ECE 3317
Applied Electromagnetic Waves

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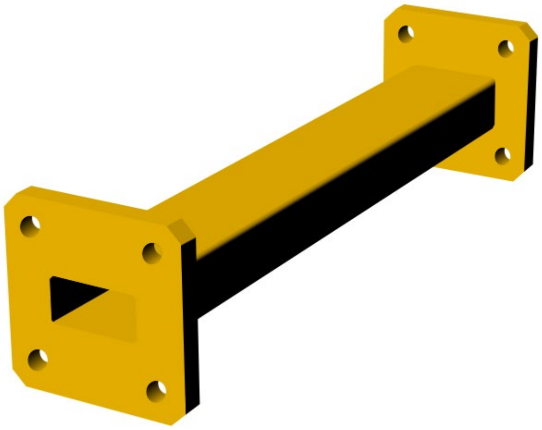
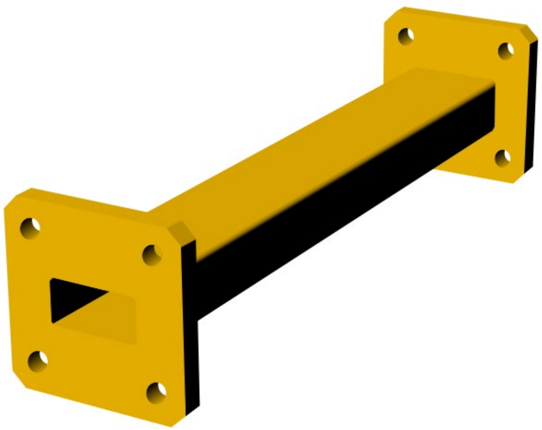
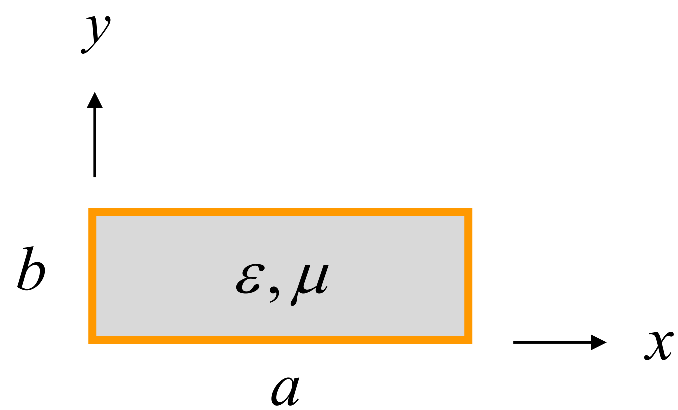
Fall 2023

Notes 20
Rectangular Waveguides



Rectangular Waveguide





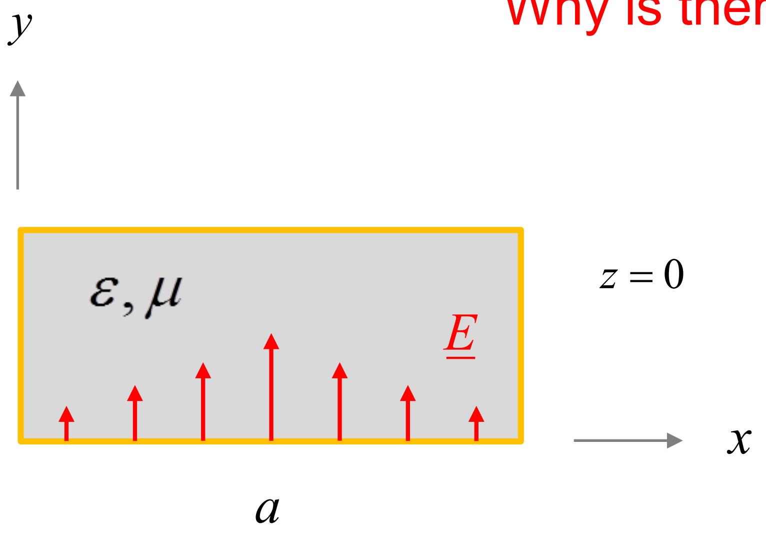
Cross section

- We assume that the boundary is a perfect electric conductor (PEC).

No TEM_z mode can exist!

Rectangular Waveguide (cont.)

Why is there no TEM$_z$ mode?



TEM$_z$ mode: $k_z = k = \omega\sqrt{\mu\varepsilon}$

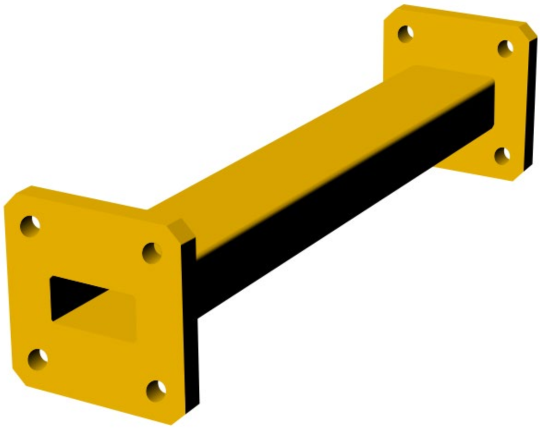
Rectangular waveguide mode (m,n): $k_z^{(m,n)} = \sqrt{k^2 - \left(\dfrac{m\pi}{a}\right)^2 - \left(\dfrac{n\pi}{b}\right)^2}$

$(m,n) \neq (0,0)$

$\Rightarrow \quad k_z^{(m,n)} \neq k$

Rectangular Waveguide (cont.)

Rectangular Waveguide

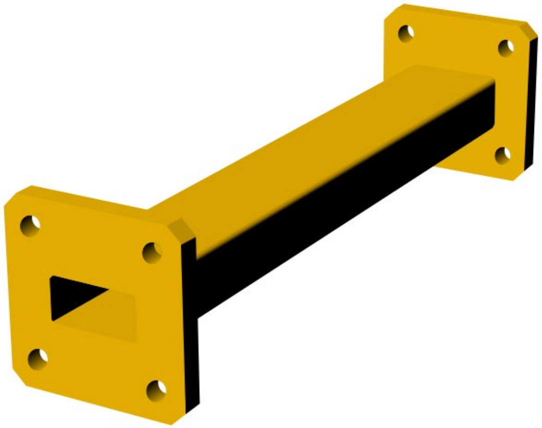


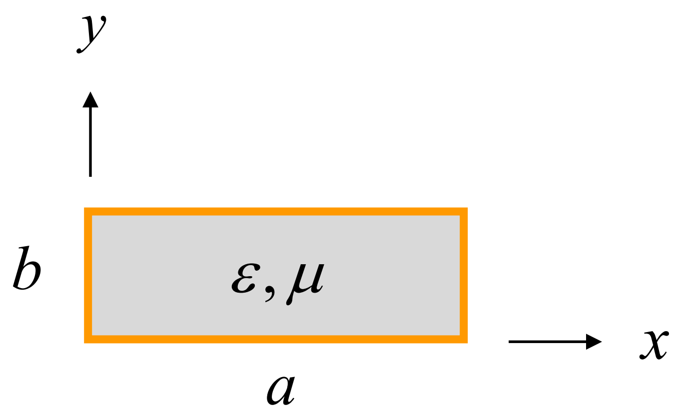
❖ Two types of modes can exist independently:

TM_z: E_z only
TE_z: H_z only

Rectangular Waveguide (cont.)

