ECE 3317
Applied Electromagnetic Waves

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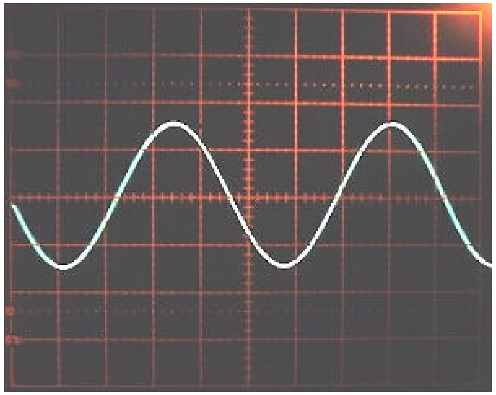
Fall 2023



Notes 2
Complex Vectors

Adapted from notes by Prof. Stuart A. Long

Notation

Circuit quantities:

- $v(t)$ is a time-varying function.
- V is a phasor (complex number).

Note:
"Handscript SF" font is used for time-domain vector quantities. (This font has been placed on Canvas for you.)

Field quantities:

- $\underline{\mathcal{E}}(t)$ is a time-varying vector function.
- $\underline{E}$ is a phasor vector (complex vector).
- $\mathcal{E}_x(t)$ is a time-varying component of a vector function.
- E_x is a phasor component of a vector function.

Appendices A, B, C, and D in the Shen & Kong text book list frequently used symbols, constants, and units. Appendix B in the Hayt & Buck book discusses units in some detail.

Complex Numbers

$$c = a + j\,b = |c|\,e^{j\phi}$$

Real part

Imaginary part

Magnitude

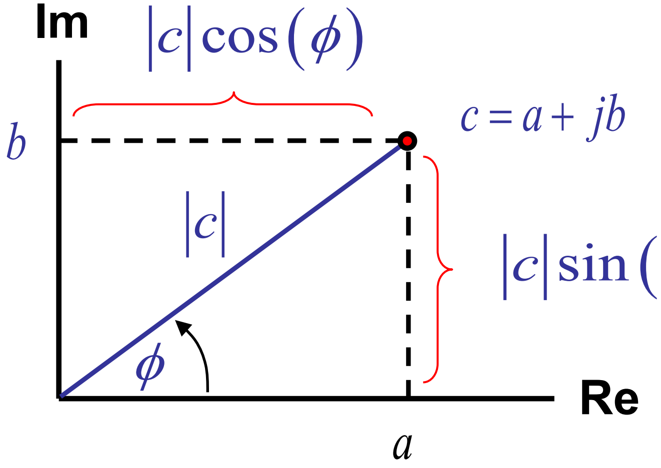
Phase (always in radians)

$$j = \sqrt{-1}$$

Euler's identity allows us to write the polar form.

Polar form:

$$|c| = \sqrt{a^2 + b^2}$$



Euler's identity:

$$e^{j\phi} = \cos\phi + j\sin\phi$$

$$\Rightarrow \quad c = |c|\left(\cos\phi + j\sin\phi\right)$$

$$a = |c|\cos\phi$$

$$b = |c|\sin\phi$$

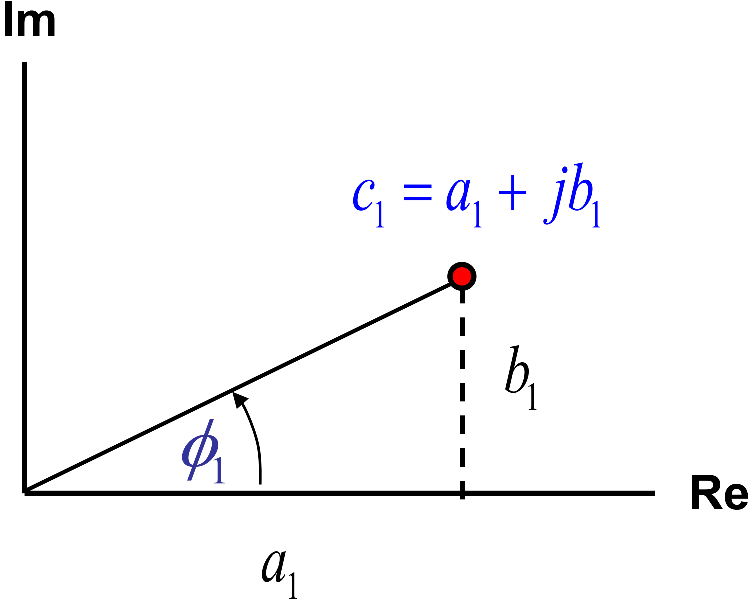
Note: The angle ϕ is measured positive going counterclockwise.

