ECE 3317
Applied Electromagnetic Waves

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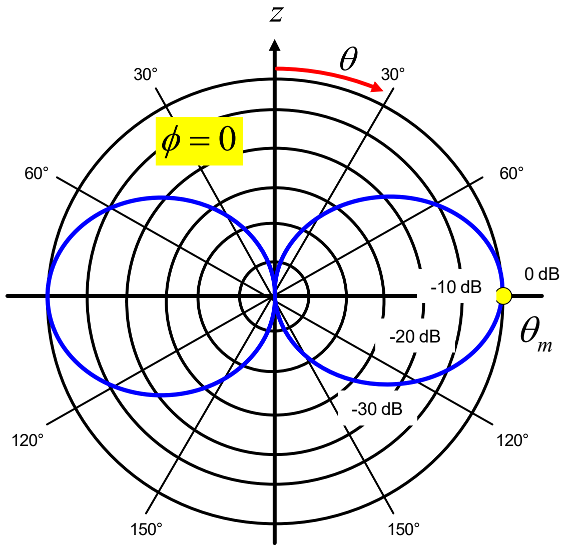
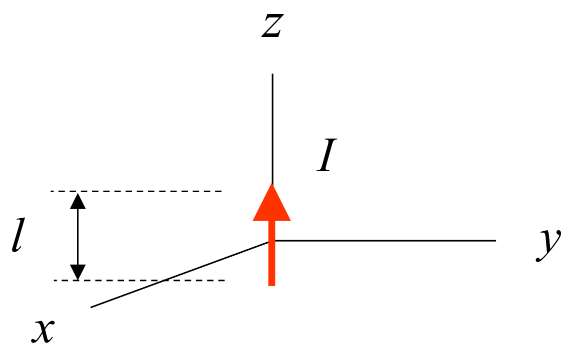
Fall 2023

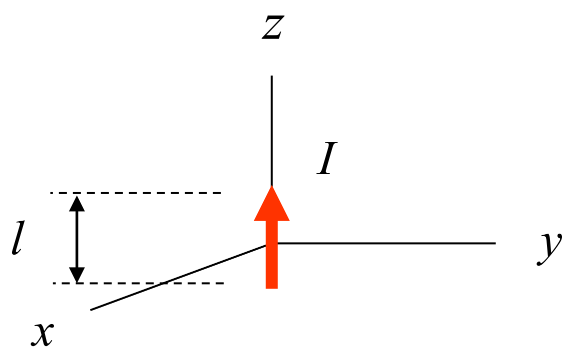
Notes 22
Antenna Patterns





Infinitesimal Dipole

The infinitesimal dipole current element is shown below.



The dipole moment (amplitude) is defined as Il.

The infinitesimal dipole is the foundation for many practical wire antennas.

From Maxwell's equations we can calculate the fields radiated by this source (e.g., see Chapter 14 of the Hayt and Buck textbook).

Infinitesimal Dipole (cont.)

The exact fields of the infinitesimal dipole in spherical coordinates are

$$E_r = \frac{Il}{2\pi}\eta_0 e^{-jk_0 r}\left(\frac{1}{r^2}\right)\left[1+\frac{1}{jk_0 r}\right]\cos\theta$$

$$E_\theta = \frac{Il}{4\pi}(j\omega\mu_0)e^{-jk_0 r}\left(\frac{1}{r}\right)\left[1+\frac{1}{jk_0 r}+\frac{1}{(jk_0 r)^2}\right]\sin\theta$$

$$H_\phi = \frac{Il}{4\pi}(jk_0)e^{-jk_0 r}\left(\frac{1}{r}\right)\left[1+\frac{1}{jk_0 r}\right]\sin\theta$$

Infinitesimal Dipole (cont.)

In the far field ($r \to \infty$) we have:

$$E_\theta(r,\theta,\phi) \sim \frac{Il}{4\pi}(j\omega\mu_0)\left(\frac{e^{-jk_0 r}}{r}\right)\sin\theta$$

$$H_\phi(r,\theta,\phi) \sim \frac{Il}{4\pi}(jk_0)\left(\frac{e^{-jk_0 r}}{r}\right)\sin\theta$$

Hence, we can identify

$$E_\theta^{\mathrm{FF}}(\theta,\phi) = \frac{Il}{4\pi}(j\omega\mu_0)\sin\theta$$

$$H_\phi^{\mathrm{FF}}(\theta,\phi) = \frac{Il}{4\pi}(jk_0)\sin\theta$$

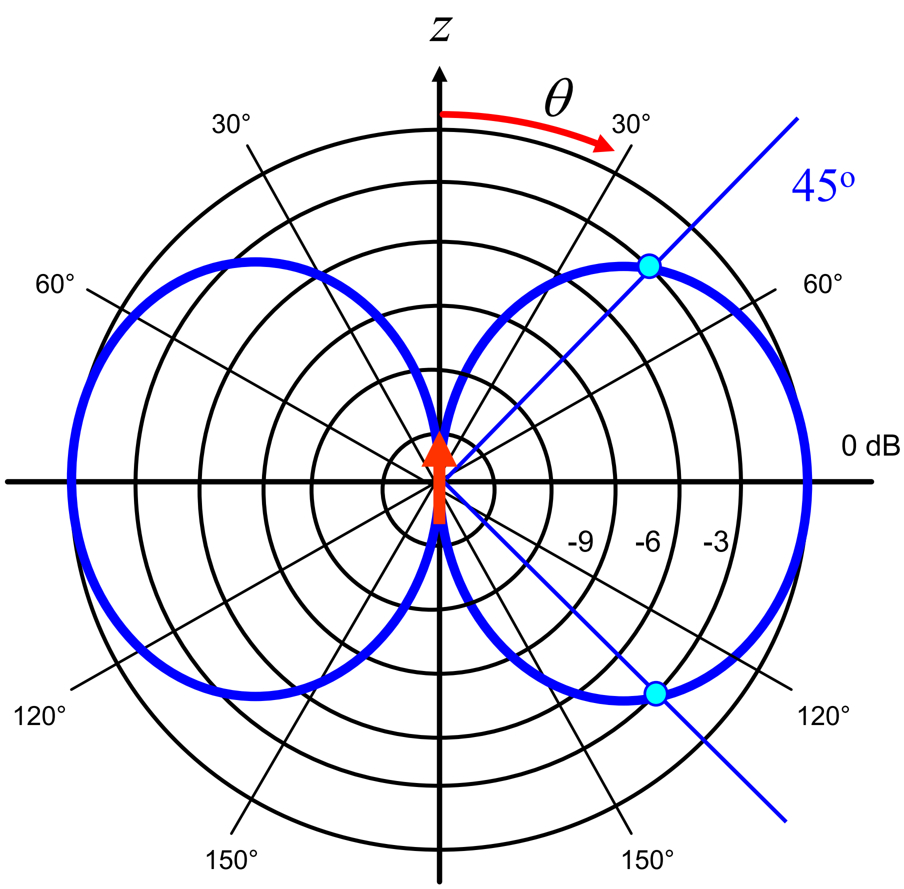
Recall:

$$\underline{E}(r,\theta,\phi) = \left(\frac{e^{-jk_0 r}}{r}\right)\underline{E}^{\mathrm{FF}}(\theta,\phi)$$