Rectangular Waveguide





Cross section

❖ We analyze the problem to solve for E_z or H_z (all other fields come from these).

TM_z: E_z only
TE_z: H_z only

TM_z Modes

$$H_z = 0, \quad E_z \neq 0$$

$$\nabla^2 E_z + k^2 E_z = 0$$ (Helmholtz equation)

$$E_z = 0$$ on boundary (PEC walls)

Guided-wave assumption: $E_z(x,y,z) = E_{z0}(x,y)\, e^{-jk_z z}$

$$\left(\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} \right) + k^2 E_z = 0$$

$$\Rightarrow \left(\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} - k_z^2 E_z \right) + k^2 E_z = 0$$

TM_z Modes (cont.)

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \left(k^2 - k_z^2\right) E_z = 0$$

Define: $k_c^2 \equiv k^2 - k_z^2$

We then have: $$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + k_c^2 E_z = 0$$

Note that k_c is an unknown at this point.

Dividing by the $\exp(-j\, k_z\, z)$ term, we have: $$\frac{\partial^2 E_{z0}}{\partial x^2} + \frac{\partial^2 E_{z0}}{\partial y^2} + k_c^2 E_{z0} = 0$$

We solve the above equation by using the method of <u>separation of variables</u>.

Please see Appendix A for the solution.

TM_z Modes (cont.)

Solution from separation of variables method: TM_{mn} mode

$$E_z(x,y,z) = A_{mn}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{-jk_z^{(m,n)}z}$$

$$k_z^{(m,n)} = \sqrt{k^2 - k_c^2} = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

$$k_c^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$$

$$k = \omega\sqrt{\mu\varepsilon} = k_0\sqrt{\mu_r\varepsilon_r}$$

$$k_0 = \omega\sqrt{\mu_0\varepsilon_0} = \frac{\omega}{c} = \frac{2\pi}{\lambda_0}$$

$$m = 1, 2, \ldots$$

$$n = 1, 2, \ldots$$

Note: If either m or n is zero, the entire field is zero.

TM$_z$ Modes (cont.)

Cutoff Frequency for Lossless Waveguide

We start with $$k_z^{(m,n)} = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

Note:
Cutoff frequency only has a clear meaning in the lossless case (k is real).

Set $k_z^{(m,n)} = 0$ This defines the cutoff frequency.

$$\Rightarrow \quad k\Big|_{f=f_c} = 2\pi f_c^{\mathrm{TM}_{m,n}} \sqrt{\mu\varepsilon} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

$$\Rightarrow \quad f_c^{\mathrm{TM}_{m,n}} = \frac{c_d}{2\pi}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

$c_d = \frac{c}{\sqrt{\varepsilon_r}}$ (nonmagnetic material)

TM$_z$ Modes (cont.)

Summary of TM$_z$ Solution: TM$_{mn}$ mode

$$E_z(x,y,z) = A_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-jk_z^{(m,n)} z}$$

$$k_z^{(m,n)} = \sqrt{k^2 - k_c^2} = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

$$m = 1, 2, \ldots$$
$$n = 1, 2, \ldots$$

Note: If <u>either</u> m or n is zero, the entire field is zero.

$$f_c^{\mathrm{TM}_{m,n}} = \frac{c_d}{2\pi} \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

(lossless waveguide)

TE_z Modes

$$E_z = 0, \quad H_z \neq 0$$

We now start with

$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + \left(k^2 - k_z^2\right) H_z = 0$$

Guided-wave assumption: $H_z\left(x, y, z\right) = H_{z0}\left(x, y\right) e^{-jk_z z}$

Define: $k_c^2 \equiv k^2 - k_z^2$

$$\Rightarrow \quad \frac{\partial^2 H_{z0}}{\partial x^2} + \frac{\partial^2 H_{z0}}{\partial y^2} + k_c^2 H_{z0} = 0$$

Please see Appendix B for the solution.

TE_z Modes (cont.)

Summary of TE_z Solution: TE_{mn} mode

$$H_z(x,y,z) = A_{mn}\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{-jk_z^{(m,n)}z}$$

$$k_z^{(m,n)} = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

$$m = 0,1,2\ldots$$
$$n = 0,1,2\ldots$$

$$(m,n) \neq (0,0)$$

$$f_c^{\mathrm{TE}_{m,n}} = \frac{c_d}{2\pi}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

(lossless waveguide)

Note: Same formula for cutoff frequency as the TM_z case!

Summary for Both Modes

TM$_{mn}$
$$E_z(x,y,z) = A_{mn}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{-jk_z^{(m,n)}z}$$

TE$_{mn}$
$$H_z(x,y,z) = A_{mn}\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{-jk_z^{(m,n)}z}$$

$$k_z^{(m,n)} = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

same formula for both modes

$$f_c^{(m,n)} = \frac{c_d}{2\pi}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