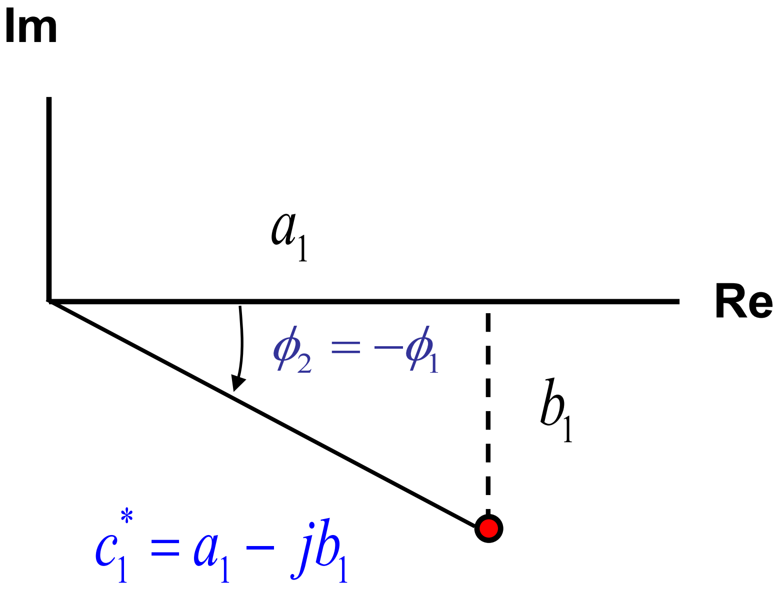
Complex Numbers (cont.)

Complex conjugate

$$c_1 = \left(a_1 + jb_1\right) = |\, c_1 \,| \, e^{j\phi_1}$$

$$c_2 = c_1^* = \left(a_1 + jb_1\right)^* \equiv a_1 - jb_1 \;\; = |\, c_1 \,| \, e^{j(-\phi_1)} \quad \left(\phi_2 = -\phi_1\right)$$





Complex Algebra

$$c_1 = a_1 + jb_1 = |c_1|e^{j\phi_1}$$
$$c_2 = a_2 + jb_2 = |c_2|e^{j\phi_2}$$

Addition

$$c_1 + c_2 = (a_1 + a_2) + j(b_1 + b_2)$$

Subtraction

$$c_1 - c_2 = (a_1 - a_2) + j(b_1 - b_2)$$

Complex Algebra (cont.)

$$c_1 = a_1 + jb_1 = |c_1|e^{j\phi_1}$$
$$c_2 = a_2 + jb_2 = |c_2|e^{j\phi_2}$$

Multiplication

$$(c_1)(c_2) = (a_1 + jb_1)(a_2 + jb_2) = (a_1a_2 - b_1b_2) + j(a_1b_2 + a_2b_1)$$

$$(c_1)(c_2) = |c_1||c_2|e^{j(\phi_1 + \phi_2)}$$

Division

$$\frac{c_1}{c_2} = \frac{a_1 + jb_1}{a_2 + jb_2} = \frac{a_1 + jb_1}{a_2 + jb_2}\left(\frac{a_2 - jb_2}{a_2 - jb_2}\right) = \frac{(a_1 + jb_1)(a_2 - jb_2)}{a_2^2 + b_2^2} = \frac{(a_1a_2 + jb_1b_2) + j(b_1a_2 - a_1b_2)}{a_2^2 + b_2^2}$$

$$\frac{c_1}{c_2} = \frac{|c_1|}{|c_2|}e^{j(\phi_1 - \phi_2)}$$

Square Root

Principal square root of a complex number (denoted by radical sign):

$$c = |c| e^{j\phi}$$

$$\sqrt{c} = \left(|c| e^{j\phi}\right)^{1/2} \qquad -\pi < \phi \leq \pi$$

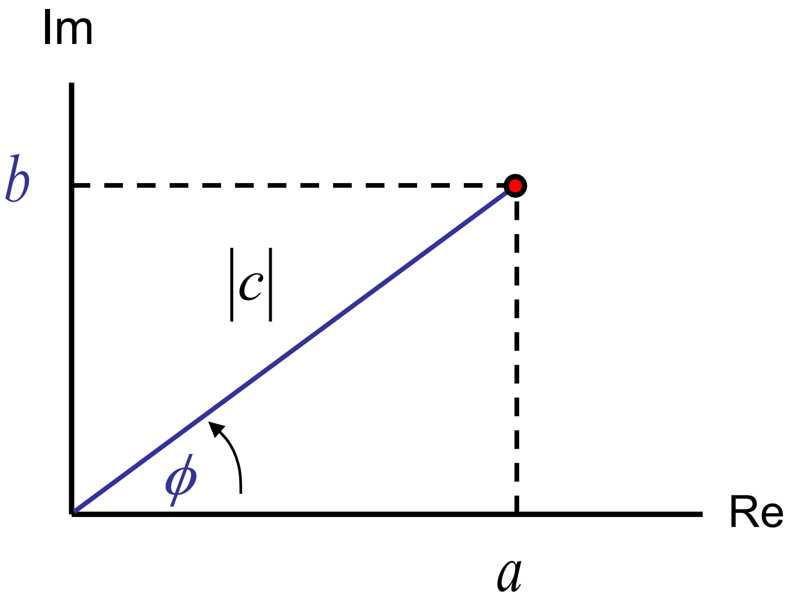
(the principal branch)

