TE}}\, E_0\, e^{-jk_{xr}x + jk_{zr}z}$$

**Transmitted Wave**

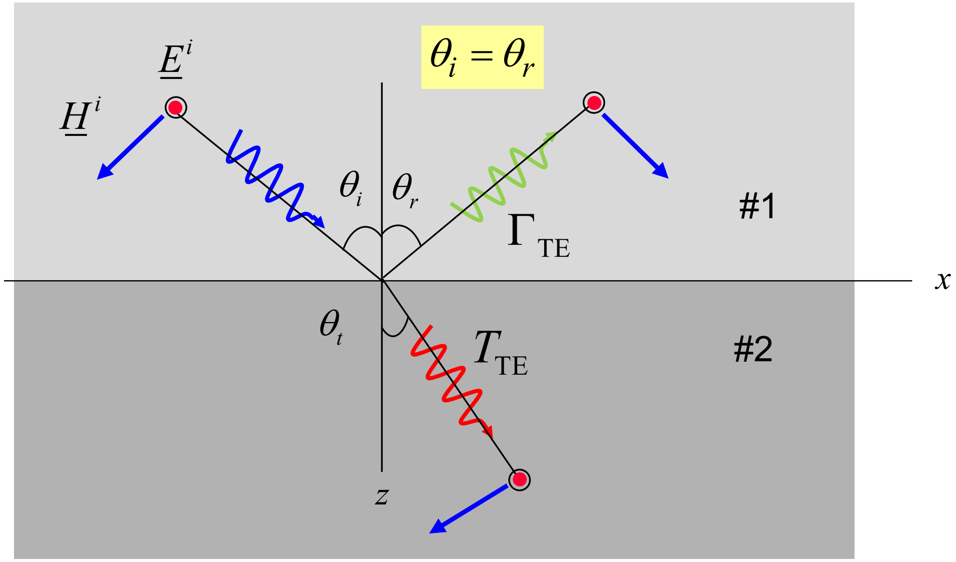
$$\underline{E}^t = \hat{y} T_{\mathrm{TE}}\, E_0\, e^{-jk_{xt}x - jk_{zt}z}$$

where

$\Gamma_{\mathrm{TE}}$ = Reflection Coefficient

$T_{\mathrm{TE}}$ = Transmission Coefficient

$$k_{xi} = k_{xr} = k_{xt} = k_1 \sin\theta_i$$




Incident Wave Vector

$$\underline{k}_i = \hat{\underline{x}}\, k_{xi} + \hat{\underline{z}}\, k_{zi} \qquad k_{zi} = k_1 \cos\theta_i$$

Reflected Wave Vector

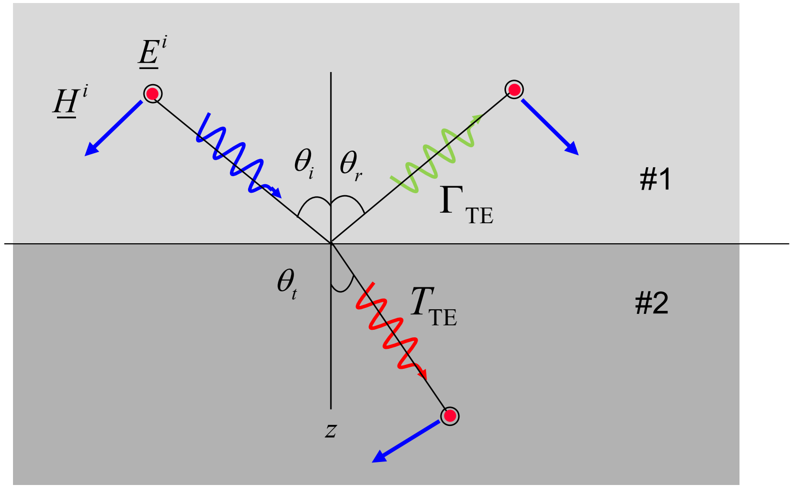
$$\underline{k}_r = \hat{\underline{x}}\, k_{xr} - \hat{\underline{z}}\, k_{zr} \qquad k_{zr} = k_1 \cos\theta_r$$

Transmitted Wave Vector

$$\underline{k}_t = \hat{\underline{x}}\, k_{xt} + \hat{\underline{z}}\, k_{zt} \qquad k_{zt} = k_2 \cos\theta_t$$

# $TE_z$ Reflection (cont.)

Recall that the tangential component of the electric field must be continuous at an interface.




Boundary condition at $z = 0$:

$$E_y^i + E_y^r = E_y^t$$

$$E_0\, e^{-jk_{xi}x - jk_{zi}z} + \Gamma_{TE}\, E_0\, e^{-jk_{xr}x + jk_{zr}z} = T_{TE}\, E_0\, e^{-jk_{xt}x - jk_{zt}z}$$

$\Downarrow$ $z = 0$

$$E_0\, e^{-jk_{xi}x} + \Gamma_{TE}\, E_0\, e^{-jk_{xr}x} = T_{TE}\, E_0\, e^{-jk_{xt}x}$$

$\Downarrow$ $k_{xi} = k_{xr} = k_{xt}$

$$E_0 + \Gamma_{TE}\, E_0 = T_{TE} E_0$$

$\Downarrow$ Divide by $E_0$

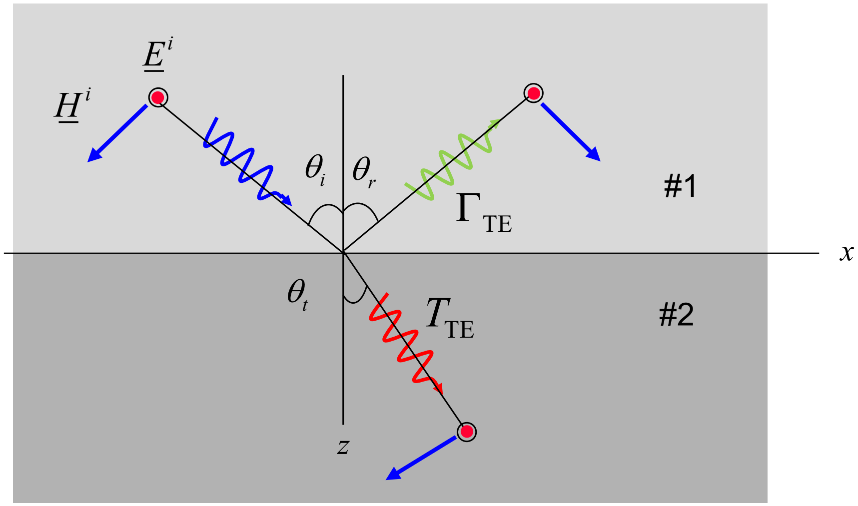
$$1 + \Gamma_{TE} = T_{TE}$$

# $\text{TE}_z$ Reflection (cont.)

We now look at the magnetic fields.

$$\nabla \times \underline{E} = -j\omega\mu\underline{H}$$

$$\Rightarrow \quad \underline{H} = \left(\frac{-1}{j\omega\mu}\right)\nabla \times \underline{E}$$




$$\underline{E}^i = \underline{\hat{y}}\,E_0\,e^{-j(k_{xi}x+k_{zi}z)}$$

$$\underline{H}^i = (-\underline{\hat{x}}\,k_{zi} + \underline{\hat{z}}\,k_{xi})\left(\frac{E_0}{\omega\mu_1}\right)e^{-j(k_{xi}x+k_{zi}z)}$$

$$\underline{E}^r = \underline{\hat{y}}\,\Gamma_{\text{TE}}E_0e^{-j(k_{xr}x-k_{zr}z)}$$

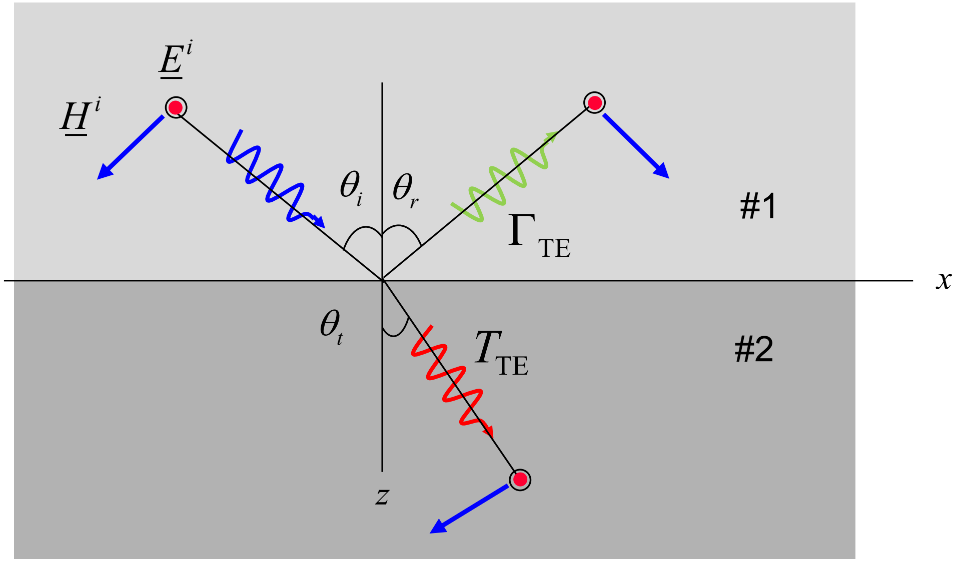
$$\underline{H}^r = (+\underline{\hat{x}}\,k_{zr} + \underline{\hat{z}}\,k_{xr})\left(\frac{\Gamma_{\text{TE}}E_0}{\omega\mu_1}\right)e^{-j(k_{xr}x-k_{zr}z)}$$

$$\underline{E}^t = \underline{\hat{y}}\,T_{\text{TE}}E_0e^{-j(k_{xt}x+k_{zt}z)}$$

$$\underline{H}^t = (-\underline{\hat{x}}\,k_{zt} + \underline{\hat{z}}\,k_{xt})\frac{T_{\text{TE}}E_0}{\omega\mu_2}e^{-j(k_{xt}x+k_{zt}z)}$$

# $TE_z$ Reflection (cont.)

Recall that the tangential component of the magnetic field must be continuous at an interface (no surface currents).