Infinitesimal Dipole (cont.)

The radiation pattern is shown below.

$$E_\theta^{\mathrm{FF}} = \frac{Il}{4\pi}(j\omega\mu_0)\sin\theta$$



$$\mathrm{dB}(\theta,\phi) = 20\log_{10}\left(\frac{\left|\underline{E}^{\mathrm{FF}}(\theta,\phi_0)\right|}{\left|\underline{E}^{\mathrm{FF}}(\theta_m,\phi_0)\right|}\right)$$

$$\mathrm{dB}(\theta,\phi) = 20\log_{10}(\sin\theta)$$

$$\sin(45^\circ) = 1/\sqrt{2}$$

$$20\log_{10}\left(1/\sqrt{2}\right) = -3 \ \ [\mathrm{dB}]$$

HPBW = 90°

Infinitesimal Dipole (cont.)

The directivity of the infinitesimal dipole is now calculated

$$D(\theta,\phi)=\frac{S_r(\theta,\phi)}{P_{rad}/\left(4\pi r^2\right)}=\frac{\left|\underline{E}^{\mathrm{FF}}(\theta,\phi)\right|^2/\left(2\eta_0\right)}{\dfrac{1}{4\pi r^2}\displaystyle\int_0^{2\pi}\int_0^{\pi}\left(\left|\underline{E}^{\mathrm{FF}}(\theta,\phi)\right|^2/\left(2\eta_0\right)\right)r^2\sin\theta\,d\theta d\phi}$$

$$=\frac{4\pi\left|\underline{E}^{\mathrm{FF}}(\theta,\phi)\right|^2}{\displaystyle\int_0^{2\pi}\int_0^{\pi}\left|\underline{E}^{\mathrm{FF}}(\theta,\phi)\right|^2\sin\theta\,d\theta d\phi}$$

Recall:

$$E_\theta^{\mathrm{FF}}=\frac{Il}{4\pi}\left(j\omega\mu_0\right)\sin\theta$$

Hence

$$D(\theta,\phi)=\frac{4\pi\sin^2\theta}{\displaystyle\int_0^{2\pi}\int_0^{\pi}\sin^2\theta\,\sin\theta\,d\theta d\phi}$$

Infinitesimal Dipole (cont.)

Evaluating the integrals, we have:

$$D(\theta,\phi) = \frac{4\pi \sin^2\theta}{\int_0^{2\pi}\int_0^{\pi} \sin^2\theta \, \sin\theta \, d\theta d\phi}$$

$$= \frac{4\pi \sin^2\theta}{(2\pi)\int_0^{\pi} \sin^2\theta \, \sin\theta \, d\theta}$$

$$= \frac{2\sin^2\theta}{\int_0^{\pi} \sin^3\theta \, d\theta}$$

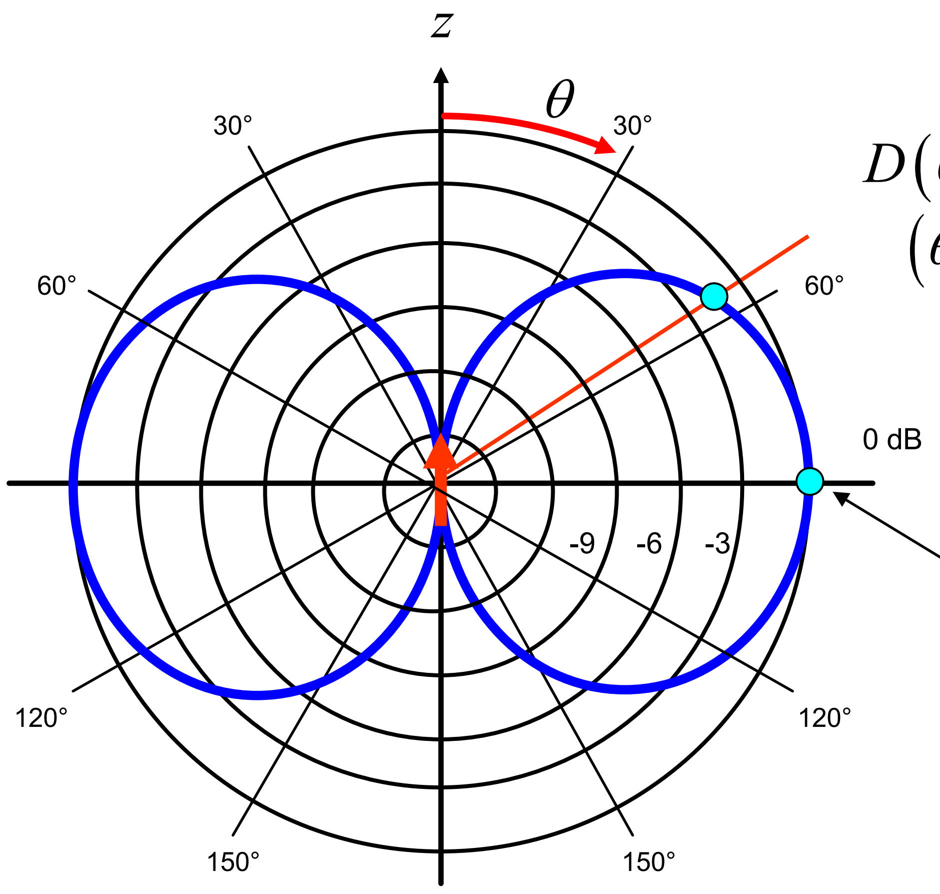
$$= \frac{2\sin^2\theta}{4/3}$$

Hence, we have

$$D(\theta,\phi) = \frac{3}{2}\sin^2\theta$$

Infinitesimal Dipole (cont.)