$$c_d = \frac{c}{\sqrt{\varepsilon_r}}$$

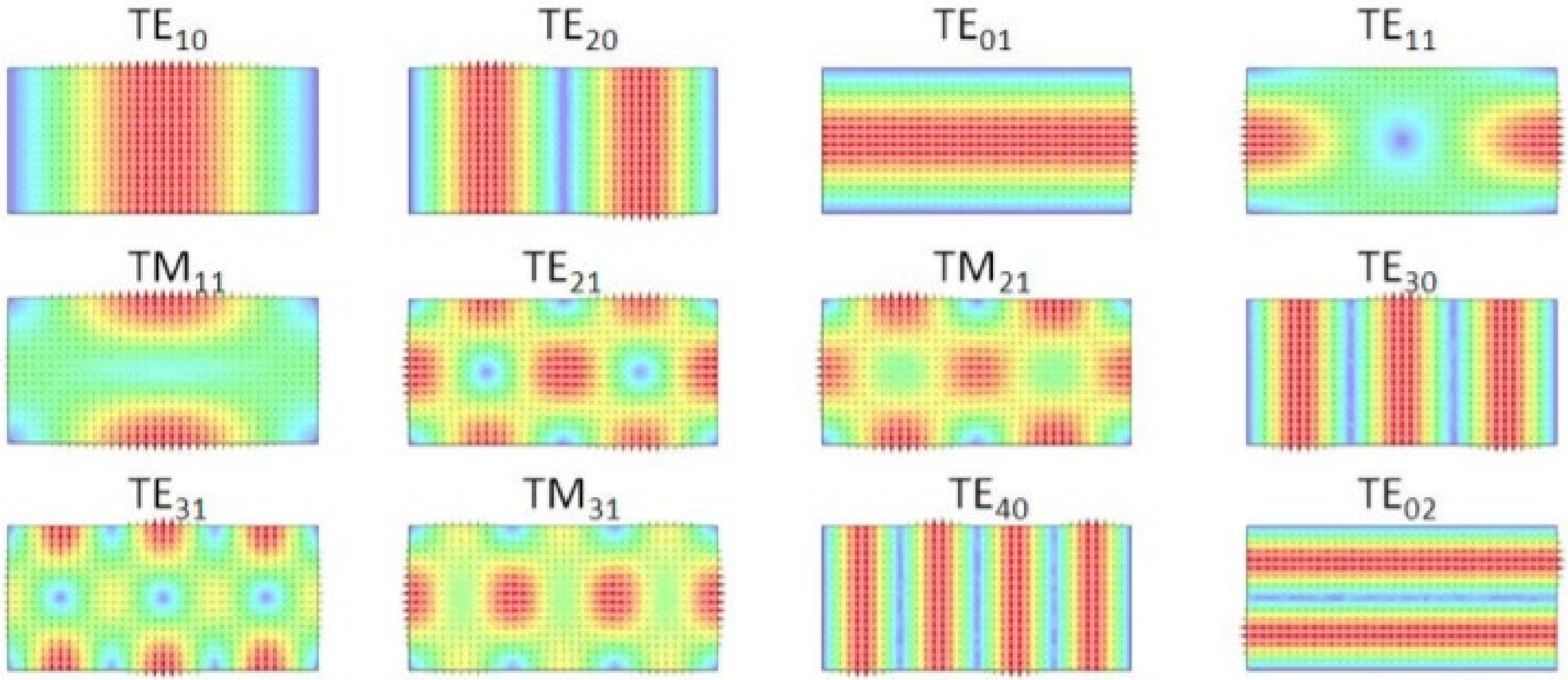
same formula for both modes

(lossless waveguide)

TM$_z$
$$m = 1,2,\ldots$$
$$n = 1,2,\ldots$$

TE$_z$
$$m = 0,1,2,\ldots$$
$$n = 0,1,2,\ldots$$
$$(m,n) \neq (0,0)$$

Field Plots



Color denotes magnitude, arrows show direction of electric field.

Wavenumber

TM$_z$ or TE$_z$ mode: $k_z = \sqrt{k^2 - k_c^2}$ with $k_c = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$

Note: The (m,n) notation is suppressed here on k_z.

Lossless waveguide:

Above cutoff: $k_z = \beta$

$$\beta = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

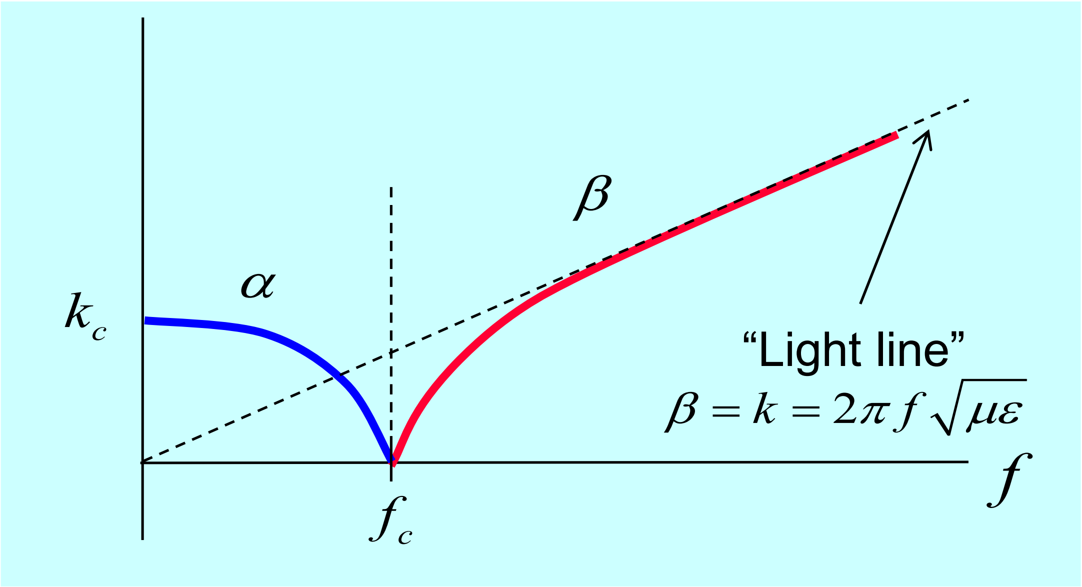
Below cutoff: $k_z = -j\alpha$

$$\alpha = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}$$

(Recall the general formula for k_z: $k_z = \beta - j\alpha$)

Wavenumber Plot

$$k_c = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$



$$\beta = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}, \qquad f > f_c$$

$$\alpha = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}, \qquad f < f_c$$

$$f_c = f_c^{(m,n)} = \frac{c_d}{2\pi}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

$$c_d = \frac{c}{\sqrt{\varepsilon_r}}$$

Guided Wavelength

Recall: The guided wavelength λ_g is the distance z that it takes for the wave to repeat itself.

$$\lambda_g = \frac{2\pi}{\beta}$$

(This assumes that we are above the cutoff frequency – otherwise guided wavelength makes no sense.)

After some algebra (see next slide):

$$\lambda_g = \frac{\lambda_d}{\sqrt{1-\left(f_c / f\right)^2}} \qquad \left(\lambda_d = \frac{\lambda_0}{\sqrt{\varepsilon_r}}\right)$$

(lossless waveguide)

$$\left(\text{Note}: \ \lambda_g > \lambda_d\right)$$

Guided Wavelength (cont.)

Derivation of wavelength formula (lossless waveguide):

$$\lambda_g = \frac{2\pi}{\beta} = \frac{2\pi}{\sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}} = \frac{2\pi}{\sqrt{k^2 - k_c^2}}$$

$$\Rightarrow \quad \lambda_g = \frac{2\pi}{\sqrt{k^2 - k_c^2}} = \frac{2\pi}{k\sqrt{1 - (k_c / k)^2}} = \frac{2\pi}{\frac{2\pi}{\lambda_d}\sqrt{1 - (k_c / k)^2}} = \frac{\lambda_d}{\sqrt{1 - (k_c / k)^2}}$$

$$\left.\begin{array}{l} k = \omega\sqrt{\mu\varepsilon} = 2\pi f\sqrt{\mu\varepsilon} \\ k_c = \omega_c\sqrt{\mu\varepsilon} = 2\pi f_c\sqrt{\mu\varepsilon} \end{array}\right\} \quad k_c / k = f_c / f$$

$$\Rightarrow \quad \lambda_g = \frac{\lambda_d}{\sqrt{1 - (f_c / f)^2}}$$

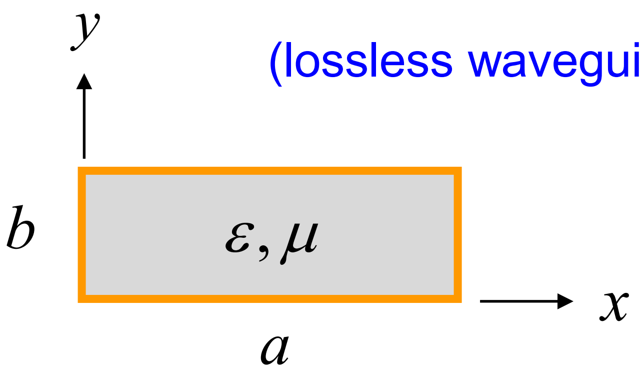
Dominant Mode

The "dominant" mode is the one with the <u>lowest cutoff frequency</u>.