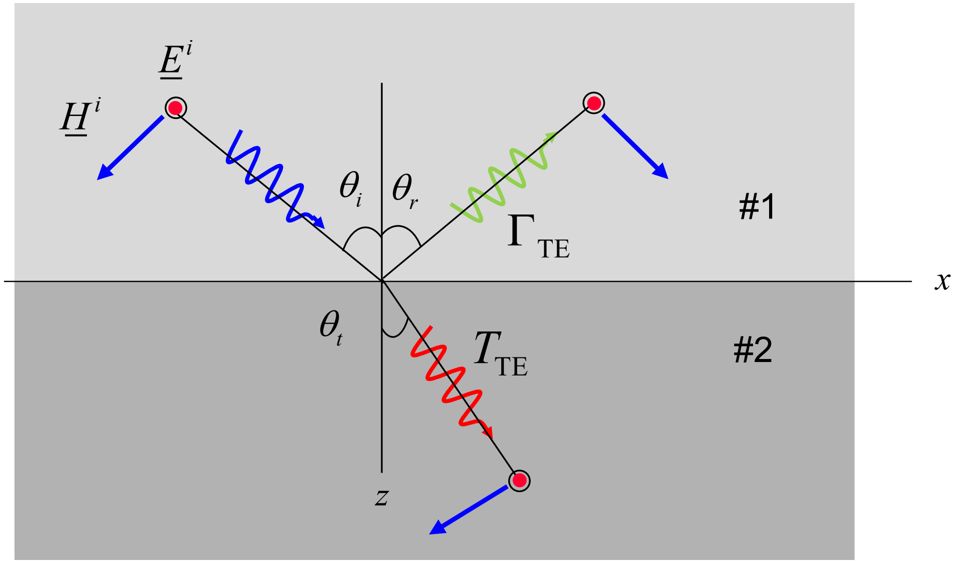

Boundary condition at $z = 0$:

$$H_x^i + H_x^r = H_x^t$$

Hence, we have:

$$-\frac{k_{zi}}{\omega\mu_1} + \frac{k_{zr}\Gamma_{\mathrm{TE}}}{\omega\mu_1} = -\frac{k_{zt}T_{\mathrm{TE}}}{\omega\mu_2}$$

# $\mathrm{TE}_z$ Reflection (cont.)




Enforcing <u>both</u> boundary conditions, we have:

$$1+\Gamma_{\mathrm{TE}}=T_{\mathrm{TE}}$$

$$\frac{k_{zi}}{\omega\mu_1}-\frac{k_{zr}\Gamma_{\mathrm{TE}}}{\omega\mu_1}=\frac{k_{zt}T_{\mathrm{TE}}}{\omega\mu_2}$$

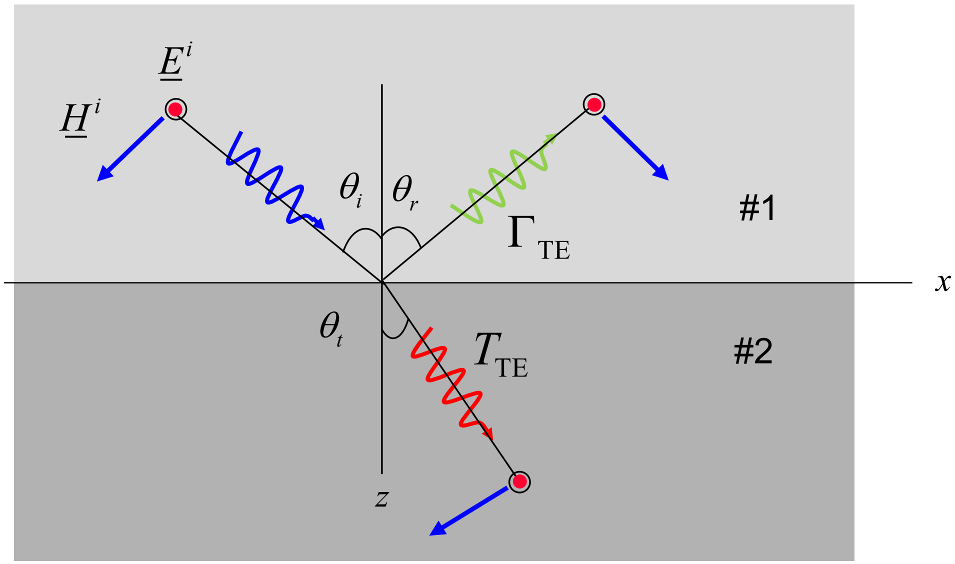
The solution is:

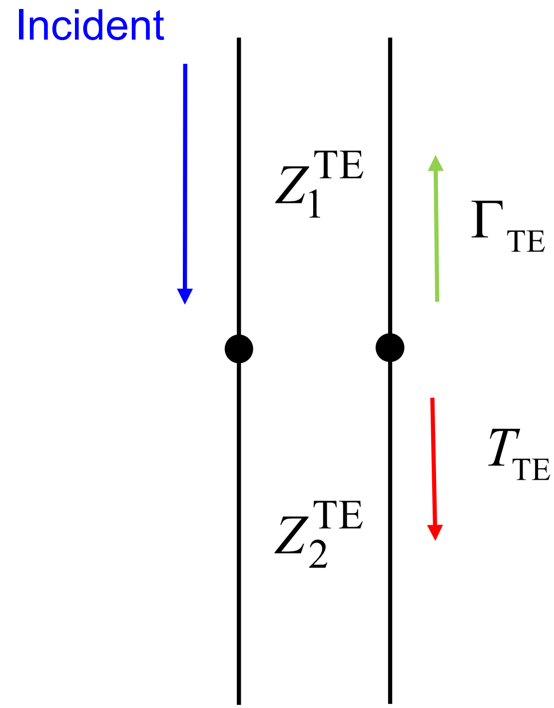
$$\Gamma_{\mathrm{TE}}=\frac{\left(\dfrac{\omega\mu_2}{k_{zt}}\right)-\left(\dfrac{\omega\mu_1}{k_{zi}}\right)}{\left(\dfrac{\omega\mu_2}{k_{zt}}\right)+\left(\dfrac{\omega\mu_1}{k_{zi}}\right)}$$

$$T_{\mathrm{TE}}=1+\Gamma_{\mathrm{TE}}$$

# $TE_z$ Reflection (cont.)

## Transmission Line Analogy



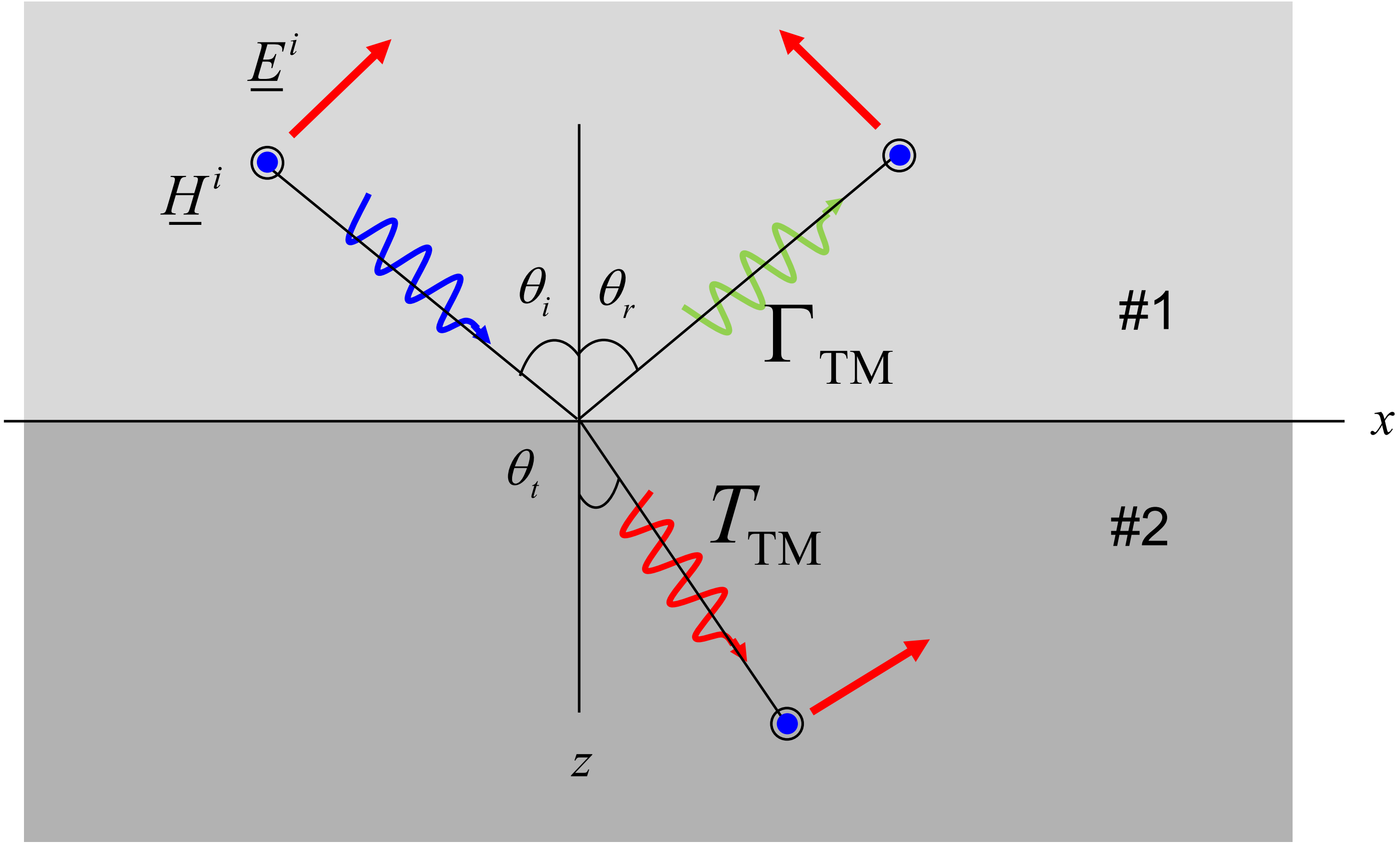




$$\Gamma_{TE} = \frac{Z_2^{TE} - Z_1^{TE}}{Z_2^{TE} + Z_1^{TE}}$$

$$T_{TE} = 1 + \Gamma_{TE}$$

$$Z_1^{TE} = \left( \frac{\omega \mu_1}{k_{zi}} \right) \qquad Z_2^{TE} = \left( \frac{\omega \mu_2}{k_{zt}} \right)$$

# $TM_z$ Reflection




Note that the electric field vector is in the $xz$ plane.