$$\sqrt{|c|}\; e^{j(\phi/2)}$$

Note:
For a positive real number x, the principal square root is positive:

$$\sqrt{x} > 0, \quad \text{for } x > 0$$

Example : $\sqrt{4} = 2$

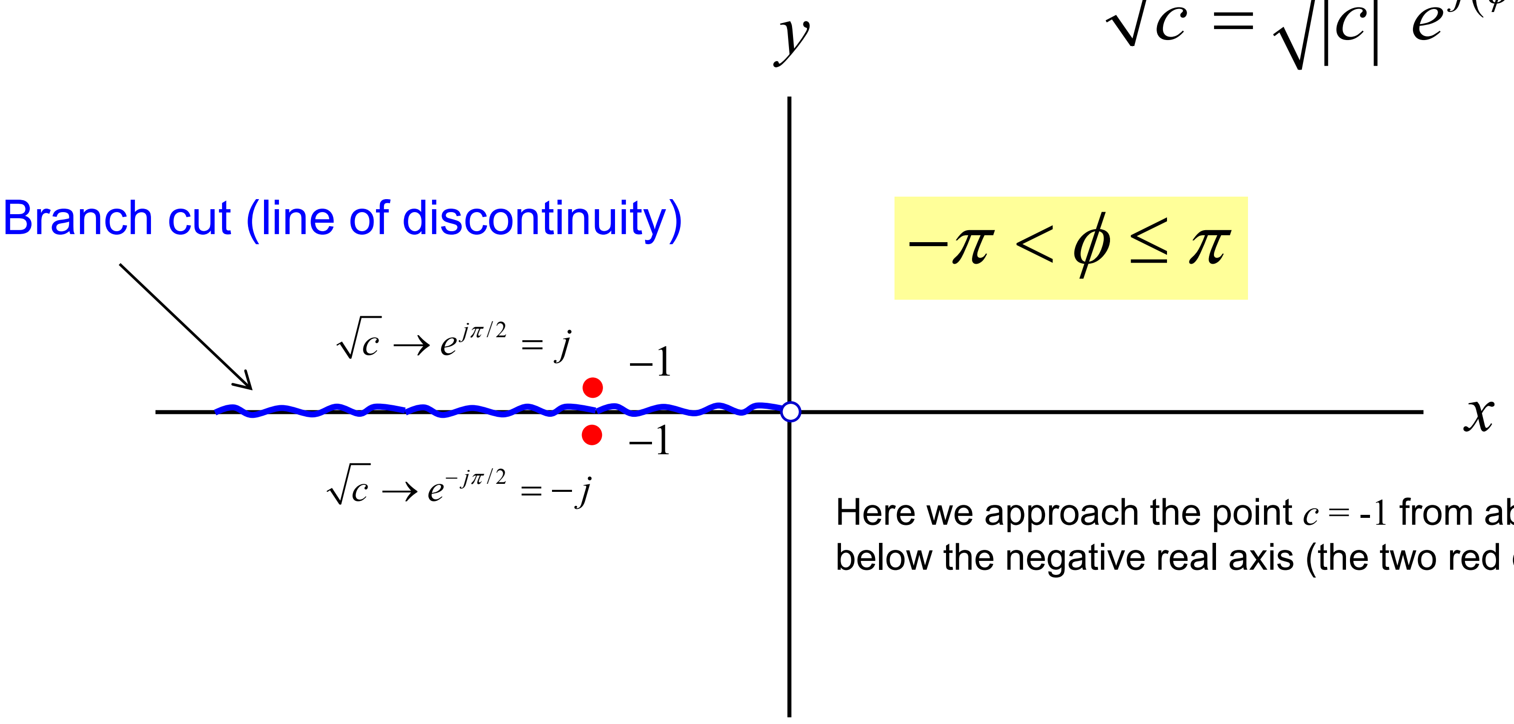


Square Root

$$c = |c|\, e^{j\phi}$$

Illustration of principal square root:

$$\sqrt{c} = \sqrt{|c|}\; e^{j(\phi/2)}$$



Branch cut (line of discontinuity)

$\sqrt{c} \to e^{j\pi/2} = j$

$\sqrt{c} \to e^{-j\pi/2} = -j$

$$-\pi < \phi \le \pi$$

Here we approach the point $c = -1$ from above or below the negative real axis (the two red dots).