Assume $b < a$

$$f_c = \frac{c_d}{2\pi}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

$$c_d = \frac{c}{\sqrt{\varepsilon_r}}$$



(lossless waveguide)

Lowest TM_z mode: TM_{11}

Lowest TE_z mode: TE_{10}

The dominant mode is the TE_{10} mode.

TM_z

$m = 1, 2, \ldots$

$n = 1, 2, \ldots$

TE_z

$m = 0, 1, 2, \ldots$

$n = 0, 1, 2, \ldots$

$(m, n) \neq (0, 0)$

Dominant Mode (cont.)

Summary (TE_{10} Mode)

$$H_z(x,y,z) = A_{10}\cos\left(\frac{\pi x}{a}\right)e^{-jk_z z}$$

$$k_z = \sqrt{k^2 - \left(\frac{\pi}{a}\right)^2} \qquad k = k_0\sqrt{\varepsilon_r}$$

$$f_c = \frac{c_d}{2a} \qquad c_d = \frac{c}{\sqrt{\varepsilon_r}}$$

$$\beta = \sqrt{k^2 - \left(\frac{\pi}{a}\right)^2}, \qquad f > f_c$$

$$\alpha = \sqrt{\left(\frac{\pi}{a}\right)^2 - k^2}, \qquad f < f_c$$

Dominant Mode (cont.)

Fields of the Dominant TE_{10} Mode

$$H_z(x,y,z) = A_{10}\cos\left(\frac{\pi x}{a}\right)e^{-jk_z z}$$

Find the other fields from these equations (Appendix A of Notes 19):

$$E_x = \left(\frac{-j\omega\mu}{k^2-k_z^2}\right)\frac{\partial H_z}{\partial y} - \left(\frac{jk_z}{k^2-k_z^2}\right)\frac{\partial E_z}{\partial x}$$

$$E_y = \left(\frac{j\omega\mu}{k^2-k_z^2}\right)\frac{\partial H_z}{\partial x} - \left(\frac{jk_z}{k^2-k_z^2}\right)\frac{\partial E_z}{\partial y}$$

$$H_x = \left(\frac{j\omega\varepsilon}{k^2-k_z^2}\right)\frac{\partial E_z}{\partial y} - \left(\frac{jk_z}{k^2-k_z^2}\right)\frac{\partial H_z}{\partial x}$$

$$H_y = \left(\frac{-j\omega\varepsilon}{k^2-k_z^2}\right)\frac{\partial E_z}{\partial x} - \left(\frac{jk_z}{k^2-k_z^2}\right)\frac{\partial H_z}{\partial y}$$

Dominant Mode (cont.)

Summary of fields for TE_{10} mode:

$$H_z(x,y,z) = A_{10}\cos\left(\frac{\pi x}{a}\right)e^{-jk_z z}$$

$$E_y(x,y,z) = E_{10}\sin\left(\frac{\pi x}{a}\right)e^{-jk_z z}$$

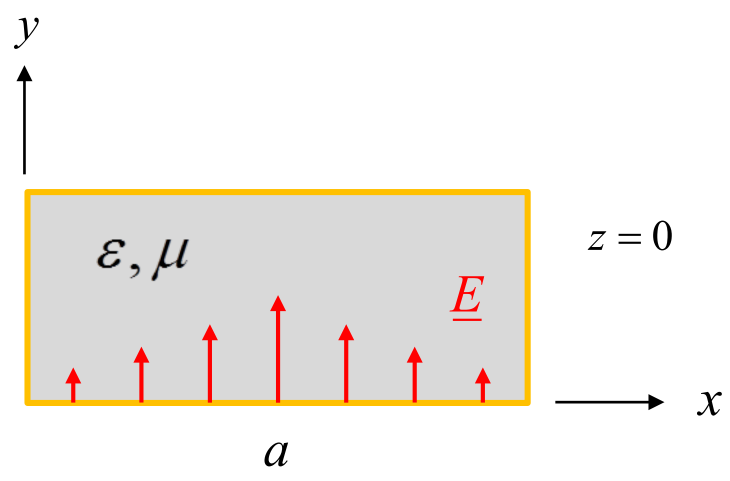
$$H_x(x,y,z) = -\left(\frac{k_z}{\omega\mu}\right)E_{10}\sin\left(\frac{\pi x}{a}\right)e^{-jk_z z}$$

where

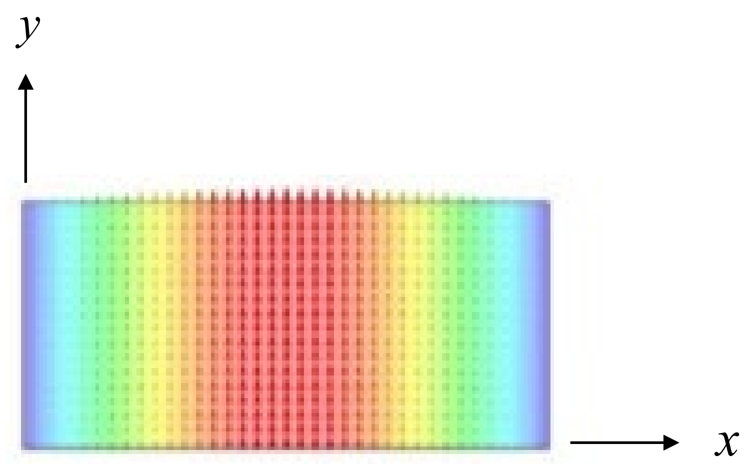
$$E_{10} = \left(\frac{j\omega\mu}{k^2 - k_z^2}\right)\left(-\frac{\pi}{a}\right)A_{10}$$

Dominant Mode (cont.)