(The wave is polarized parallel to the plane of incidence.)

**Word of caution:** The notation used for the reflection coefficient in the $TM_z$ case is different from what is in the Shen & Kong book. (We use reflection coefficient to represent the reflection of the tangential electric field, not the tangential magnetic field.)

# $\text{TM}_z$ Reflection (cont.)

Incident Wave

$$\underline{H}^i = \underline{\hat{y}}\, H_0 \left(\frac{\omega\varepsilon_1}{k_{zi}}\right) e^{-jk_{xi}x - jk_{zi}z}$$

Reflected Wave

$$\underline{H}^r = \underline{\hat{y}}\left(-\Gamma_{\text{TM}}\right) H_0 \left(\frac{\omega\varepsilon_1}{k_{zr}}\right) e^{-jk_{xr}x + jk_{zr}z}$$

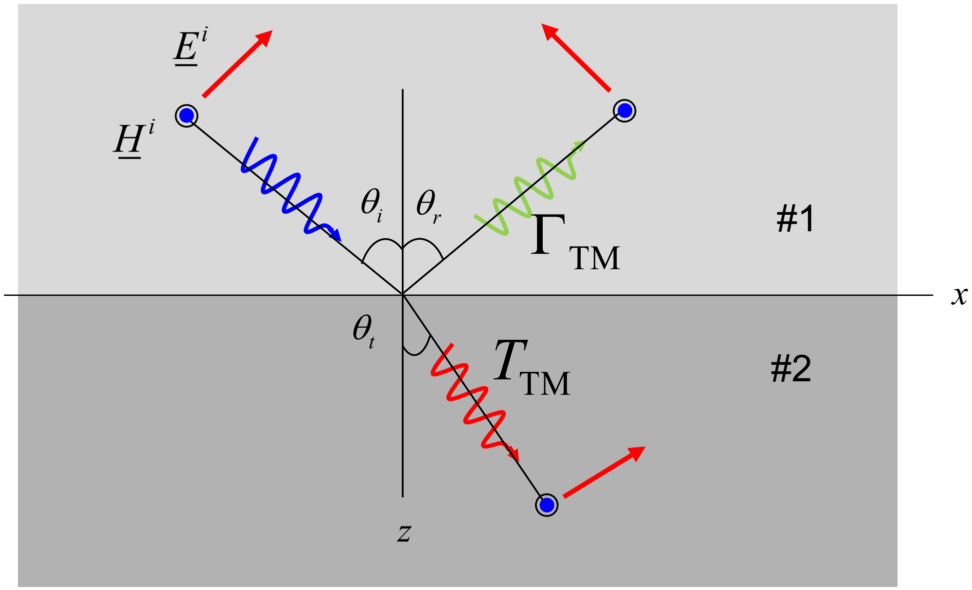
Transmitted Wave

$$\underline{H}^t = \underline{\hat{y}}\, T_{\text{TM}} H_0 \left(\frac{\omega\varepsilon_2}{k_{zt}}\right) e^{-jk_{xt}x - jk_{zt}z}$$

where

$\Gamma_{\text{TE}}$ = Reflection Coefficient

$T_{\text{TE}}$ = Transmission Coefficient




Incident Wave Vector

$$\underline{k}_i = \underline{\hat{x}}\, k_{xi} + \underline{\hat{z}}\, k_{zi}$$

Reflected Wave Vector

$$\underline{k}_r = \underline{\hat{x}}\, k_{xr} - \underline{\hat{z}}\, k_{zr}$$

Transmitted Wave Vector

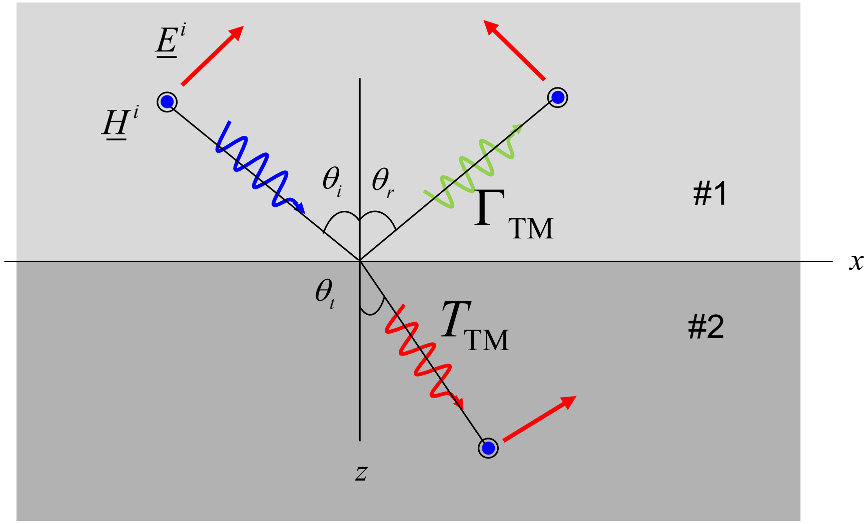
$$\underline{k}_t = \underline{\hat{x}}\, k_{xt} + \underline{\hat{z}}\, k_{zt}$$

# TM$_z$ Reflection (cont.)

We now look at the electric fields.

$$\nabla \times \underline{H} = j\omega\varepsilon \underline{E}$$

$$\Rightarrow \quad \underline{E} = \left(\frac{1}{j\omega\varepsilon}\right)\nabla \times \underline{H}$$




$$\underline{H}^i = \underline{\hat{y}}\, H_0 \left(\frac{\omega\varepsilon_1}{k_{zi}}\right) e^{-jk_{xi}x - jk_{zi}z}$$

$$\underline{E}^i = (\underline{\hat{x}} - \underline{\hat{z}}\, k_{xi} / k_{zi}) H_0\, e^{-j(k_{xi}x + k_{zi}z)}$$

$$\underline{H}^r = \underline{\hat{y}}(-\Gamma_{\mathrm{TM}})\left(\frac{\omega\varepsilon_1}{k_{zr}}\right) H_0\, e^{-jk_{xr}x + jk_{zr}z}$$

$$\underline{E}^r = (\underline{\hat{x}} + \underline{\hat{z}}\, k_{xr} / k_{zr})\Gamma_{\mathrm{TM}} H_0\, e^{-j(k_{xr}x - k_{zr}z)}$$

$$\underline{H}^t = \underline{\hat{y}}\, T_{\mathrm{TM}}\left(\frac{\omega\varepsilon_2}{k_{zt}}\right) H_0\, e^{-jk_{xt}x - jk_{zt}z}$$

$$\underline{E}^t = (\underline{\hat{x}} - \underline{\hat{z}}\, k_{xt} / k_{zt}) T_{\mathrm{TM}} H_0\, e^{-j(k_{xt}x + k_{zt}z)}$$

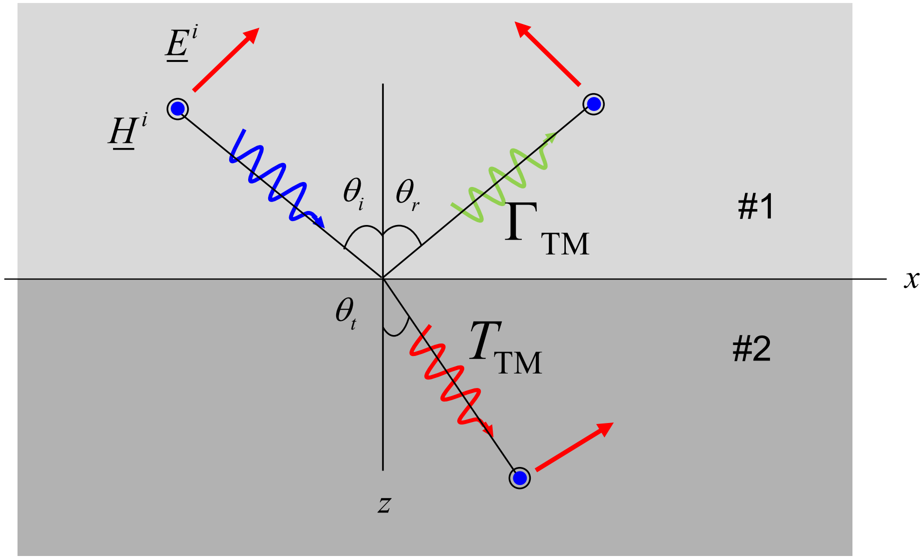
Note that $\Gamma_{\mathrm{TM}}$ is the reflection coefficient for the tangential electric field ($E_x$).