$$E_\theta^{\mathrm{FF}} = \frac{Il}{4\pi}(j\omega\mu_0)\sin\theta \qquad D(\theta,\phi) = \frac{3}{2}\sin^2\theta$$



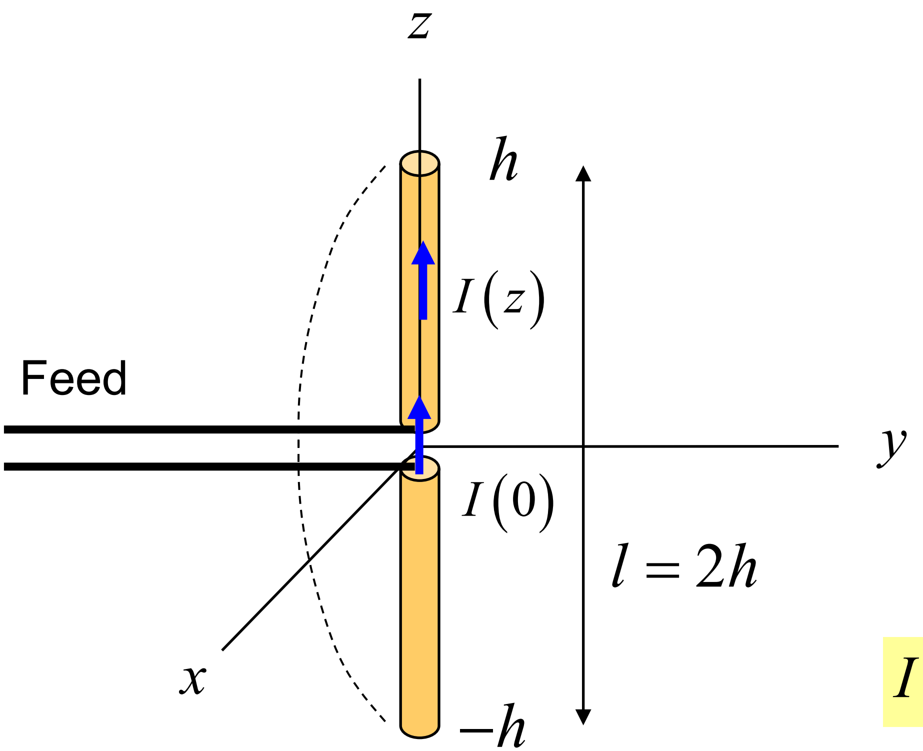
$$D(\theta,\phi) = 1.0$$
$$(\theta = 54.7^\circ)$$

$$D(\theta,\phi) = 1.5$$
$$(\theta = 90^\circ)$$

The far-field pattern is shown, with the directivity labeled at two points.

Wire Antenna

A center-fed wire antenna is shown below.



$$I(0) = I_0$$

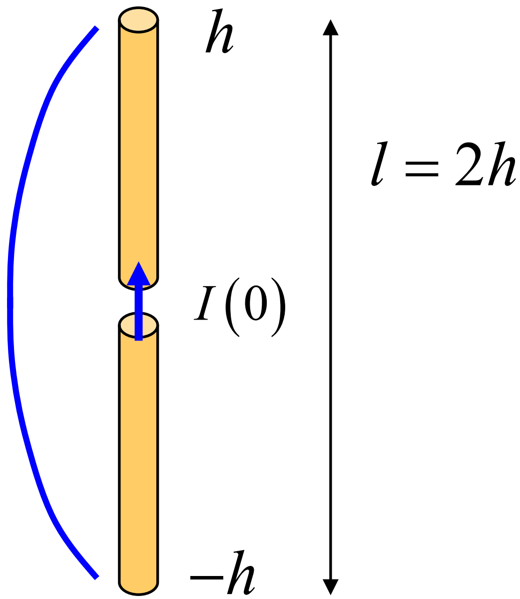
A good approximation to the current is:

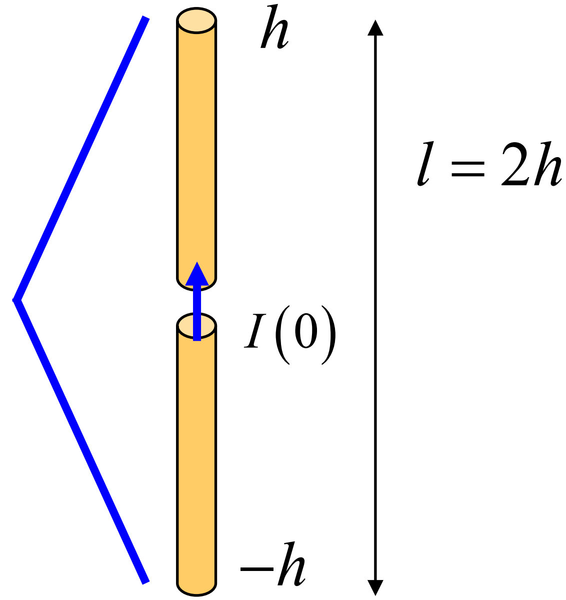
$$I(z) = \left(\frac{I_0}{\sin(k_0 h)} \right) \sin\left[k_0 \left(h - |z| \right) \right]$$

Wire Antenna (cont.)