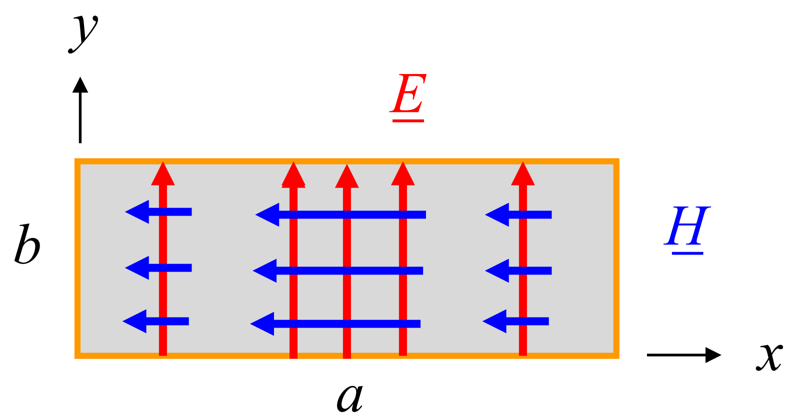
TE_{10} Mode



Length of arrows denotes magnitude of field



Color denotes magnitude of field

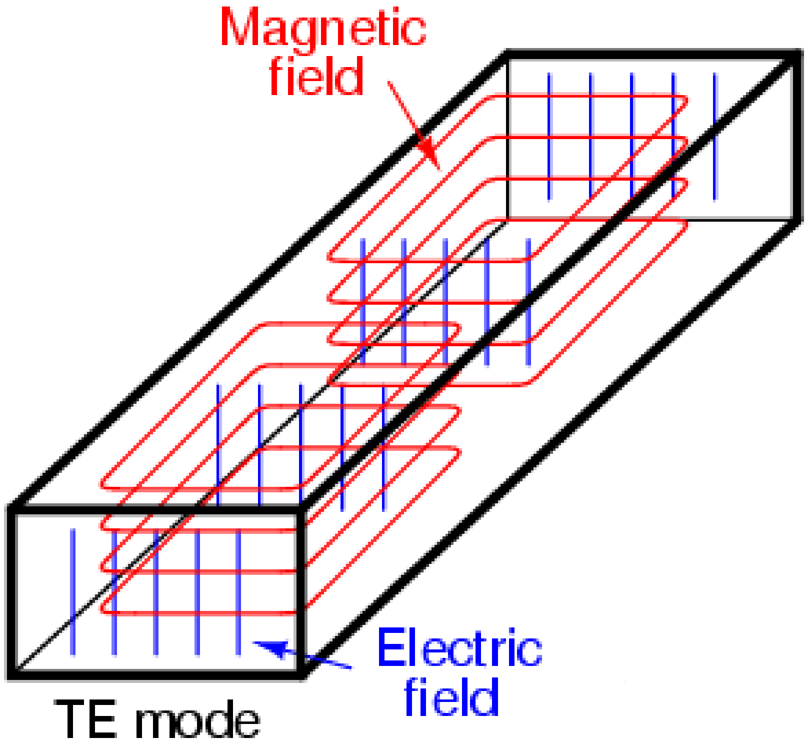


Spacing between arrows denotes magnitude of field

Dominant Mode (cont.)

TE_{10} Mode

3D View



Dominant Mode (cont.)

What is the mode with the next highest cutoff frequency?

$$f_c = \frac{c_d}{2\pi}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

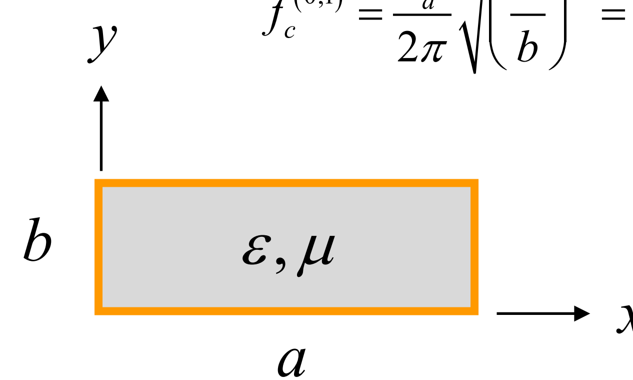
$$f_c^{(1,0)} = \frac{c_d}{2a}$$

$$f_c^{(2,0)} = \frac{c_d}{2\pi}\sqrt{\left(\frac{2\pi}{a}\right)^2} = \frac{c_d}{2}\left(\frac{1}{a/2}\right)$$

$$f_c^{(0,1)} = \frac{c_d}{2\pi}\sqrt{\left(\frac{\pi}{b}\right)^2} = \frac{c_d}{2}\left(\frac{1}{b}\right)$$

$$c_d = \frac{c}{\sqrt{\varepsilon_r}}$$

Assume $b < a/2$

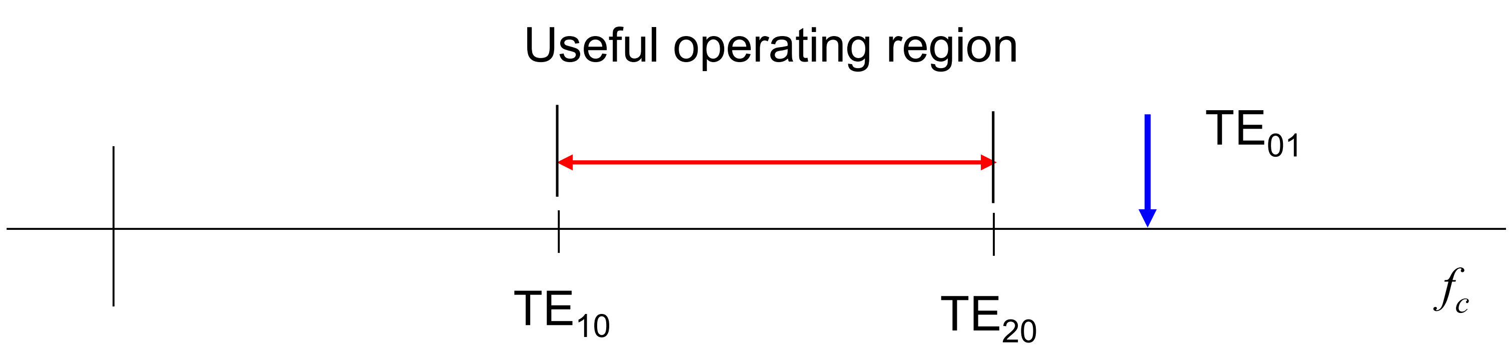


(lossless waveguide)

Then the next highest is the TE_{20} mode.

$$f_c^{(2,0)} = 2 f_c^{(1,0)}$$

A 2:1 operating band!



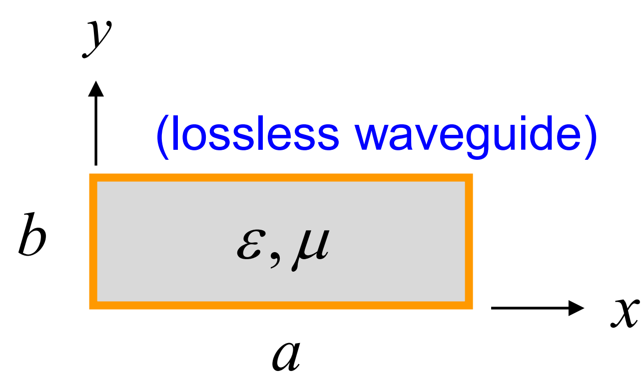
Dominant Mode (cont.)