Note: Matlab uses the principal square root: $\sqrt{-1} = +j$

Square Root (cont.)

$$c = |c|\, e^{j\phi}$$

$$\sqrt{c} = \sqrt{|c|}\; e^{j(\phi/2)}$$

$$-\pi < \phi \leq \pi$$

Property of principal square root:

$$-\pi < \phi \leq \pi$$

$$\Downarrow$$

$$-\pi/2 < \phi/2 \leq \pi/2$$

$$\Downarrow$$

$$\boxed{\mathrm{Re}\sqrt{c} \geq 0}$$

Examples :

$$\sqrt{4} = 2$$

$$\sqrt{j} = \sqrt{1e^{j\pi/2}} = \sqrt{1}\, e^{j\pi/4} = \frac{1+j}{\sqrt{2}}$$

Square Root (cont.)

$$c = |c| e^{j\phi}$$

General square root of a complex number:

Denotes the principal branch

$$\begin{aligned}
c^{1/2} &= \left(|c| e^{j\phi}\right)^{1/2} \\
&= \left(|c| e^{j\left(\phi_p + 2\pi n\right)}\right)^{1/2} \quad \left(-\pi < \phi_p \le \pi, \quad n \text{ is any integer}\right) \\
&= \sqrt{|c|}\; e^{j\left(\phi_p/2 + 2\pi n/2\right)} \\
&= \left(\sqrt{|c|}\; e^{j\phi_p/2}\right) e^{jn\pi} \\
&= \pm\sqrt{|c|}\; e^{j\phi_p/2} \quad \left(+ \text{ for } n \text{ even, - for } n \text{ odd}\right) \\
&= \pm\sqrt{c}
\end{aligned}$$

Examples:

$$4^{1/2} = \pm 2$$

$$j^{1/2} = \pm\left(\frac{1+j}{\sqrt{2}}\right)$$

The general square root has two possible values.

Time-Harmonic Quantities

$$v(t) = A\cos(\omega t + \phi)$$

Amplitude

Angular Frequency

Phase

$$\boxed{f = \frac{\omega}{2\pi}}$$

Frequency [Hz]

$$\boxed{T = \frac{1}{f}}$$

Period [s]

From Euler's identity:

$$v(t) = \mathrm{Re}\left\{A e^{j\phi} e^{j\omega t}\right\}$$

Define the phasor : $V \equiv A e^{j\phi}$

We then have

$$v(t) = \mathrm{Re}\left\{V e^{j\omega t}\right\}$$

Time-Harmonic Quantities (cont.)

$$v(t) = A\cos(\omega t + \phi)$$

$$\begin{cases} V \equiv A e^{j\phi} \\ v(t) = \mathrm{Re}\left\{ V e^{j\omega t} \right\} \end{cases}$$

$V \equiv A e^{j\phi}$ going from time domain to phasor domain

$v(t) = \mathrm{Re}\left\{ V e^{j\omega t} \right\}$ going from phasor domain to time domain

Time-domain ⇔ Phasor domain

$$v(t) \Leftrightarrow V$$

Time-Harmonic Quantities (cont.)