# $\mathrm{TM}_z$ Reflection (cont.)

Boundary conditions:

$$H_y^i + H_y^r = H_y^t$$

$$E_x^i + E_x^r = E_x^t$$




Enforcing both boundary conditions, we have

$$\frac{\omega\varepsilon_1}{k_{zi}} - \Gamma_{\mathrm{TM}}\left(\frac{\omega\varepsilon_1}{k_{zr}}\right) = T_{\mathrm{TM}}\left(\frac{\omega\varepsilon_2}{k_{zt}}\right)$$

$$1 + \Gamma_{\mathrm{TM}} = T_{\mathrm{TM}}$$

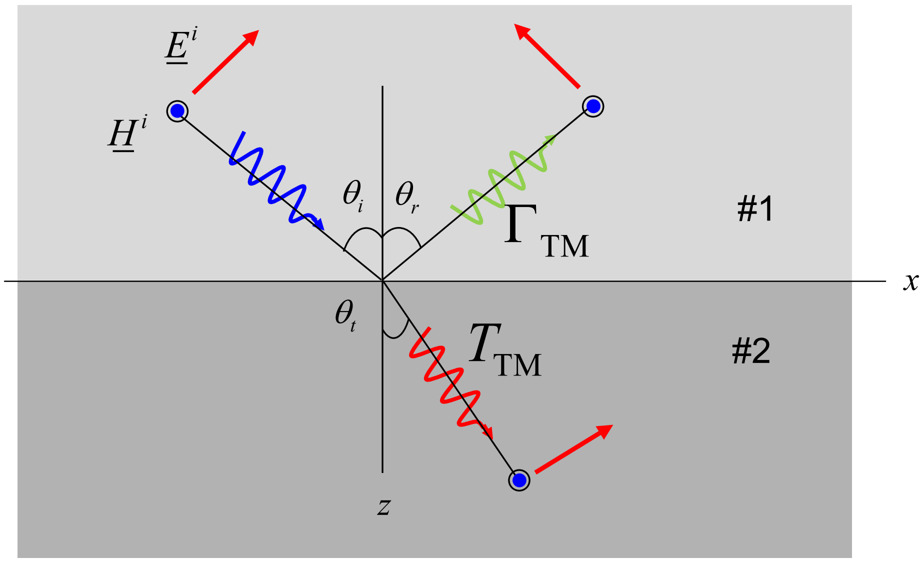
The solution is:

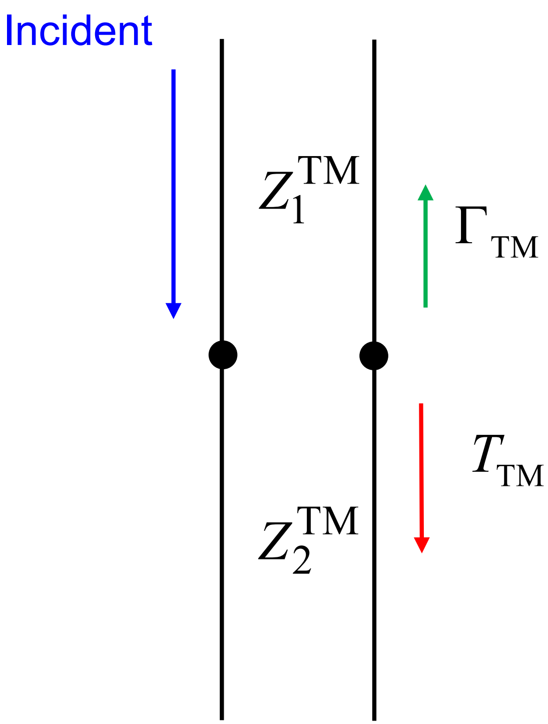
$$\Gamma_{\mathrm{TM}} = \frac{\left(\dfrac{k_{zt}}{\omega\varepsilon_2}\right) - \left(\dfrac{k_{zi}}{\omega\varepsilon_1}\right)}{\left(\dfrac{k_{zt}}{\omega\varepsilon_2}\right) + \left(\dfrac{k_{zi}}{\omega\varepsilon_1}\right)}$$

$$T_{\mathrm{TM}} = 1 + \Gamma_{\mathrm{TM}}$$

# $TM_z$ Reflection (cont.)

## Transmission Line Analogy



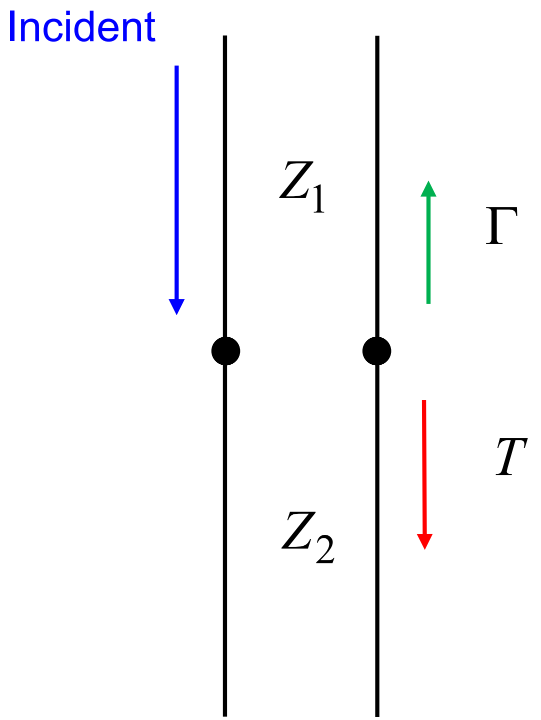

$$\Gamma_{TM} = \frac{Z_2^{TM} - Z_1^{TM}}{Z_2^{TM} + Z_1^{TM}}$$

$$T_{TM} = 1 + \Gamma_{TM}$$

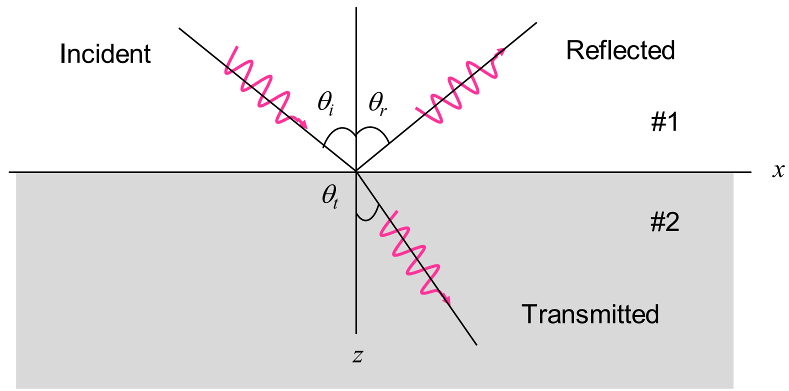
$$Z_1^{TM} = \left(\frac{k_{zi}}{\omega\varepsilon_1}\right) \qquad Z_2^{TM} = \left(\frac{k_{zt}}{\omega\varepsilon_2}\right)$$

# $TM_z$ Reflection (cont.)

## Summary of Transmission Line Modeling Equations




$$Z_m^{\mathrm{TM}} = \left( \frac{k_{zm}}{\omega \varepsilon_m} \right) \qquad Z_m^{\mathrm{TE}} = \left( \frac{\omega \mu_m}{k_{zm}} \right) \qquad m = 1, 2$$

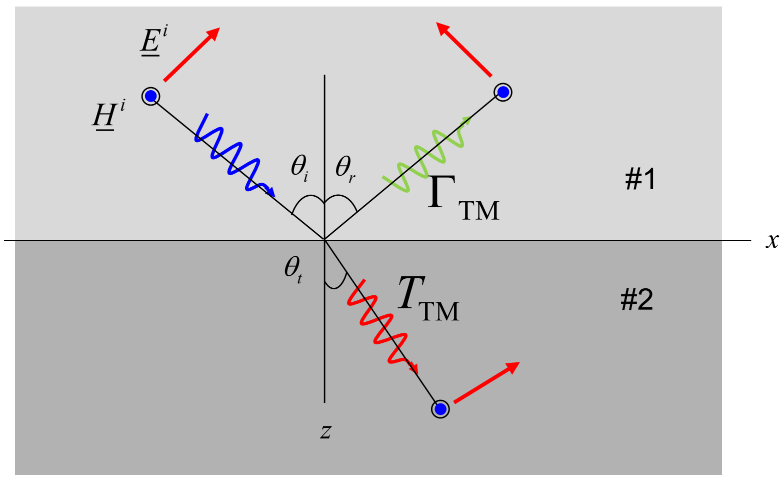


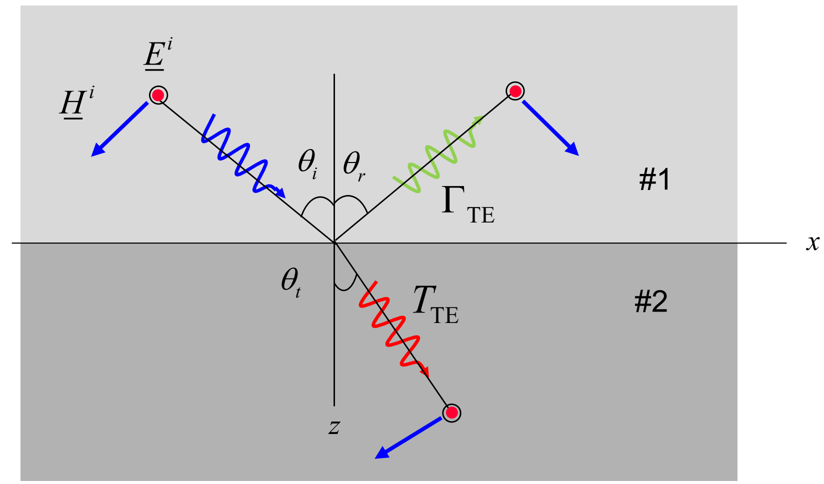

$$k_{z1} = k_{zi} = \sqrt{k_1^2 - k_{xi}^2} = \sqrt{k_1^2 - \left(k_1 \sin\theta_i\right)^2} = k_1 \cos\theta_i$$

$$k_{z2} = k_{zt} = \sqrt{k_2^2 - k_{xt}^2} = \sqrt{k_2^2 - k_{xi}^2} = \sqrt{k_2^2 - \left(k_1 \sin\theta_i\right)^2} = \sqrt{k_2^2 - \left(k_2 \sin\theta_t\right)^2} = k_2 \cos\theta_t$$