What is the mode with the next highest cutoff frequency?

$$f_c^{(2,0)} = \frac{c_d}{2\pi}\sqrt{\left(\frac{2\pi}{a}\right)^2} = \frac{c_d}{2}\left(\frac{1}{a/2}\right) \qquad f_c^{(0,1)} = \frac{c_d}{2\pi}\sqrt{\left(\frac{\pi}{b}\right)^2} = \frac{c_d}{2}\left(\frac{1}{b}\right)$$

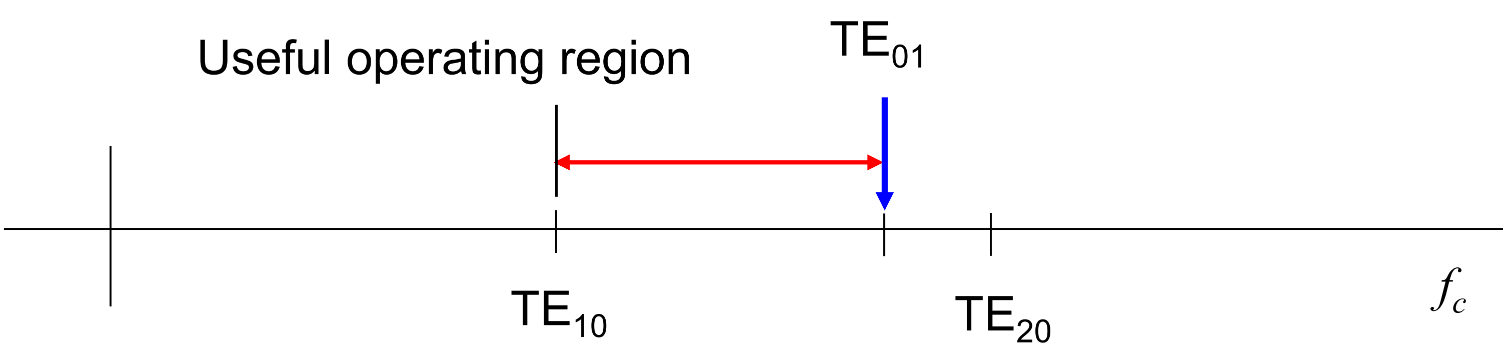
$$c_d = \frac{c}{\sqrt{\varepsilon_r}}$$

Assume $b > a/2$



Then the next highest is the TE_{01} mode.

The useable bandwidth is now lower than before.



Dominant Mode (cont.)

$$P_z = \mathrm{Re}\int_0^b\int_0^a \frac{1}{2}\left(\underline{E}\times\underline{H}^*\right)\cdot\underline{\hat{z}}\,dxdy$$
$$= \mathrm{Re}\int_0^b\int_0^a -\frac{1}{2}\left(E_y H_x^*\right)dxdy$$
$$= -\frac{1}{2}b\,\mathrm{Re}\int_0^a E_y H_x^*\,dx$$

Power flow in lossless waveguide ($f > f_c$):

$$E_y(x,y,z) = E_{10}\sin\left(\frac{\pi x}{a}\right)e^{-jk_z z}$$

$$P_z = \left(\frac{ab}{4\omega\mu}\right)\beta\left|E_{10}\right|^2 \quad [\mathrm{W}]$$ (watts flowing down the waveguide)

(The derivation is omitted, but please see the formula box above.)

Note: Above cutoff, there is only watts flowing (no vars). Below cutoff there is no watts flowing (only vars).

- Make b larger to get more power flow for a given value of a.
- Keep b smaller than $a/2$ to get maximum bandwidth.

➡ The optimum dimension for b is $a/2$
(gives maximum power flow without sacrificing bandwidth).

Dominant Mode (cont.)

Plane wave interpretation of TE_{10} mode

$$E_y(x,y,z) = E_{10}\sin\left(\frac{\pi x}{a}\right)e^{-jk_z z} = E_{10}\sin(k_x x)e^{-jk_z z} \qquad \left(k_x \equiv \frac{\pi}{a}\right)$$

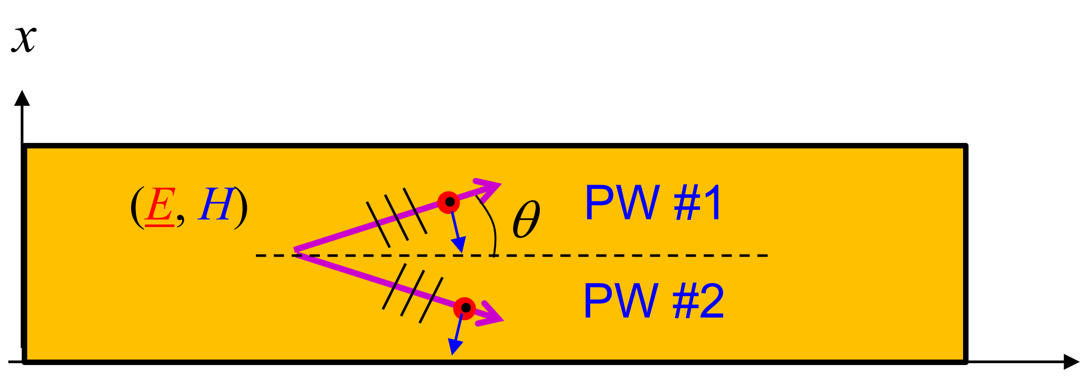
$$= E_{10}\left(\frac{e^{jk_x x} - e^{-jk_x x}}{2j}\right)e^{-jk_z z}$$

Note: $\sin z = \left(\dfrac{e^{jz} - e^{-jz}}{2j}\right)$

$$E_y(x,y,z) = E'_{10}\, e^{-jk_x x} e^{-jk_z z} + E''_{10}\, e^{jk_x x} e^{-jk_z z}$$

$$\left(\begin{aligned} E'_{10} &\equiv -E_{10}/(2j) \\ E''_{10} &\equiv +E_{10}/(2j) \end{aligned}\right)$$

PW #1 PW #2



$$\tan\theta = \frac{k_x}{k_z} = \frac{\pi / a}{\sqrt{k^2 - \left(\dfrac{\pi}{a}\right)^2}}$$

Dominant Mode (cont.)

Losses in Waveguide ($f > f_c$)

$$\alpha \approx \alpha_d + \alpha_c$$

Dielectric loss:

$$k_z = \sqrt{k_0^2 \varepsilon_r \left(1 - j \tan \delta_d\right) - \left(\frac{\pi}{a}\right)^2} = \beta - j\alpha_d$$

Recall: $\varepsilon_{rc} = \varepsilon_r \left(1 - j \tan \delta\right)$

$$\Rightarrow \quad \alpha_d = -\mathrm{Im} \sqrt{k_0^2 \varepsilon_r \left(1 - j \tan \delta_d\right) - \left(\frac{\pi}{a}\right)^2}$$

Note:
If we are below cutoff, attenuation is mainly due to evanescence, so we don't worry about conductor and dielectric loss then.

Conductor loss:

$$\alpha_c = \frac{R_s}{b\eta_0} \frac{\sqrt{\varepsilon_r}}{\sqrt{1 - \left(f_c / f\right)^2}} \left(1 + \frac{2b}{a}\left(\frac{f_c}{f}\right)^2\right) \quad [\text{np/m}]$$

(This is derived in ECE 5317.)

Example

Find the single-mode operating frequency region for air-filled X-band waveguide.