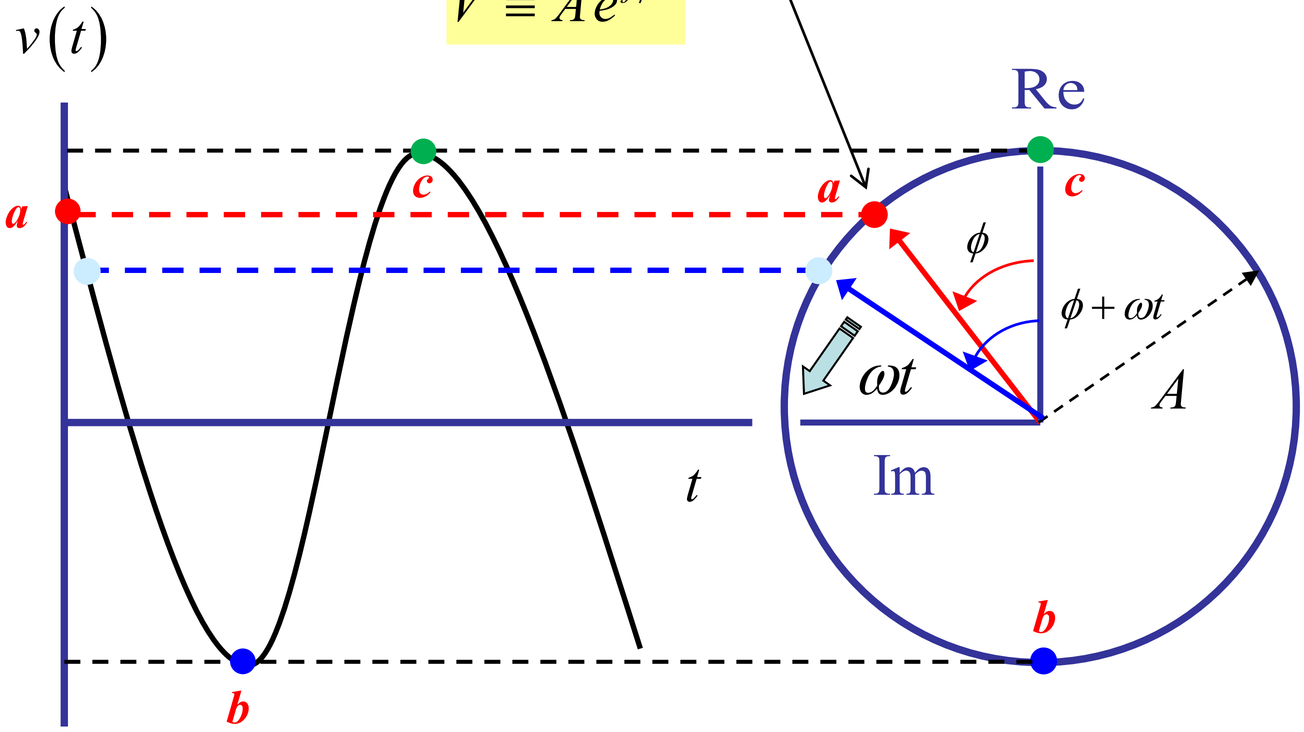
$$v(t) = A\cos(\omega t + \phi)$$

$$v(t) = \mathrm{Re}\left\{ V e^{j\omega t} \right\}$$

$$= \mathrm{Re}\left\{ A e^{j\phi}\, e^{j\omega t} \right\}$$

The complex number V

$$V \equiv A e^{j\phi}$$



Time-Harmonic Quantities (cont.)

$$v(t) \Leftrightarrow V$$

Note:

$$u(t) + v(t) \Leftrightarrow U + V$$

This assumes that the two sinusoidal signals are at the same frequency.

$$\frac{\partial}{\partial t} v(t) \Leftrightarrow j\omega V$$

There are no time derivatives in the phasor domain!

However:

$$u(t)\,v(t) \not\Leftrightarrow UV \quad \left(\text{i.e., } u(t)\,v(t) \neq \mathrm{Re}\left(UVe^{j\omega t}\right)\right)$$

All phasors are complex numbers, but not all complex numbers are phasors!

Complex Vectors

Vectors in the Phasor Domain

$$\underline{\mathscr{E}}(t) = \underline{\hat{x}}\mathscr{E}_x(t) + \underline{\hat{y}}\mathscr{E}_y(t) + \underline{\hat{z}}\mathscr{E}_z(t) \quad \text{(assumed to be sinusoidal)}$$

$$= \underline{\hat{x}}A_x \cos(\omega t + \phi_x) + \underline{\hat{y}}A_y \cos(\omega t + \phi_y) + \underline{\hat{z}}A_z \cos(\omega t + \phi_z)$$

Convert to phasor domain:

$$\underline{\mathscr{E}}(t) = \mathrm{Re}\left(\underline{\hat{x}}A_x e^{j\phi_x} e^{j\omega t}\right) + \mathrm{Re}\left(\underline{\hat{y}}A_y e^{j\phi_y} e^{j\omega t}\right) + \mathrm{Re}\left(\underline{\hat{z}}A_z e^{j\phi_z} e^{j\omega t}\right)$$

$$= \mathrm{Re}\left\{\left(\underline{\hat{x}}A_x e^{j\phi_x} + \underline{\hat{y}}A_y e^{j\phi_y} + \underline{\hat{z}}A_z e^{j\phi_z}\right) e^{j\omega t}\right\}$$

$$= \mathrm{Re}\left\{\left(\underline{\hat{x}}E_x + \underline{\hat{y}}E_y + \underline{\hat{z}}E_z\right) e^{j\omega t}\right\}$$

$$= \mathrm{Re}\left\{\underline{E}\, e^{j\omega t}\right\}$$

Question:
Why does the frequency have to be the same for all components?

where

$$\underline{E} \equiv \underline{\hat{x}}E_x + \underline{\hat{y}}E_y + \underline{\hat{z}}E_z$$

Complex vector!

Complex Vectors (cont.)

$$\underline{\mathscr{E}}(t) = \underline{\hat{x}}\mathscr{E}_x(t) + \underline{\hat{y}}\mathscr{E}_y(t) + \underline{\hat{z}}\mathscr{E}_z(t) \quad \text{(sinusoidal)}$$

We have proven:

$$\underline{\mathscr{E}}(t) = \operatorname{Re}\left\{\underline{E}\, e^{j\omega t}\right\}$$

So we work with phasor vectors the same way as we do with phasor scalars!

where

$$\underline{E} \equiv \underline{\hat{x}}E_x + \underline{\hat{y}}E_y + \underline{\hat{z}}E_z$$

Complex vector!

Notation:

$$\underline{\mathscr{E}}(t) \Leftrightarrow \underline{E}$$

$$\mathscr{E}_x(t) \Leftrightarrow E_x$$

$$\mathscr{E}_y(t) \Leftrightarrow E_y$$

$$\mathscr{E}_z(t) \Leftrightarrow E_z$$

Example 1.15 (Shen & Kong)

Assume $\underline{A} = \underline{\hat{x}} + j\underline{\hat{y}}$

Find the corresponding time-domain vector.