A sketch of the current is shown below for two cases.

$$I(z)=\left(\frac{I_0}{\sin(k_0 h)}\right)\sin\left[k_0\left(h-|z|\right)\right]$$





Resonant dipole ($l=\lambda_0/2,\ k_0 h=\pi/2$)

$$I(z)=I_0\cos\left(k_0|z|\right)=I_0\cos\left(\frac{\pi}{2}\frac{z}{h}\right)$$

Short dipole ($l \ll \lambda_0$)

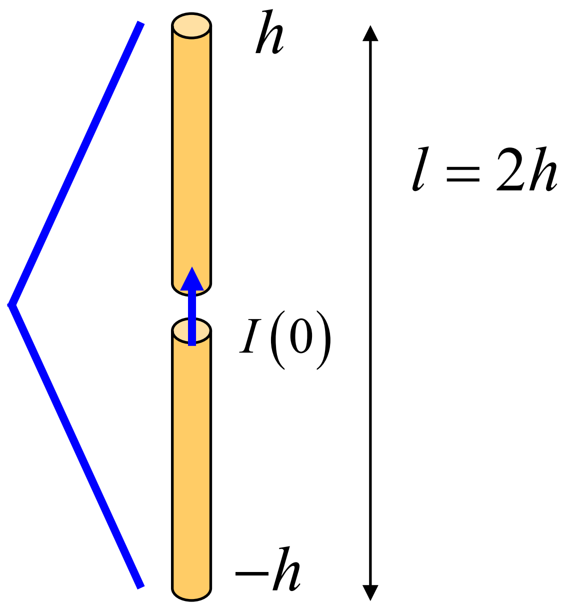
$$I(z)\approx I_0\left[1-\frac{|z|}{h}\right]$$

Use

$$\sin(x)\approx x$$

Wire Antenna (cont.)

Short Dipole



Short dipole ($l << \lambda_0 / 2$)

$$I(z) \approx I_0 \left[1 - \frac{|z|}{h} \right]$$

The average value of the current is $I_0 / 2$.

$$(Il)_{\text{eff}} = \frac{1}{2}(I_0 l)$$

Infinitesimal dipole:

$$E_\theta^{\text{FF}} = \frac{Il}{4\pi}(j\omega\mu_0)\sin\theta$$

$$H_\phi^{\text{FF}} = \frac{Il}{4\pi}(jk_0)\sin\theta$$

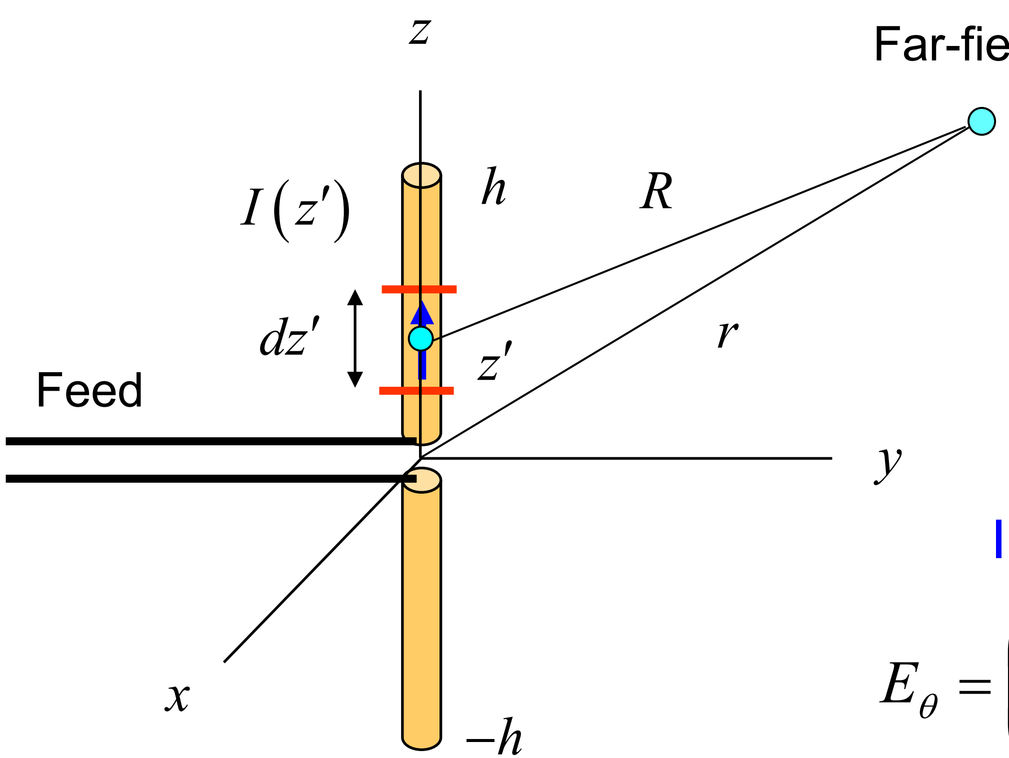
Short dipole:

$$E_\theta^{\text{FF}} = \frac{(Il)_{\text{eff}}}{4\pi}(j\omega\mu_0)\sin\theta$$

$$H_\phi^{\text{FF}} = \frac{(Il)_{\text{eff}}}{4\pi}(jk_0)\sin\theta$$

Wire Antenna (cont.)

For an arbitrary length dipole wire antenna, we need to consider the radiation by each differential piece of the current.



$$\underline{r} = (x, y, z)$$

$$R = \left[x^2 + y^2 + (z - z')^2 \right]^{\frac{1}{2}}$$

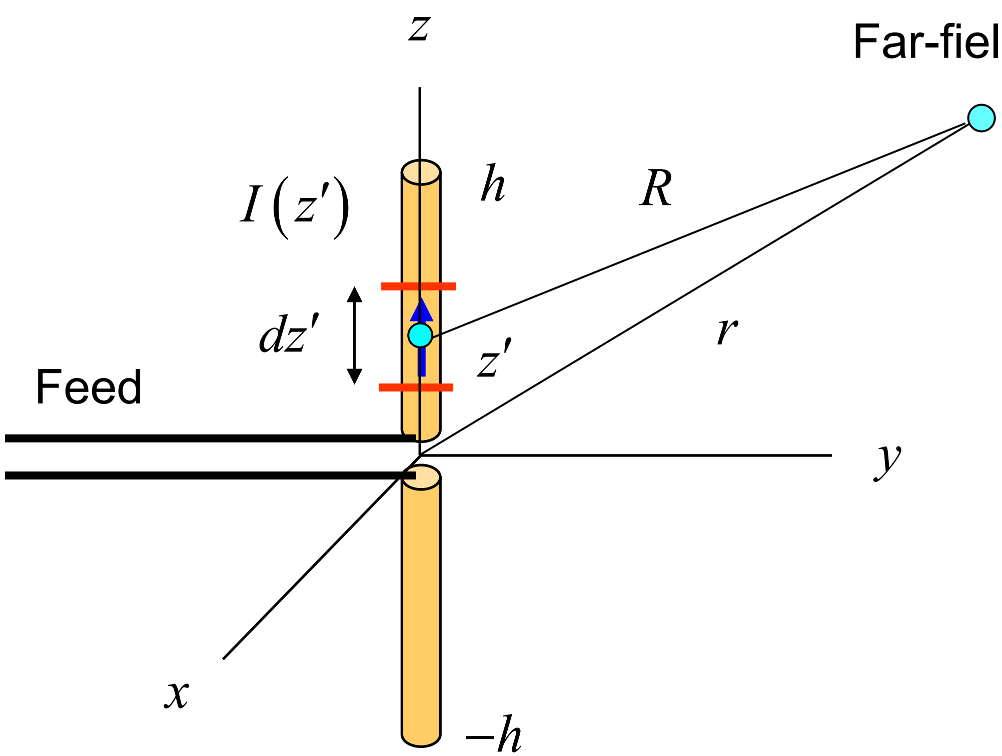
Infinitesimal dipole:

$$E_\theta = \left(\frac{e^{-jk_0 r}}{r} \right) \frac{Il}{4\pi} (j\omega\mu_0) \sin\theta$$

Wire antenna:

$$E_\theta = \frac{1}{4\pi} (j\omega\mu_0) \sin\theta \int_{-h}^{h} \frac{e^{-jk_0 R}}{R} I(z') \, dz'$$

Wire Antenna (cont.)



$$\underline{r} = (x, y, z)$$

$$R = \sqrt{x^2 + y^2 + (z - z')^2}$$

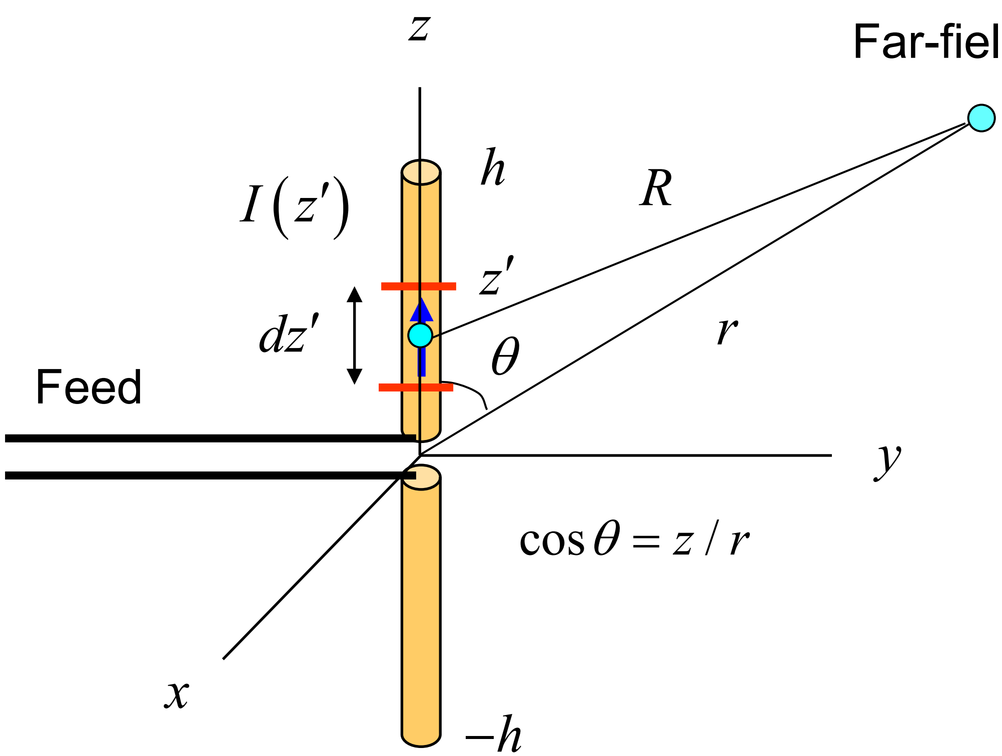
$$= \sqrt{(x^2 + y^2 + z^2) + (z'^2) - 2(zz')}$$

$$= \sqrt{r^2 + z'^2 - 2zz'}$$

$$= r\sqrt{1 + \left(\frac{z'}{r}\right)^2 - 2\left(\frac{zz'}{r^2}\right)}$$

$$= r\sqrt{1 + \left(\frac{z'}{r}\right)^2 - 2\left(\frac{z}{r}\right)\left(\frac{z'}{r}\right)}$$

Wire Antenna (cont.)