(**Recall:** $k_{xt} = k_{xi}$) (**Recall:** $k_1 \sin\theta_i = k_2 \sin\theta_t$)

# Power Reflection



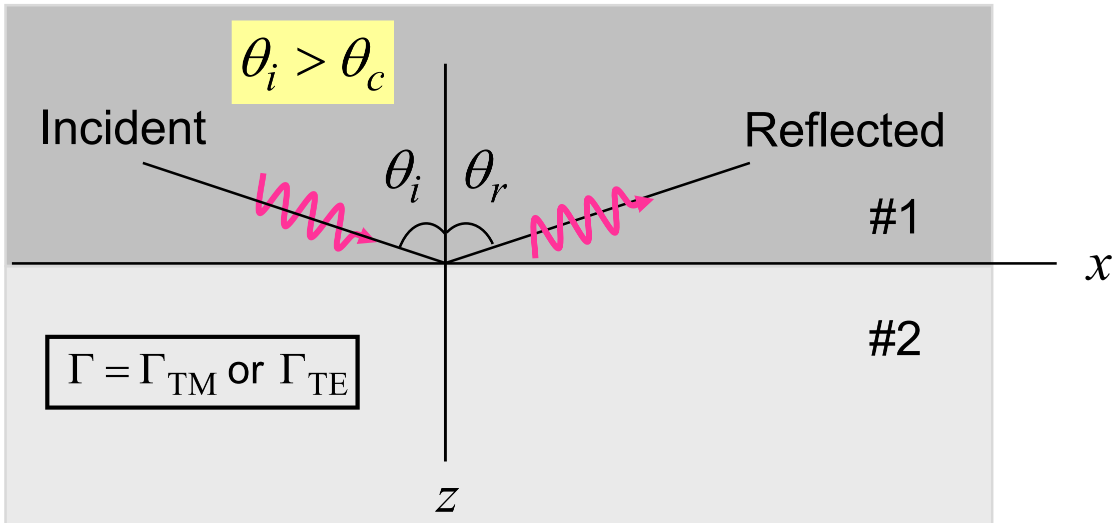




$$\% \text{ power reflected} = 100\,|\Gamma|^2$$

$$\% \text{ power transmitted} = 100\left(1-|\Gamma|^2\right)$$

$$\Gamma = \Gamma_{TM} \text{ or } \Gamma_{TE}$$

# Power Reflection Beyond Critical Angle




$$\% \text{ power reflected} = 100\,|\Gamma|^2$$

$$\Gamma = \frac{Z_2 - Z_1}{Z_2 + Z_1}$$

$$\left.\begin{array}{l} Z_2^{TM} = \dfrac{k_{zt}}{\omega\varepsilon_2} \\ Z_2^{TE} = \dfrac{\omega\mu_2}{k_{zt}} \end{array}\right\} \quad k_{zt} = -j\alpha_{zt}$$

$\Rightarrow$ $Z_2 =$ imaginary ("reactive load")

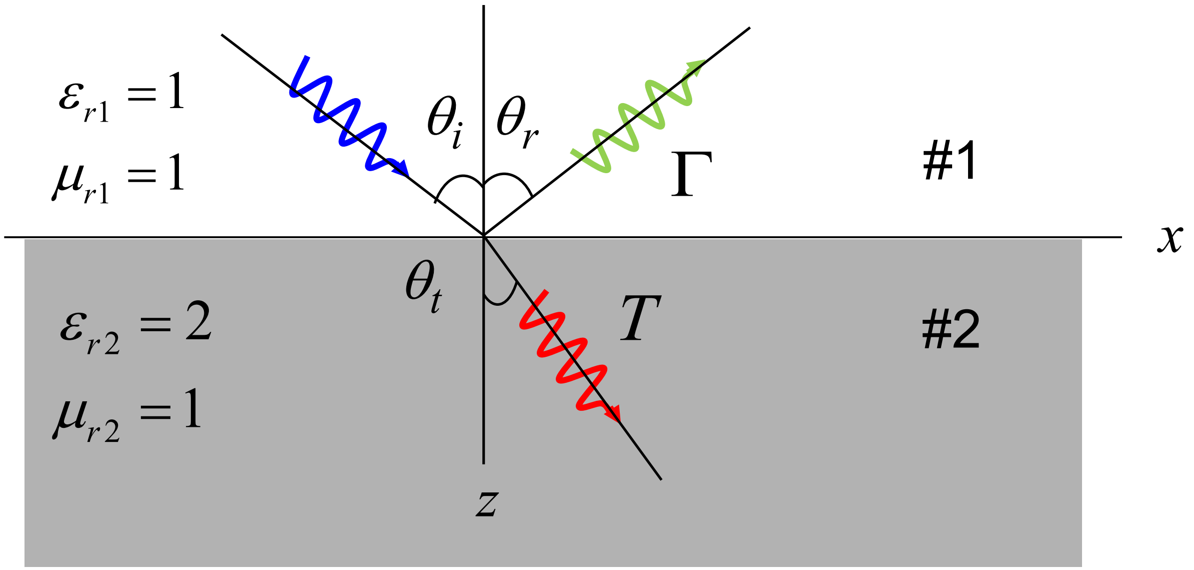
$\Rightarrow$ $|\Gamma| = 1$

All of the incident power is reflected.

# Example

Given:

$\theta_i = 30^\circ$




Find:

% power reflected and transmitted for a $TM_z$ wave

% power reflected and transmitted for a $TE_z$ wave

Snell's law: $n_1 \sin\theta_i = n_2 \sin\theta_t$

$$\sqrt{1}\sin 30^\circ = \sqrt{2}\sin\theta_t$$

$$\theta_t = 20.70^\circ$$

Recall:

$$n_i \equiv \sqrt{\varepsilon_{ri}\mu_{ri}}$$

# Example (cont.)

First look at the TM$_z$ case:

$$\Gamma_{\mathrm{TM}} = \frac{Z_2^{\mathrm{TM}} - Z_1^{\mathrm{TM}}}{Z_2^{\mathrm{TM}} + Z_1^{\mathrm{TM}}}$$

$$Z_2^{\mathrm{TM}} = \frac{k_{zt}}{\omega\varepsilon_2} = \frac{k_2\cos\theta_t}{\omega\varepsilon_2} = \frac{\cancel{\omega}\sqrt{\mu_2\varepsilon_2}\cos\theta_t}{\cancel{\omega}\varepsilon_2}$$

$$= \sqrt{\frac{\mu_2}{\varepsilon_2}}\cos\theta_t = \frac{\eta_0}{\sqrt{\varepsilon_{r2}}}\cos\theta_t$$

$$= \frac{376.7303}{\sqrt{2}}\cos\left(20.70^\circ\right)$$

$$= 249.8 \;\; [\Omega]$$

$$Z_1^{\mathrm{TM}} = \eta_0\cos\theta_i$$

$$= 327.1 \;\; [\Omega]$$

$$\Gamma_{\mathrm{TM}} = \frac{Z_2^{\mathrm{TM}} - Z_1^{\mathrm{TM}}}{Z_2^{\mathrm{TM}} + Z_1^{\mathrm{TM}}}$$

$$T_{\mathrm{TM}} = 1 + \Gamma_{\mathrm{TM}}$$

$$\Gamma_{\mathrm{TM}} = -0.1339$$

$$T_{\mathrm{TM}} = 0.8661$$

Note that these values do not depend on frequency for a lossless material!

# Example (cont.)

## Summary for TM$_z$ Case

$$\% \text{ power reflected} = 100\,|\Gamma|^2$$

$$\% \text{ power transmitted} = 100\left(1-|\Gamma|^2\right)$$

$$\Gamma^{\text{TM}} = -0.1339$$

$$\% \text{ power reflected} = 1.79$$

$$\% \text{ power transmitted} = 98.21$$

# Example (cont.)

Next, look at the $\mathrm{TE}_z$ case: $\Gamma_{\mathrm{TE}} = \dfrac{Z_2^{\mathrm{TE}} - Z_1^{\mathrm{TE}}}{Z_2^{\mathrm{TE}} + Z_1^{\mathrm{TE}}}$

$$Z_2^{\mathrm{TE}} = \frac{\omega\mu_2}{k_{zt}} = \frac{\cancel{\omega}\,\mu_2}{\cancel{\omega}\sqrt{\mu_2\varepsilon_2}\cos\theta_t}$$

$$= \sqrt{\frac{\mu_2}{\varepsilon_2}}\frac{1}{\cos\theta_t} = \frac{\eta_0}{\sqrt{\varepsilon_{r2}}}\sec\theta_t$$

$$= 285.5 \ [\Omega]$$

$$Z_1^{\mathrm{TE}} = \eta_0 \sec\theta_i$$

$$= 436.2 \ [\Omega]$$

$$\Gamma_{\mathrm{TE}} = \frac{Z_2^{\mathrm{TE}} - Z_1^{\mathrm{TE}}}{Z_2^{\mathrm{TE}} + Z_1^{\mathrm{TE}}}$$

$$T_{\mathrm{TE}} = 1 + \Gamma_{TE}$$

$$\Gamma_{\mathrm{TE}} = -0.2088$$

$$T_{\mathrm{TE}} = 0.7912$$

Note that these values do not depend on frequency for a lossless material!