Note : $\left|\underline{\mathscr{A}}(t)\right| = \sqrt{\mathscr{A}_x^2(t) + \mathscr{A}_y^2(t)} = \sqrt{\cos^2 \omega t + \sin^2 \omega t} = 1$

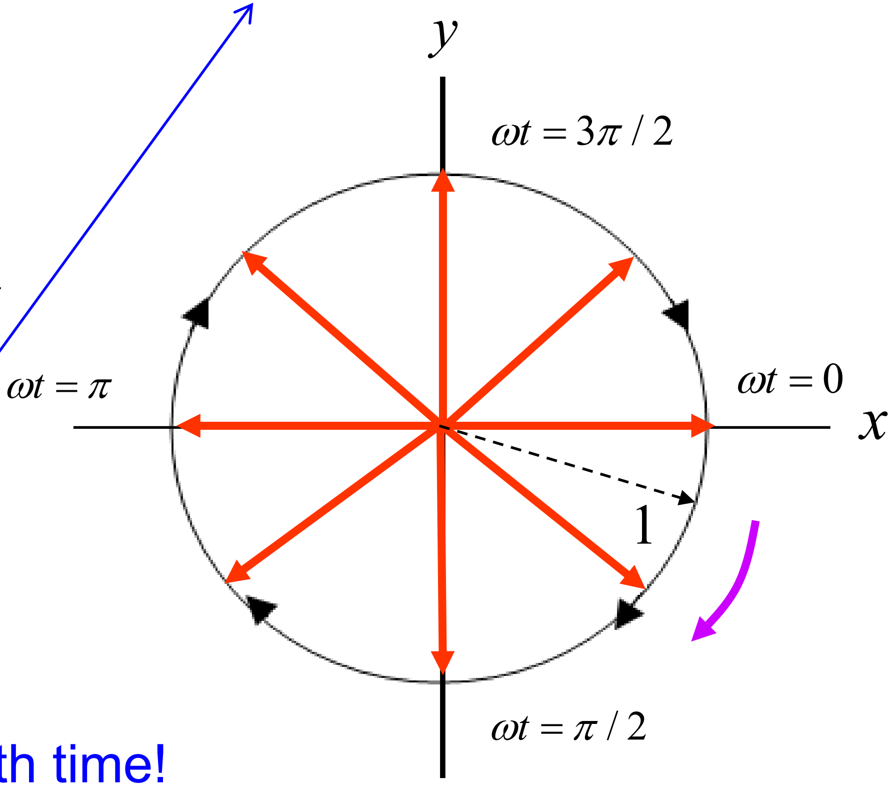
$$\underline{\mathscr{A}}(t) = \mathrm{Re}\left\{\underline{A}\, e^{j\omega t}\right\}$$

$$= \mathrm{Re}\left\{(\underline{\hat{x}} + j\underline{\hat{y}})\, e^{j\omega t}\right\}$$

$$= \mathrm{Re}\left\{(\underline{\hat{x}} + j\underline{\hat{y}})(\cos \omega t + j \sin \omega t)\right\}$$

$$= \underline{\hat{x}} \cos \omega t - \underline{\hat{y}} \sin \omega t$$

$$\underline{\mathscr{A}}(t) = \underline{\hat{x}} \cos \omega t - \underline{\hat{y}} \sin \omega t$$



The vector rotates (clockwise) with time!

Example 1.15 (cont.)

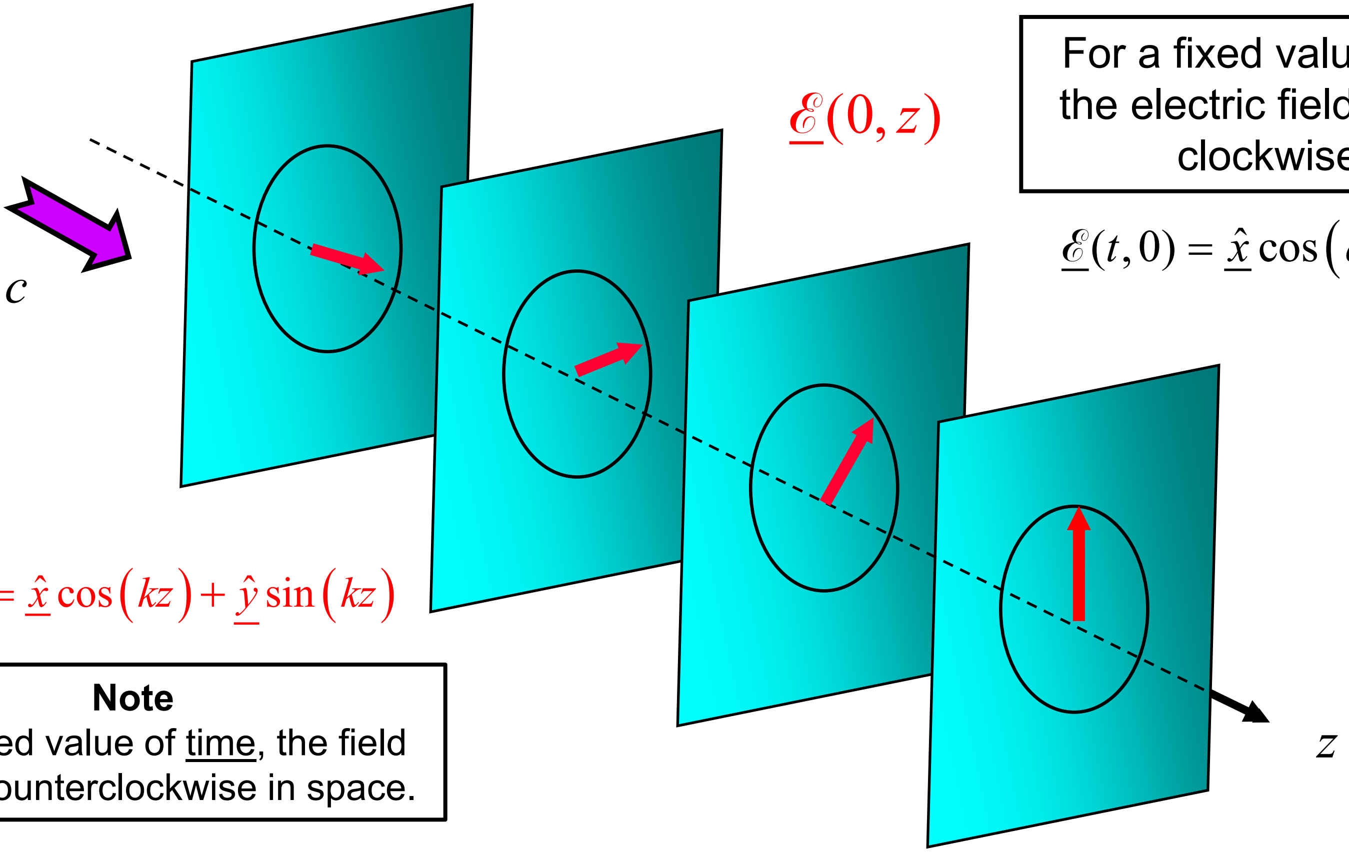
Practical application:

A *circular-polarized plane wave* (discussed later)

$$\underline{\mathcal{E}}(t,z)=\underline{\hat{x}}\cos\left(\omega t-kz\right)-\underline{\hat{y}}\sin\left(\omega t-kz\right)$$

For a fixed value of position z, the electric field vector rotates clockwise in time.

$$\underline{\mathcal{E}}(t,0)=\underline{\hat{x}}\cos\left(\omega t\right)-\underline{\hat{y}}\sin\left(\omega t\right)$$



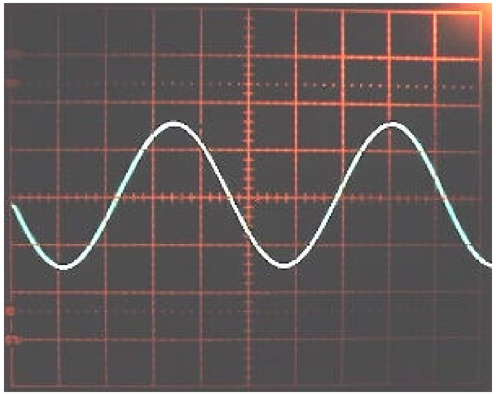
$$\underline{\mathcal{E}}(0,z)=\underline{\hat{x}}\cos\left(kz\right)+\underline{\hat{y}}\sin\left(kz\right)$$

Note

For a fixed value of time, the field rotates counterclockwise in space.

Time Average of Time-Harmonic Quantities

$$v(t) = A\cos(\omega t + \phi)$$



$$\langle v(t) \rangle \equiv \frac{1}{T}\int_0^T A\cos(\omega t + \phi)\, dt = 0$$

$$\left(T = \frac{1}{f} = \frac{2\pi}{\omega} \right)$$

Note : $\cos^2 x = \left[1 + \cos\left(2x\right)\right]/2$

$$\left\langle v^2(t) \right\rangle = \frac{1}{T}\int_0^T A^2 \cos^2(\omega t + \phi)\, dt$$

$$\left\langle v^2(t) \right\rangle = \frac{A^2}{T}\int_0^T \left\{ \frac{1 + \cos[2(\omega t + \phi)]}{2} \right\} dt$$

$\uparrow$

Sinusoidal (time ave $= 0$)

$$\left\langle v^2(t) \right\rangle = \frac{A^2}{T}\left(\frac{T}{2}\right) = \frac{A^2}{2}$$

Hence

$$\left\langle v^2(t) \right\rangle = \frac{A^2}{2}$$

(Note: The average value of $\cos^2$ is $1/2$.)

Time Average of Time-Harmonic Quantities (cont.)

Next, consider the time average of a product of sinusoids:

$$\left\langle v(t)i\left(t\right)\right\rangle = \frac{1}{T}\int_0^T \left[A\cos(\omega t+\alpha)\right]\left[B\cos(\omega t+\beta\right]dt$$

Note : $\cos x \cos y = \left[\cos\left(x-y\right)+\cos\left(x+y\right)\right]/2$

$$\left\langle v(t)i(t)\right\rangle = \frac{1}{T}AB\int_0^T \cos(\omega t+\alpha)\cos(\omega t+\beta)dt$$

$$= \frac{AB}{T}\int_0^T\left[\frac{\cos\left(\alpha-\beta\right)+\cos\left(2\omega t+\alpha+\beta\right)}{2}\right]dt$$

$$= \frac{AB}{T}\left[T\,\frac{\cos\left(\alpha-\beta\right)}{2}\right]$$

$$= AB\left[\frac{\cos\left(\alpha-\beta\right)}{2}\right]$$

Sinusoidal (time ave $=0$)

Time Average of Time-Harmonic Quantities (cont.)