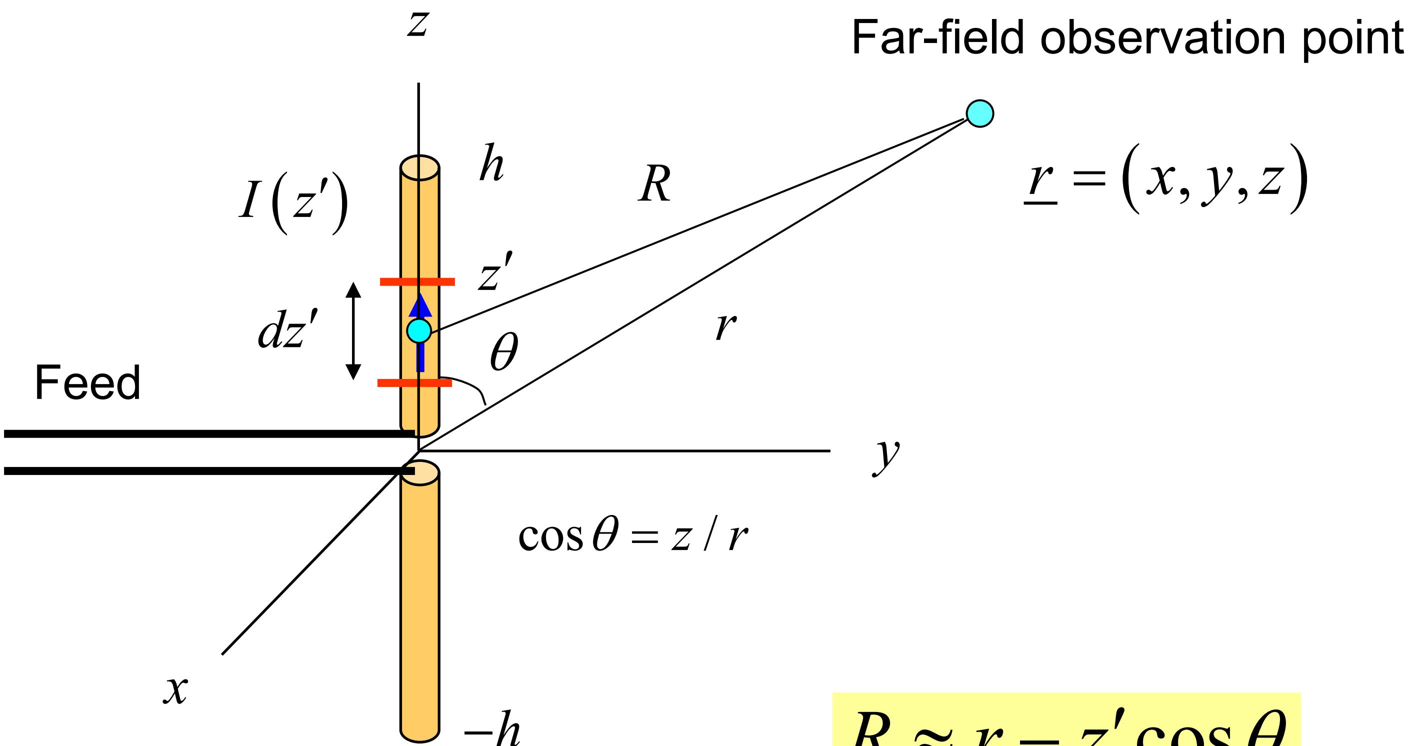
$$\underline{r} = (x, y, z)$$

$$R = r\sqrt{1 + \left(\frac{z'}{r}\right)^2 - 2\left(\frac{z}{r}\right)\left(\frac{z'}{r}\right)}$$

$$R = r\sqrt{1 + \left(\frac{z'}{r}\right)^2 - 2(\cos\theta)\left(\frac{z'}{r}\right)} \approx r\sqrt{1 - 2(\cos\theta)\left(\frac{z'}{r}\right)} \approx r\left[1 - (\cos\theta)\left(\frac{z'}{r}\right)\right] = r - z'\cos\theta$$

Note: $\sqrt{1+x} \approx 1 + \frac{x}{2}, \quad |x| << 1$

Wire Antenna (cont.)



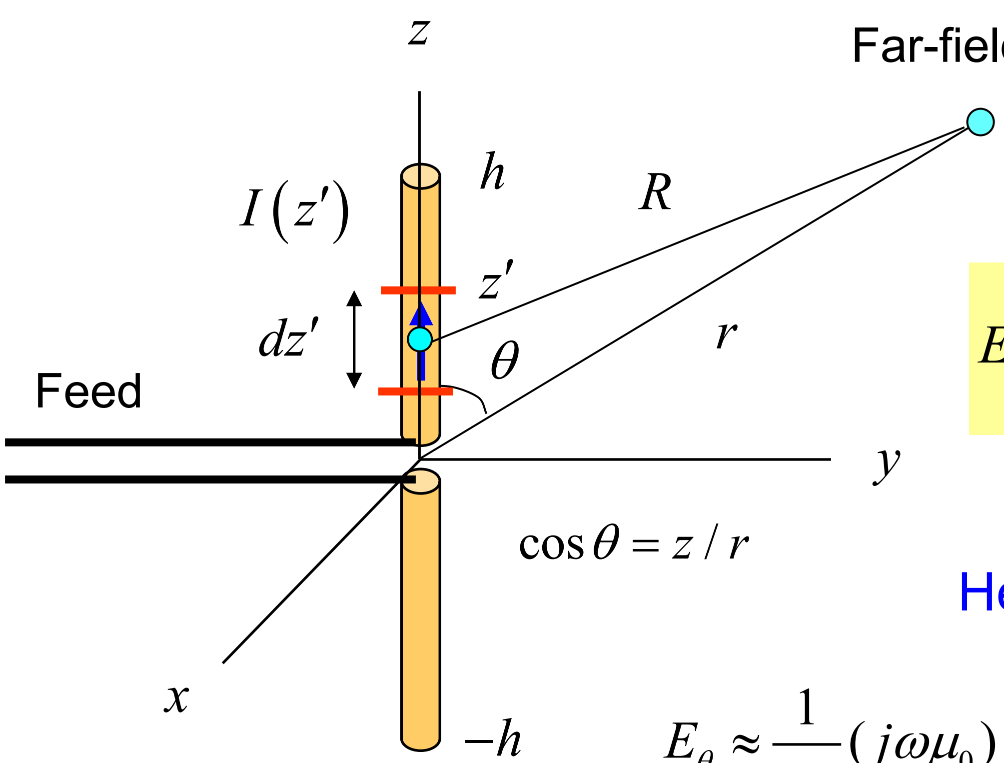
$$\underline{r} = (x, y, z)$$

$$R \approx r - z' \cos\theta$$

It can be shown that this approximation is accurate when

$$r > \frac{2D^2}{\lambda_0} \qquad (D = 2h)$$

Wire Antenna (cont.)



$$\underline{r} = (x, y, z)$$

$$E_\theta = \frac{1}{4\pi}(j\omega\mu_0)\sin\theta \int_{-h}^{h} \frac{e^{-jk_0 R}}{R} I(z')\,dz'$$

Hence, we have:

$$E_\theta \approx \frac{1}{4\pi}(j\omega\mu_0)\sin\theta \int_{-h}^{h} \frac{e^{-jk_0(r - z'\cos\theta)}}{r - z'\cos\theta} I(z')\,dz'$$

$$= \frac{1}{4\pi}(j\omega\mu_0)\sin\theta \left(\frac{e^{-jk_0 r}}{r}\right) \int_{-h}^{h} \frac{e^{+jk_0 z'\cos\theta}}{1 - (z'/r)\cos\theta} I(z')\,dz'$$

$$\approx \frac{1}{4\pi}(j\omega\mu_0)\sin\theta \left(\frac{e^{-jk_0 r}}{r}\right) \int_{-h}^{h} e^{+jk_0 z'\cos\theta} I(z')\,dz'$$

Wire Antenna (cont.)

We define the space factor of the wire antenna:

$$\mathrm{SF}(\theta) \equiv \int_{-h}^{h} I(z')\, e^{+jk_0 z' \cos\theta}\, dz'$$

We then have the following result for the far-field pattern of the wire antenna:

$$E_\theta \approx \frac{1}{4\pi}(j\omega\mu_0)\sin\theta\left(\frac{e^{-jk_0 r}}{r}\right)\mathrm{SF}(\theta)$$

Note:
The term in front of the array factor in the above equation is the far-field pattern of the unit-amplitude infinitesimal dipole.