# Example (cont.)

## Summary for TE$_z$ Case

$$\%\ \text{power reflected} = 100\ |\Gamma|^2$$

$$\%\ \text{power transmitted} = 100\left(1 - |\Gamma|^2\right)$$

$$\Gamma^{\mathrm{TE}} = -0.2088$$

$$\%\ \text{power reflected} = 4.36$$

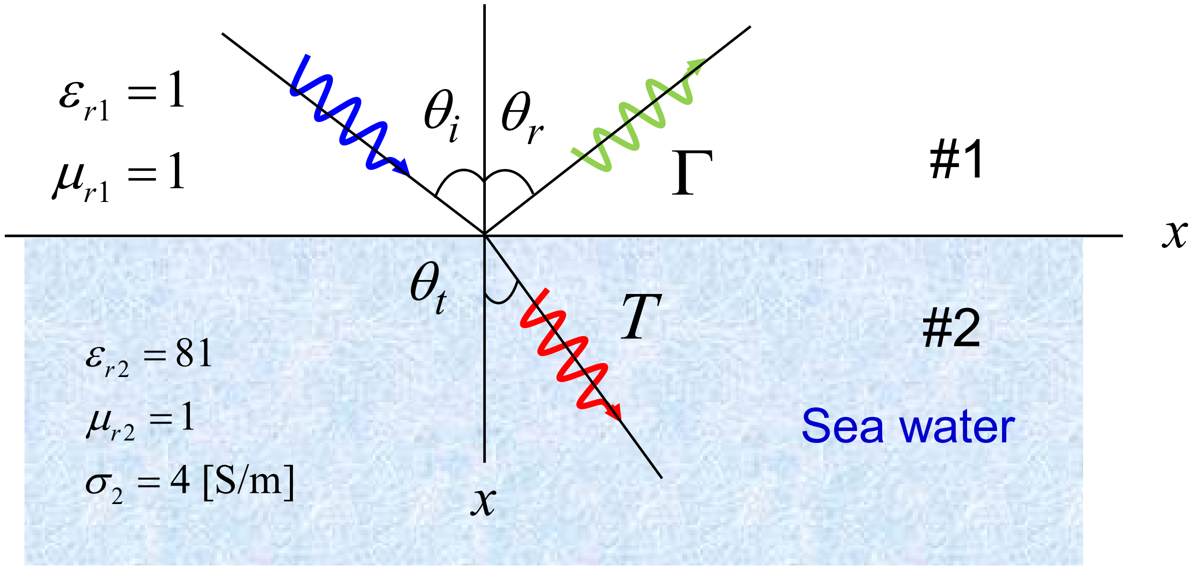
$$\%\ \text{power transmitted} = 95.64$$

# Example

The lower region is now lossy.

Given:

$$\theta_i = 30^\circ$$
$$f = 1 \text{ GHz}$$




Find:

% power reflected and transmitted for a $TM_z$ wave

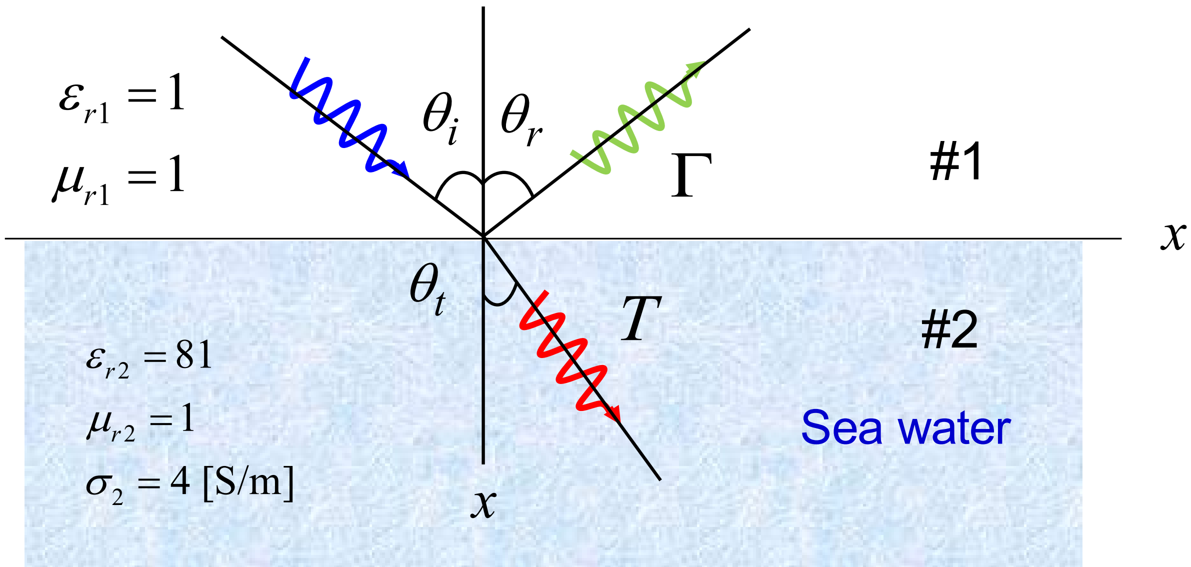
% power reflected and transmitted for a $TE_z$ wave

# Example (cont.)

Given:

$\theta_i = 30^\circ$

$f = 1$ GHz




$$\varepsilon_{c2} = \varepsilon_2 - j\frac{\sigma_2}{\omega} = \varepsilon_0 \varepsilon_{r2} - j\frac{\sigma_2}{\omega} \quad \Rightarrow \quad \varepsilon_{rc2} = \varepsilon_{r2} - j\frac{\sigma_2}{\omega \varepsilon_0} = 81 - j(71.90)$$

We avoid using Snell's law now, since it will give us a complex angle $\theta_t$ in region 2!

(**Recall:** $n_1 \sin\theta_i = n_2 \sin\theta_t$)

**Recommendation:** Work with the wavenumber $k_{zt}$ only (avoid using $\theta_t$).

# Example (cont.)

First look at the TM$_z$ case: $$\Gamma_{\text{TM}} = \frac{Z_2^{\text{TM}} - Z_1^{\text{TM}}}{Z_2^{\text{TM}} + Z_1^{TM}}$$

$$Z_2^{\text{TM}} = \frac{k_{zt}}{\omega \varepsilon_{c2}} = \frac{\sqrt{k_2^2 - k_{xt}^2}}{\omega \varepsilon_{c2}} = \frac{\sqrt{k_2^2 - k_{xi}^2}}{\omega \varepsilon_{c2}} = \frac{\sqrt{k_2^2 - k_1^2 \sin^2 \theta_i}}{\omega \varepsilon_{c2}}$$

$$= \frac{\sqrt{\omega^2 \mu_2 \varepsilon_{c2} - \omega^2 \mu_1 \varepsilon_1 \sin^2 \theta_i}}{\omega \varepsilon_{c2}}$$

$$= \frac{\sqrt{\mu_0 \varepsilon_{c2} - \mu_0 \varepsilon_1 \sin^2 \theta_i}}{\varepsilon_{c2}}$$

$$= \eta_0 \frac{\sqrt{\varepsilon_{rc2} - \varepsilon_{r1} \sin^2 \theta_i}}{\varepsilon_{rc2}}$$

$$= \eta_0 \frac{\sqrt{\varepsilon_{rc2} - \sin^2 \theta_i}}{\varepsilon_{rc2}}$$

**Note :** $k_{zt} = k_2 \cos \theta_t$
(This is difficult to work with!)

$$\varepsilon_{rc2} = 81 - j(71.90)$$

(at 1.0 [GHz])

$$Z_1^{\text{TM}} = \eta_0 \cos \theta_i$$

$$Z_1^{\text{TM}} = 326.3 \ [\Omega]$$

$$\Gamma_{\text{TM}} = \frac{Z_2^{\text{TM}} - Z_1^{\text{TM}}}{Z_2^{\text{TM}} + Z_1^{\text{TM}}}$$

$$T_{\text{TM}} = 1 + \Gamma_{\text{TM}}$$

$$Z_2^{\text{TM}} = 33.82 + j(12.82) \ [\Omega]$$

$$\Gamma_{\text{TM}} = -0.8099 + j(0.0644)$$

The reflection coefficient now depends on frequency.