Standard X-band* waveguide:

$$a = 0.900 \text{ inches } (2.286 \text{ cm})$$
$$b = 0.400 \text{ inches } (1.016 \text{ cm})$$

Note: $b < a/2$

Use

$$f_c^{(1,0)} = \frac{c}{2a}$$

X-band waveguide

Hence, we have:

$$f_c^{(1,0)} = 6.56 \ \text{[GHz]}$$

$$f_c^{(2,0)} = 13.11 \ \text{[GHz]}$$

* X-band: from 8.0 to 12 GHz.

$$6.56 < f < 13.11 \ \text{[GHz]}$$

Example (cont.)

- Find the phase constant of the TE_{10} mode at 9.00 GHz.
- Find the attenuation in dB/m at 5.00 GHz

Recall: $f_c^{(1,0)} = 6.56\ [\text{GHz}]$

X-band waveguide

$$\beta = \sqrt{k^2 - \left(\frac{\pi}{a}\right)^2}, \qquad f > f_c$$

$$\alpha = \sqrt{\left(\frac{\pi}{a}\right)^2 - k^2}, \qquad f < f_c$$

$$k = k_0 = \omega\sqrt{\mu_0 \varepsilon_0} = 2\pi f / c = 2\pi / \lambda_0$$

At 9.0 GHz: $k = 188.62\ [\text{rad/m}]$

At 5.0 GHz: $k = 104.79\ [\text{rad/m}]$

$$k_c = \pi / a = 137.43\ [\text{rad/m}]$$

At 9.00 GHz: $\beta = 129.13\ [\text{rad/m}]$

At 5.00 GHz: $\alpha = 88.91\ [\text{nepers/m}]$

Example (cont.)

At 5.0 GHz:

$$\alpha = 88.91 \left[\text{nepers/m}\right]$$

X-band waveguide

Recall:

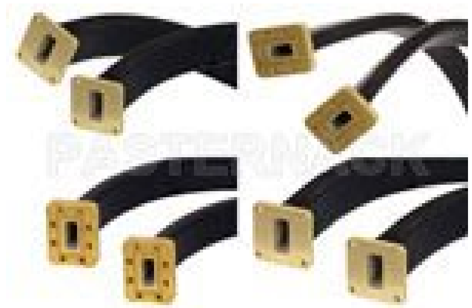
$$\text{dB/m} = 8.68589\,\alpha$$

$$\text{Attenuation} = 772 \text{ dB/m}$$

This is a very rapid attenuation!

Waveguide Components

Straight sections



Flexible waveguides

Waveguide bends

Waveguide adapters

Waveguide couplers

Waveguide terminations

https://www.pasternack.com

Waveguide Modes in Transmission Lines

- A transmission line normally operates in the TEM_z mode, where the two conductors have equal and opposite currents.

- At high frequencies, waveguide modes can also propagate on transmission lines.

- This is undesirable, and it limits the high-frequency range of operation for the transmission line.

Waveguide Modes in Coax

Dominant waveguide mode in coax (derivation omitted):

TE_{11} mode:

$$f_c^{\mathrm{TE}_{11}} \approx \frac{c}{a\sqrt{\varepsilon_r}}\left(\frac{1}{\pi}\right)\left(\frac{1}{1+b/a}\right)$$

Note:
In this notation, the "11" subscript refers to the angular and radial variation.

Example: RG 142 coax

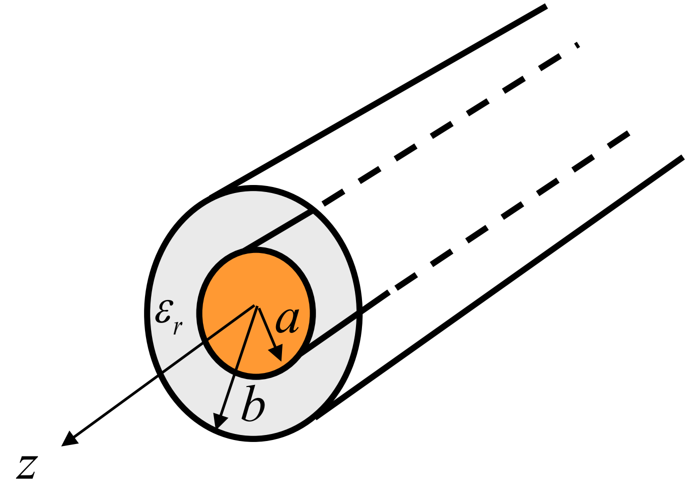
$$a = 0.035 \text{ inches} = 8.89\times10^{-4} \text{ [m]}$$

$$b = 0.116 \text{ inches} = 29.46\times10^{-4} \text{ [m]}$$

$$\varepsilon_r = 2.2$$

$$b/a = 3.31 \Rightarrow Z_0 = 48.4[\Omega]$$

$$f_c^{\mathrm{TE}_{11}} \approx 16.8 \text{ [GHz]}$$



$$Z_0 = \sqrt{\frac{L}{C}} = \frac{\eta_0}{2\pi\sqrt{\varepsilon_r}}\ln\left(\frac{b}{a}\right)$$

This coax cannot be used above 16.8 [GHz]

Appendix A: TM_z Modes

We want to solve: $$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \left(k^2 - k_z^2\right) E_z = 0$$

Define: $$k_c^2 \equiv k^2 - k_z^2$$

We then have: $$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + k_c^2 E_z = 0$$

Note that k_c is an unknown at this point.

Dividing by the $\exp(-jk_z z)$ term, we have: $$\frac{\partial^2 E_{z0}}{\partial x^2} + \frac{\partial^2 E_{z0}}{\partial y^2} + k_c^2 E_{z0} = 0$$

We solve the above equation by using the method of separation of variables.

We assume: $$E_{z0}(x, y) = X(x)\, Y(y)$$

Appendix A (cont.)

$$\frac{\partial^2 E_{z0}}{\partial x^2} + \frac{\partial^2 E_{z0}}{\partial y^2} + k_c^2 E_{z0} = 0$$

$$E_{z0}(x,y) = X(x)\,Y(y)$$

Hence $$X''Y + X\,Y'' = -k_c^2 XY$$

Divide by XY : $$\frac{X''}{X} + \frac{Y''}{Y} = -k_c^2$$

Hence $$\frac{X''}{X} = -k_c^2 - \frac{Y''}{Y}$$

This has the form $$F(x) = G(y)$$

Both sides of the equation must be a constant!