Wire Antenna (cont.)

Using our assumed approximate current function, we have:

$$I(z) = \left(\frac{I_0}{\sin(k_0 h)} \right) \sin\left[k_0 \left(h - |z| \right) \right]$$

Hence

$$\mathrm{SF}(\theta) = \int_{-h}^{h} \left(\frac{I_0}{\sin(k_0 h)} \right) \sin\left[k_0 \left(h - |z'| \right) \right] e^{+jk_0 z' \cos\theta} \, dz'$$

The result is (derivation omitted):

$$\mathrm{SF}(\theta) = 2 \left(\frac{I_0}{\sin(k_0 h)} \right) \left[\frac{\cos(k_0 h \cos\theta) - \cos(k_0 h)}{k_0 \sin^2\theta} \right]$$

Wire Antenna (cont.)

In summary, we have:

$$E_\theta \approx \frac{1}{4\pi}(j\omega\mu_0)\sin\theta\left(\frac{e^{-jk_0 r}}{r}\right)\mathrm{SF}(\theta)$$

$$\mathrm{SF}(\theta) = 2\left(\frac{I_0}{\sin(k_0 h)}\right)\left[\frac{\cos(k_0 h\cos\theta)-\cos(k_0 h)}{k_0\sin^2\theta}\right]$$

Thus, we have:

$$E_\theta \approx \left(\frac{e^{-jk_0 r}}{r}\right)\left(\frac{1}{2\pi}(j\omega\mu_0 h)\right)\left(\frac{I_0}{\sin(k_0 h)}\right)\left[\frac{\cos(k_0 h\cos\theta)-\cos(k_0 h)}{k_0 h\sin\theta}\right]$$

Wire Antenna (cont.)

For a resonant half-wave dipole antenna:

$$h = \lambda_0 / 4$$

$$k_0 h = \pi / 2$$

$$E_\theta \approx \left(\frac{e^{-jk_0 r}}{r} \right) \left(\frac{1}{2\pi} (j\omega\mu_0 h) \right) (I_0) \left[\frac{\cos\left(\frac{\pi}{2} \cos\theta \right)}{(\pi / 2) \sin\theta} \right]$$

or

$$E_\theta \approx \left(\frac{e^{-jk_0 r}}{r} \right) \left(\frac{j\omega\mu_0}{\pi^2} \right) (I_0 h) \left[\frac{\cos\left(\frac{\pi}{2} \cos\theta \right)}{\sin\theta} \right]$$

Wire Antenna (cont.)

$$E_\theta \approx \left(\frac{e^{-jk_0 r}}{r}\right)\left(\frac{j\omega\mu_0}{\pi^2}\right)\left(I_0 h\right)\left[\frac{\cos\left(\dfrac{\pi}{2}\cos\theta\right)}{\sin\theta}\right]$$

$$h = \lambda_0 / 4$$
$$k_0 h = \pi / 2$$

The directivity is:

$$D(\theta,\phi) = \frac{4\pi\left|\underline{E}^{\mathrm{FF}}(\theta,\phi)\right|^2}{\int_0^{2\pi}\int_0^{\pi}\left|\underline{E}^{\mathrm{FF}}(\theta,\phi)\right|^2 \sin\theta\, d\theta d\phi} = \frac{4\pi\left|\underline{E}^{\mathrm{FF}}(\theta,\phi)\right|^2}{2\pi\int_0^{\pi}\left|\underline{E}^{\mathrm{FF}}(\theta,\phi)\right|^2 \sin\theta\, d\theta d\phi}$$

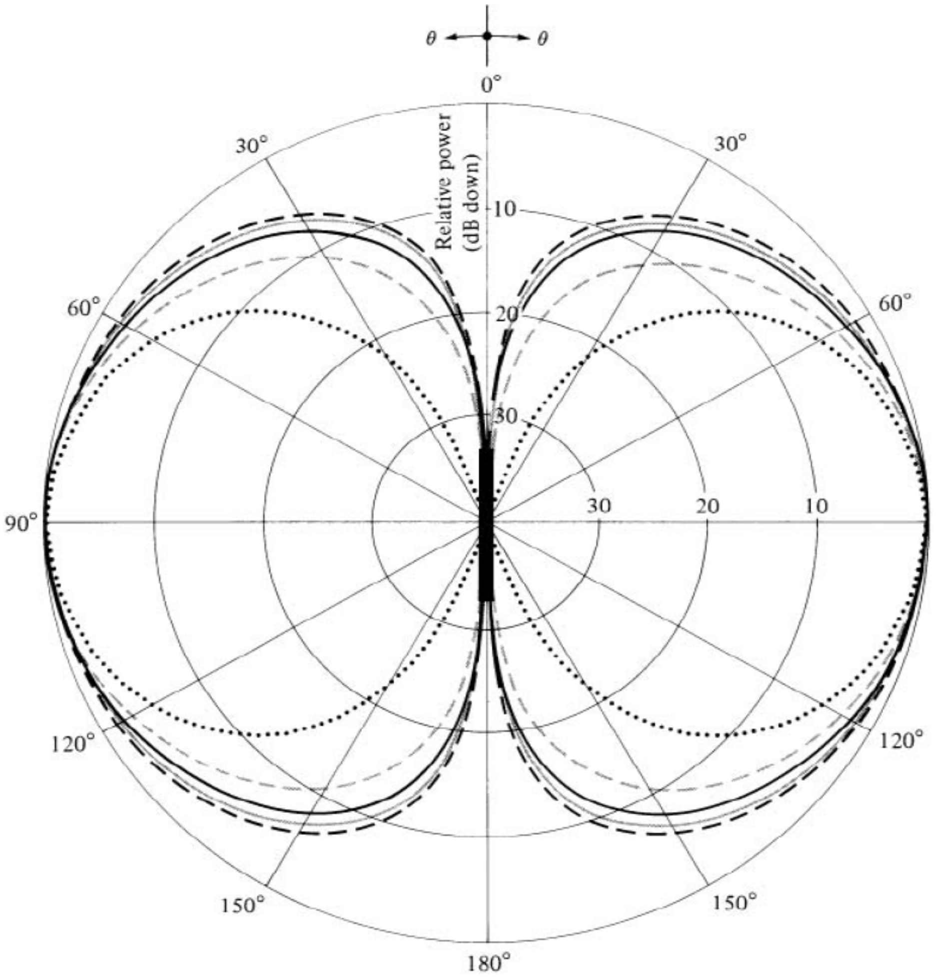
The result (from numerical calculations) is:

$$D(\pi/2,\phi) = 1.643$$

Wire Antenna (cont.)