# Example (cont.)

## Summary for $\text{TM}_z$ Case

$$\%\ \text{power reflected} = 100\ |\Gamma|^2$$

$$\%\ \text{power transmitted} = 100\left(1 - |\Gamma|^2\right)$$

$$\Gamma^{\text{TM}} = -0.8099 + j\left(0.0644\right)$$

$$\%\ \text{power reflected} = 66.0$$

$$\%\ \text{power transmitted} = 34.0$$

# Example (cont.)

Next, look at the TE$_z$ case: $\Gamma_{\text{TE}} = \dfrac{Z_2^{\text{TE}} - Z_1^{\text{TE}}}{Z_2^{\text{TE}} + Z_1^{\text{TE}}}$

$$Z_2^{\text{TE}} = \frac{\omega\mu_2}{k_{zt}} = \frac{\omega\mu_2}{\sqrt{k_2^2 - k_{xt}^2}} = \frac{\omega\mu_2}{\sqrt{k_2^2 - k_{xi}^2}} = \frac{\omega\mu_2}{\sqrt{k_2^2 - k_1^2 \sin^2\theta_i}}$$

$$= \frac{\omega\mu_2}{\sqrt{\omega^2\mu_2\varepsilon_{c2} - \omega^2\mu_1\varepsilon_1 \sin^2\theta_i}}$$

$$= \frac{\mu_0}{\sqrt{\mu_0\varepsilon_{c2} - \mu_0\varepsilon_1 \sin^2\theta_i}}$$

$$= \eta_0 \frac{1}{\sqrt{\varepsilon_{rc2} - \varepsilon_{r1} \sin^2\theta_i}}$$

$$= \eta_0 \frac{1}{\sqrt{\varepsilon_{rc2} - \sin^2\theta_i}}$$

$$\varepsilon_{rc2} = 81 - j(71.90)$$

(at 1.0 [GHz])

$$Z_1^{\text{TE}} = \eta_0 \sec\theta_i$$

$$Z_1^{\text{TE}} = 435.0\ [\Omega]$$

$$\Gamma_{\text{TE}} = \frac{Z_2^{\text{TE}} - Z_1^{\text{TE}}}{Z_2^{\text{TE}} + Z_1^{\text{TE}}}$$

$$T_{\text{TE}} = 1 + \Gamma_{\text{TE}}$$

$$Z_2^{\text{TE}} = 33.86 + j(12.89)$$

$$\Gamma_{\text{TE}} = -0.8542 + j(0.0510)$$

The reflection coefficient now depends on frequency.

# Example (cont.)

## Summary for TE$_z$ Case

$$\%\ \text{power reflected} = 100\ |\Gamma|^2$$

$$\%\ \text{power transmitted} = 100\left(1 - |\Gamma|^2\right)$$

$$\Gamma^{TE} = -0.8542 + j(0.0510)$$

$$\%\ \text{power reflected} = 73.2$$

$$\%\ \text{power transmitted} = 26.8$$

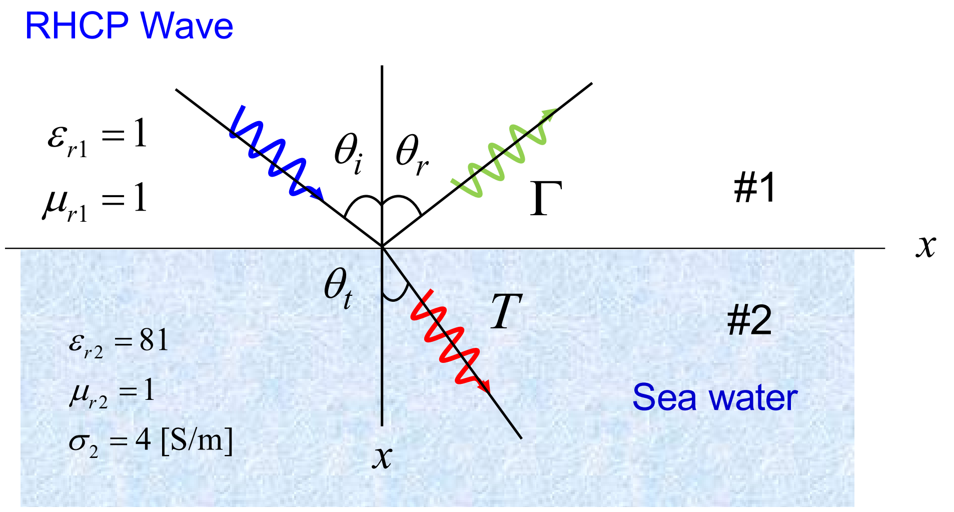
# Example (cont.)

**Now consider an incident RHCP Wave**

Given:

$\theta_i = 30^\circ$

$f = 1\ \text{GHz}$




Assume incident power density is 1 [W/m$^2$].

$$P_i^{\text{TM}} = P_i^{\text{TE}} = 0.5 \left[\text{W/m}^2\right]$$

$$P_r^{\text{TM}} = P_i^{\text{TM}} \left|\Gamma_{\text{TM}}\right|^2 = 0.5\left|-0.8099 + j(0.0644)\right|^2 = 0.3300 \left[\text{W/m}^2\right]$$

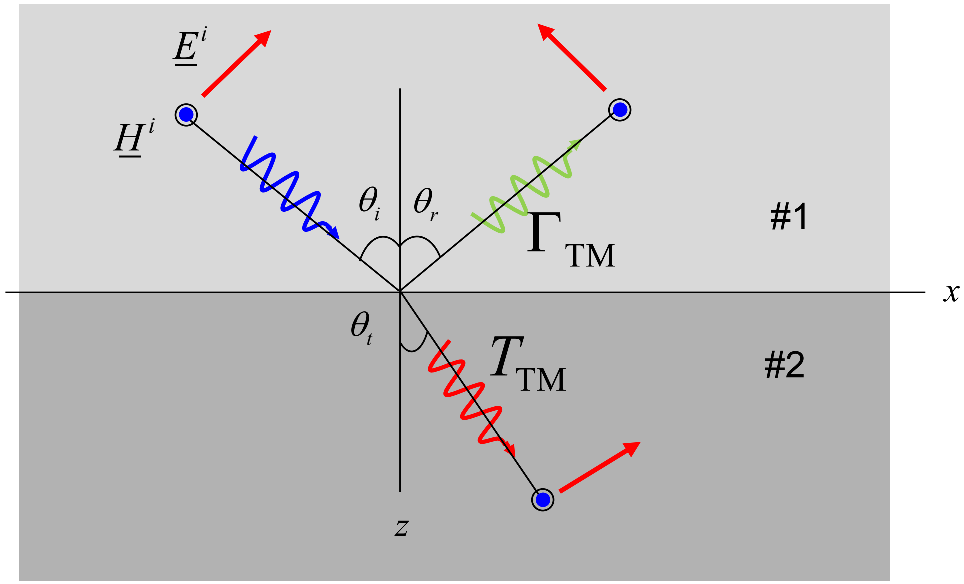
$$P_r^{\text{TE}} = P_i^{\text{TE}} \left|\Gamma_{\text{TE}}\right|^2 = 0.5\left|-0.8542 + j(0.0510)\right|^2 = 0.3661 \left[\text{W/m}^2\right]$$

- The reflected power is 69.6% of the incident power
- The reflected wave has 47.4% of its power in the TM polarization.

# Brewster Angle

Assume $TM_z$ polarization

Assume lossless regions




$$\Gamma_{TM} = \frac{Z_2^{TM} - Z_1^{TM}}{Z_2^{TM} + Z_1^{TM}}$$

Set $\Gamma_{TM} = 0$ ⇒ $Z_2^{TM} = Z_1^{TM}$

Matched load!

# Brewster Angle (cont.)

$$Z_2^{\mathrm{TM}} = Z_1^{\mathrm{TM}} \qquad Z_1^{\mathrm{TM}} = \left(\frac{k_{z1}}{\omega\varepsilon_1}\right) = \left(\frac{k_{zi}}{\omega\varepsilon_1}\right) \qquad Z_2^{\mathrm{TM}} = \left(\frac{k_{z2}}{\omega\varepsilon_2}\right) = \left(\frac{k_{zt}}{\omega\varepsilon_2}\right)$$

Hence, we have:

$$\frac{k_{zi}}{\omega\varepsilon_1} = \frac{k_{zt}}{\omega\varepsilon_2} \qquad \left(k_{zt} = k_2 \cos\theta_t = \sqrt{k_2^2 - k_1^2 \sin^2\theta_i}\right)$$

$$\Rightarrow \frac{k_1 \cos\theta_i}{\varepsilon_1} = \frac{\sqrt{k_2^2 - k_1^2 \sin^2\theta_i}}{\varepsilon_2}$$

$$\Rightarrow \frac{k_1^2 (1 - \sin^2\theta_i)}{\varepsilon_1^2} = \frac{k_2^2 - k_1^2 \sin^2\theta_i}{\varepsilon_2^2}$$

# Brewster Angle (cont.)

$$\frac{k_1^2(1-\sin^2\theta_i)}{\varepsilon_1^2} = \frac{k_2^2 - k_1^2\sin^2\theta_i}{\varepsilon_2^2}$$

Assume $\mu_1 = \mu_2$:

$$\frac{\varepsilon_1\left(1-\sin^2\theta_i\right)}{\varepsilon_1^2} = \frac{\varepsilon_2 - \varepsilon_1\sin^2\theta_i}{\varepsilon_2^2} \Longrightarrow \frac{1-\sin^2\theta_i}{\varepsilon_1} = \frac{\varepsilon_2 - \varepsilon_1\sin^2\theta_i}{\varepsilon_2^2}$$

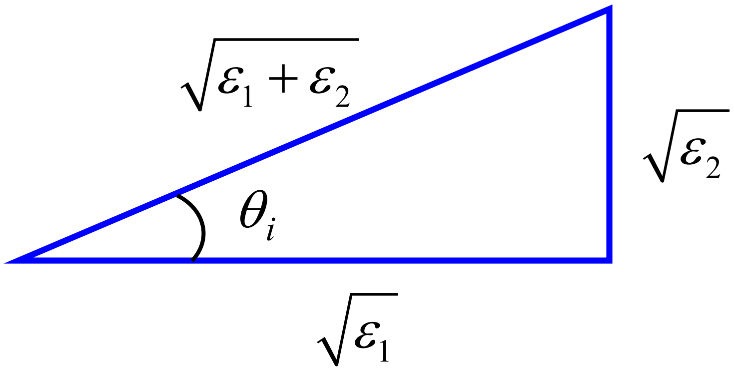
$$\Longrightarrow \varepsilon_2^2 - \varepsilon_2^2\sin^2\theta_i = \varepsilon_1\varepsilon_2 - \varepsilon_1^2\sin^2\theta_i$$

$$\Longrightarrow \sin^2\theta_i = \frac{\varepsilon_2^2 - \varepsilon_2\varepsilon_1}{\varepsilon_2^2 - \varepsilon_1^2} = \frac{\varepsilon_2(\varepsilon_2 - \varepsilon_1)}{\varepsilon_2^2 - \varepsilon_1^2} = \frac{\varepsilon_2(\varepsilon_2 - \varepsilon_1)}{\left(\varepsilon_2 - \varepsilon_1\right)\left(\varepsilon_2 + \varepsilon_1\right)} = \frac{\varepsilon_2}{\varepsilon_1 + \varepsilon_2}$$

# Brewster Angle (cont.)

$$\sin\theta_i = \sqrt{\frac{\varepsilon_2}{\varepsilon_1 + \varepsilon_2}}$$

Geometrical angle picture:




Hence

$$\theta_i = \tan^{-1}\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$$

# Brewster Angle (cont.)

This special angle is called the Brewster angle $\theta_b$.