Next, consider

$$VI^* = \left(A\,e^{j\alpha}\right)\left(B\,e^{j\beta}\right)^*$$

$$= \left(A\,e^{j\alpha}\right)\left(Be^{-j\beta}\right)$$

$$= AB\,e^{j(\alpha-\beta)}$$

Hence,

$$\mathrm{Re}\left(VI^*\right) = AB\cos\left(\alpha-\beta\right)$$

Recall that

$$\left\langle v(t)i(t)\right\rangle = AB\left[\frac{\cos\left(\alpha-\beta\right)}{2}\right]$$ (from previous slide)

Hence

$$\left\langle v(t)i(t)\right\rangle = \frac{1}{2}\mathrm{Re}\left(VI^*\right)$$

Question:
Can we put the conjugate on the V instead of the I?

Time Average of Time-Harmonic Quantities (cont.)

The results directly extend to vectors that vary sinusoidally in time.

Consider: $\underline{\mathscr{D}}(t)\cdot\underline{\mathscr{E}}(t)=\left(\mathscr{D}_x\mathscr{E}_x+\mathscr{D}_y\mathscr{E}_y+\mathscr{D}_z\mathscr{E}_z\right)$

$\mathscr{D}_{x,y,z}(t)=\mathrm{Re}\left[D_{x,y,z}e^{j\omega t}\right]$ etc.

$$\begin{aligned}\left\langle\underline{\mathscr{D}}\cdot\underline{\mathscr{E}}\right\rangle&=\left\langle\mathscr{D}_x\mathscr{E}_x\right\rangle+\left\langle\mathscr{D}_y\mathscr{E}_y\right\rangle+\left\langle\mathscr{D}_z\mathscr{E}_z\right\rangle\\&=\frac{1}{2}\mathrm{Re}\left(D_xE_x^*\right)+\frac{1}{2}\mathrm{Re}\left(D_yE_y^*\right)+\frac{1}{2}\mathrm{Re}\left(D_zE_z^*\right)\\&=\frac{1}{2}\mathrm{Re}\left(D_xE_x^*+D_yE_y^*+D_zE_z^*\right)\end{aligned}$$

Hence $\left\langle\underline{\mathscr{D}}(t)\cdot\underline{\mathscr{E}}(t)\right\rangle=\frac{1}{2}\mathrm{Re}\left(\underline{D}\cdot\underline{E}^*\right)$

Time Average of Time-Harmonic Quantities (cont.)

The result holds for both dot product and cross products.

$$\left\langle \underline{\mathscr{D}}(t)\cdot\underline{\mathscr{E}}(t)\right\rangle = \frac{1}{2}\,\mathrm{Re}\left(\underline{D}\cdot\underline{E}^{*}\right)$$

$$\left\langle \underline{\mathscr{E}}(t)\times\underline{\mathscr{H}}(t)\right\rangle = \frac{1}{2}\,\mathrm{Re}\left(\underline{E}\times\underline{H}^{*}\right)$$

where

$$\underline{\mathscr{D}}(x,y,z,t) = \mathrm{Re}\left[\underline{D}(x,y,z)\,e^{j\omega t}\right]$$ etc.

Time Average of Time-Harmonic Quantities (cont.)

To illustrate, consider the time-average stored electric energy density [J/m^3] for a sinusoidal electric field.

$$U_E(t) = \frac{1}{2}\underline{\mathscr{D}}(t) \cdot \underline{\mathscr{E}}(t)$$

(from ECE 3318)

$$\langle U_E(t) \rangle = \frac{1}{2}\langle \underline{\mathscr{D}}(t) \cdot \underline{\mathscr{E}}(t) \rangle$$

$$= \frac{1}{2}\left[\frac{1}{2}\mathrm{Re}\left(\underline{D} \cdot \underline{E}^*\right)\right]$$

$$\boxed{\langle U_E(t) \rangle = \frac{1}{4}\mathrm{Re}\left(\underline{D} \cdot \underline{E}^*\right)}$$

“Stored electric energy density”

Time Average of Time-Harmonic Quantities (cont.)

Similarly,

$$U_H(t) = \frac{1}{2}\underline{\mathscr{B}}(t)\cdot\underline{\mathscr{H}}(t)$$

"Stored magnetic energy density"

$$\left\langle U_H(t)\right\rangle = \frac{1}{4}\mathrm{Re}\left(\underline{B}\cdot\underline{H}^*\right)$$

$$\underline{\mathscr{S}}(t) \equiv \underline{\mathscr{E}}(t)\times\underline{\mathscr{H}}(t)$$

"Poynting vector"

$$\left\langle \underline{\mathscr{S}}(t)\right\rangle = \frac{1}{2}\mathrm{Re}\left(\underline{E}\times\underline{H}^*\right)$$