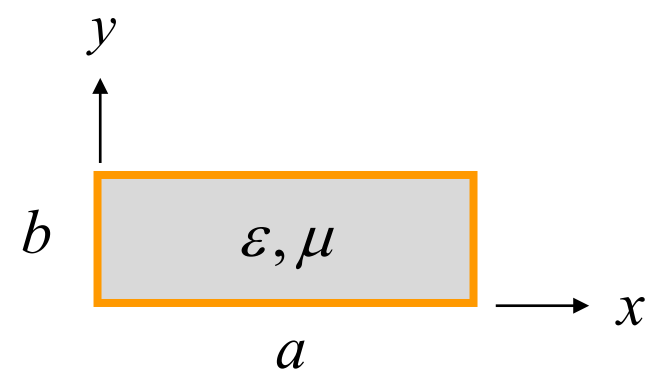
Appendix A (cont.)

$$\frac{X''}{X} = -k_c^2 - \frac{Y''}{Y} = \text{constant}$$

Denote $$\frac{X''}{X} = -k_x^2 = \text{constant}$$

General solution: $X(x) = A\sin(k_x x) + B\cos(k_x x)$

Boundary conditions: $$\begin{cases} X(0) = 0 & (1) \\ X(a) = 0 & (2) \end{cases}$$



(1) ⇨ $B = 0 \quad \Rightarrow \quad X(x) = A\sin(k_x x)$

(2) ⇨ $\sin(k_x a) = 0$

Appendix A (cont.)

From the last slide:

$$\sin(k_x a) = 0$$

This gives us the following result:

$$k_x a = m\pi, \quad m = 1, 2 \ldots$$

$$\Rightarrow \quad k_x = \frac{m\pi}{a}$$

Hence $\quad X(x) = A\sin\left(\frac{m\pi x}{a}\right)$

Now we turn our attention to the $Y(y)$ function.

Appendix A (cont.)

We have

$$\frac{X''}{X} = -k_c^2 - \frac{Y''}{Y} = -k_x^2$$

Hence $$\frac{Y''}{Y} = k_x^2 - k_c^2$$

Denote $$k_y^2 = k_c^2 - k_x^2$$

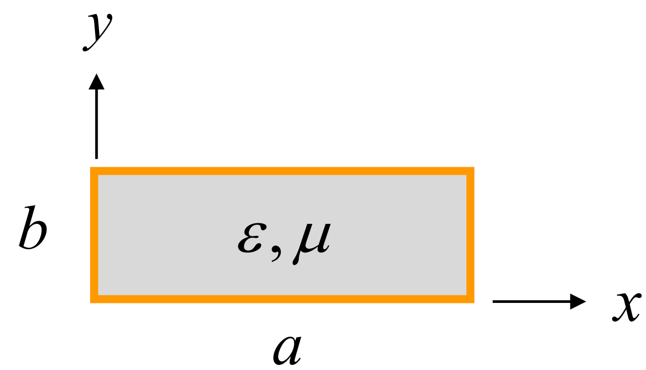
Then we have $$\frac{Y''}{Y} = -k_y^2$$

General solution: $$Y(y) = C\sin(k_y y) + D\cos(k_y y)$$

Appendix A (cont.)

$$Y(y) = C\sin(k_y y) + D\cos(k_y y)$$

Boundary conditions: $\begin{cases} Y(0) = 0 & (3) \\ Y(b) = 0 & (4) \end{cases}$



(3) ⇨ $D = 0 \quad \Rightarrow \quad Y(y) = C\sin(k_y y)$

(4) ⇨ $\sin(k_y b) = 0$

Equation (4) gives us the following result: $k_y b = n\pi, \;\; n = 1, 2\ldots$

⇨ $$k_y = \frac{n\pi}{b}$$

Appendix A (cont.)

The $Y(y)$ function is then $Y(y) = C\sin\left(\frac{n\pi y}{b}\right)$

Therefore, we have

$$E_{z0}(x,y) = X(x)Y(y) = AC\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

New notation:

$$E_{z0}(x,y) = A_{mn}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

The E_z field inside the waveguide thus has the following form:

$$E_z(x,y,z) = A_{mn}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{-jk_z^{(m,n)}z}$$

Appendix A (cont.)

Recall that $k_y^2 = k_c^2 - k_x^2$

Hence, $k_c^2 = k_x^2 + k_y^2$

Therefore, the solution for k_c is given by

$$k_c^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$$

Next, recall that $k_c^2 = k^2 - k_z^2$

Hence $k_z^2 = k^2 - k_c^2$

$$\Rightarrow \quad k_z = k_z^{(m,n)} = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

Appendix B: TE_z Modes

$$E_z = 0, \quad H_z \neq 0$$

We now start with

$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + \left(k^2 - k_z^2\right) H_z = 0$$

Using the separation of variables method again, we have

$$H_{z0}\left(x, y\right) = X\left(x\right) Y\left(y\right)$$

where

$$X(x) = A\sin(k_x x) + B\cos(k_x x)$$

$$Y(y) = C\sin(k_y y) + D\cos(k_y y)$$

and

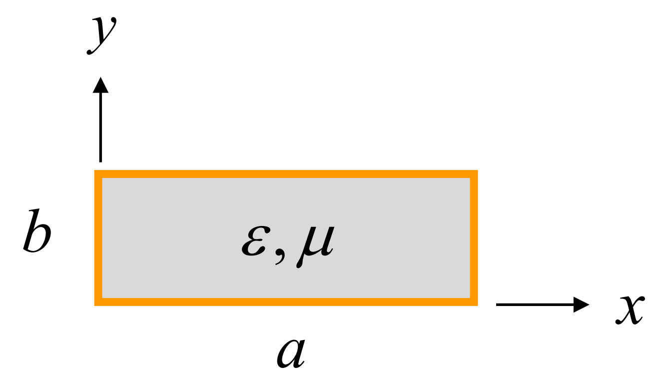
$$k_c^2 = k_x^2 + k_y^2 \qquad k_z^2 = k^2 - k_c^2$$

Appendix B (cont.)

Boundary conditions:

$$E_x(x,0)=0 \qquad E_y(0,y)=0$$

$$E_x(x,b)=0 \qquad E_y(a,y)=0$$



The result is

$$H_{z0}(x,y)=A_{mn}\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$

This can be shown by using the following equations:

$$E_x=\left(\frac{-j\omega\mu}{k^2-k_z^2}\right)\frac{\partial H_z}{\partial y}-\left(\frac{jk_z}{k^2-k_z^2}\right)\cancel{\frac{\partial E_z}{\partial x}} \quad \Rightarrow \quad \frac{\partial H_z}{\partial y}=0, \quad y=0,\, b$$

$$E_y=\left(\frac{j\omega\mu}{k^2-k_z^2}\right)\frac{\partial H_z}{\partial x}-\left(\frac{jk_z}{k^2-k_z^2}\right)\cancel{\frac{\partial E_z}{\partial y}} \quad \Rightarrow \quad \frac{\partial H_z}{\partial x}=0, \quad x=0,\, a$$

Appendix B (cont.)

The H_z field inside the waveguide thus has the following form:

$$H_z(x,y,z) = A_{mn}\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{-jk_z^{(m,n)}z}$$

$$k_z^{(m,n)} = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

Same formula for cutoff frequency as the TE_z case!

$$m = 0,1,2\ldots \qquad (m,n) \neq (0,0)$$
$$n = 0,1,2\ldots$$

Note: The (0,0) TE_z mode is not valid, since it violates the magnetic Gauss law:

$$\underline{H}(x,y,z) = \underline{\hat{z}}\,A_{00}\,e^{-jkz} \;\Rightarrow\; \nabla\cdot\underline{H}(x,y,z) \neq 0$$