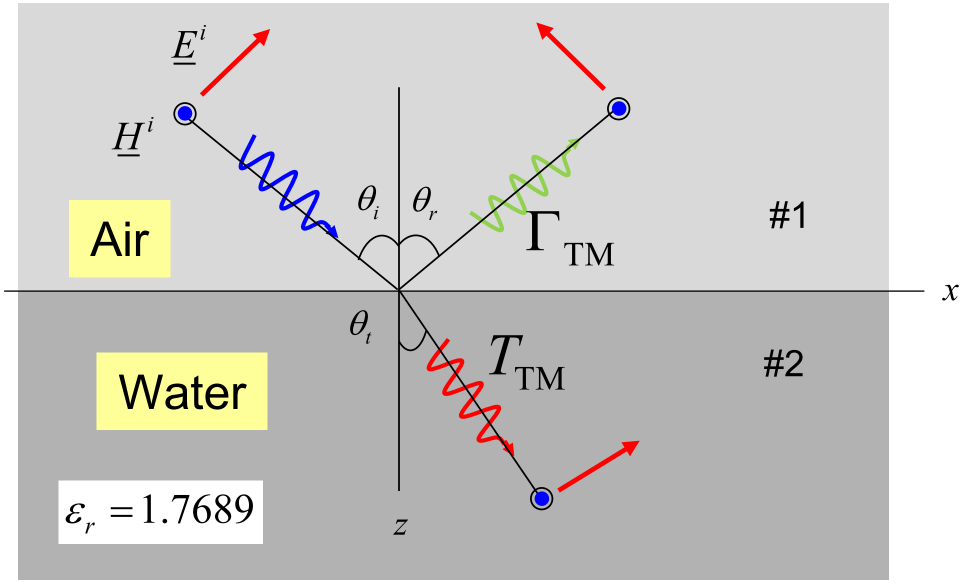
$$\theta_i = \theta_b = \tan^{-1}\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$$

- For non-magnetic media ($\mu = \mu_0$), only the $TM_z$ polarization (parallel polarization) has a Brewster angle.
- A Brewster angle exists for any material contrast ratio (it doesn't matter which side is optically denser).

# Brewster Angle (cont.)

Example

Light goes from air to water.

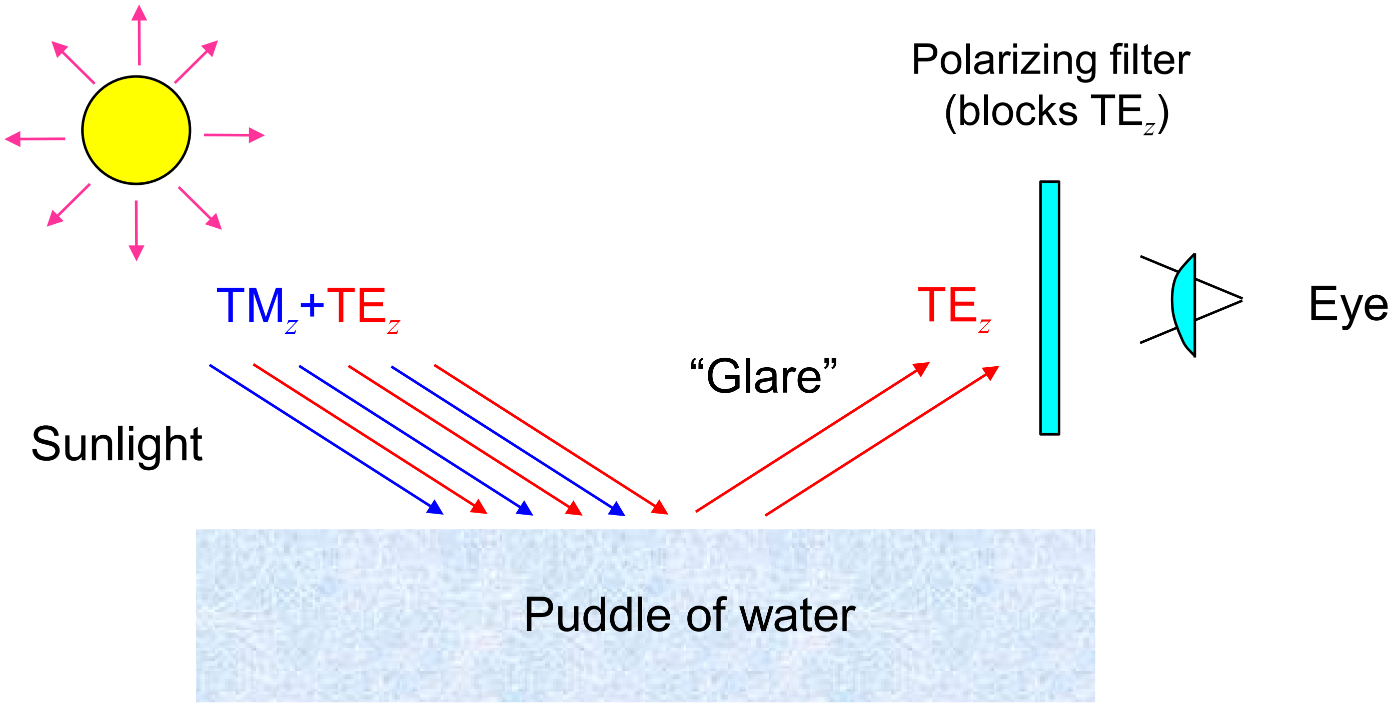



$$\theta_i = \theta_b = \tan^{-1}\sqrt{\frac{\varepsilon_2}{\varepsilon_1}} = \tan^{-1}\sqrt{\frac{\varepsilon_{r2}}{\varepsilon_{r1}}} = \tan^{-1}\sqrt{1.7689}$$

$$\theta_b = 53.06^{\circ}$$

# Brewster Angle (cont.)

## Polaroid Sunglasses




The reflection from the puddle of water (the "glare") is reduced.

# Polarization Effects

A building seen through polaroid sunglasses

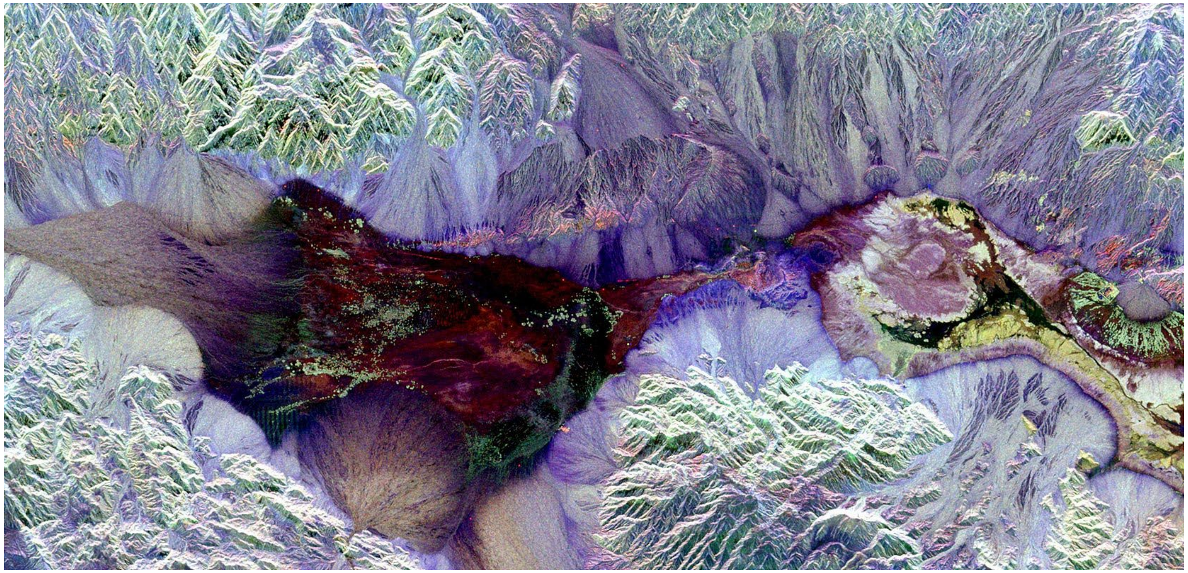
https://en.wikipedia.org/wiki/Polaroid_(polarizer)

By Jason7825 at English Wikipedia

# Polarization Effects (cont.)

“**Polarimetry** is the measurement and interpretation of the polarization of transverse waves, most notably electromagnetic waves, such as radio or light waves. Typically polarimetry is done on electromagnetic waves that have traveled through or have been reflected, refracted or diffracted by some material in order to characterize that object.”

https://en.wikipedia.org/wiki/Polarimetry



Synthetic aperture radar image of Death Valley colored using polarimetry.