Results

$l = 2h$



$l = \lambda/50$	$l = \lambda/50$	3-dB beamwidth = 90°
$l = \lambda/4$	$l = \lambda/4$	3-dB beamwidth = 87°
$l = \lambda/2$	$l = \lambda/2$	3-dB beamwidth = 78°
$l = 3\lambda/4$	$l = 3\lambda/4$	3-dB beamwidth = 64°
$l = \lambda$	$l = \lambda$	3-dB beamwidth = 47.8°

Wire Antenna (cont.)

Radiated Power:

$$P_{rad} = \frac{1}{2\eta_0} \int_0^{2\pi} \int_0^{\pi} \left| \underline{E}^{\mathrm{FF}}(\theta,\phi) \right|^2 \sin\theta \, d\theta d\phi$$

$$= \frac{1}{2\eta_0} \int_0^{2\pi} \int_0^{\pi} \left| \left(\frac{1}{2\pi} (j\omega\mu_0 h) \right) \left(\frac{I_0}{\sin(k_0 h)} \right) \left[\frac{\cos(k_0 h \cos\theta) - \cos(k_0 h)}{k_0 h \sin\theta} \right] \right|^2 \sin\theta \, d\theta d\phi$$

Simplify using $\quad \dfrac{\omega\mu_0}{k_0} = \dfrac{\omega\mu_0}{\omega\sqrt{\mu_0\varepsilon_0}} = \sqrt{\dfrac{\mu_0}{\varepsilon_0}} = \eta_0$

$$P_{rad} = \frac{1}{2\eta_0} \int_0^{2\pi} \int_0^{\pi} \left| \left(\frac{1}{2\pi} (j\eta_0) \right) \left(\frac{I_0}{\sin(k_0 h)} \right) \left[\frac{\cos(k_0 h \cos\theta) - \cos(k_0 h)}{\sin\theta} \right] \right|^2 \sin\theta \, d\theta d\phi$$

Wire Antenna (cont.)

Performing the ϕ integral gives us a factor of 2π:

$$P_{rad} = (2\pi)\frac{1}{2\eta_0}\int_0^{\pi}\left|\left(\frac{1}{2\pi}(j\eta_0)\right)\left(\frac{I_0}{\sin(k_0 h)}\right)\left[\frac{\cos(k_0 h\cos\theta)-\cos(k_0 h)}{\sin\theta}\right]\right|^2 \sin\theta\, d\theta$$

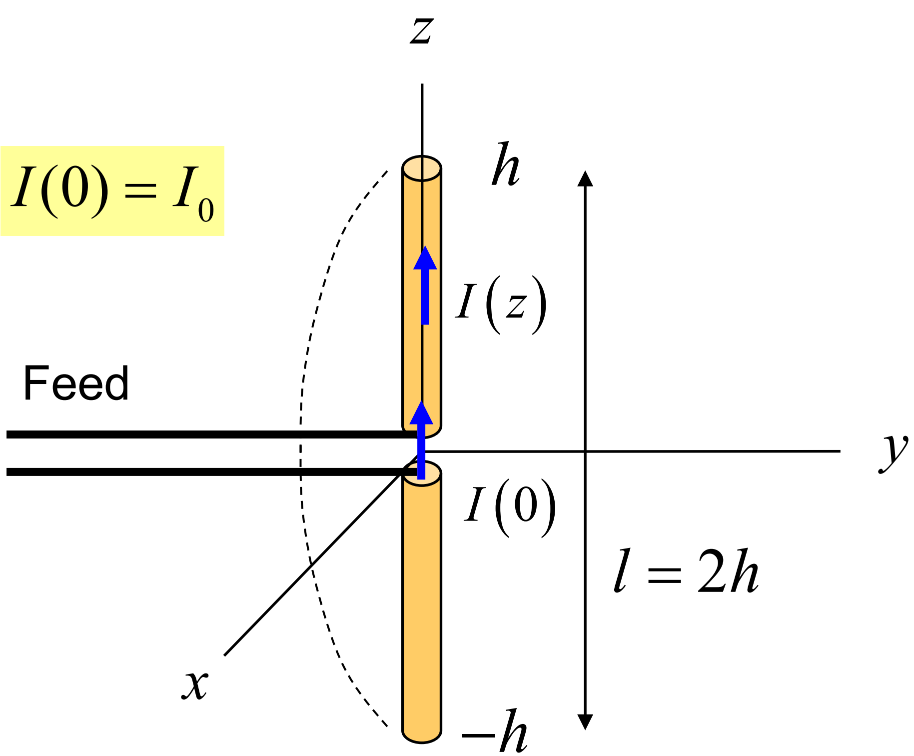
After simplifying, the result is then

$$P_{rad} = \left(\frac{\eta_0}{4\pi}\right)\left(\frac{I_0}{\sin(k_0 h)}\right)^2 \int_0^{\pi}\left[\frac{\cos(k_0 h\cos\theta)-\cos(k_0 h)}{\sin\theta}\right]^2 \sin\theta\, d\theta$$

Wire Antenna (cont.)

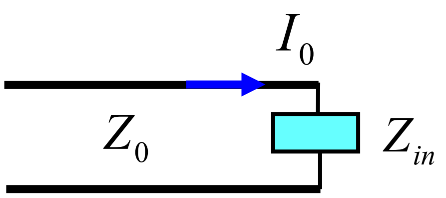
The radiation resistance is defined from $P_{rad} = \frac{1}{2} R_{rad} \left| I_0 \right|^2$

$$I(0) = I_0$$



$$\Rightarrow \quad R_{rad} = \frac{2 P_{rad}}{\left| I_0 \right|^2}$$

Circuit Model



$$Z_{in} = R_{in} + jX_{in}$$

$$R_{in} = R_{rad}$$

For a resonant antenna ($l \approx \lambda_0 / 2$), $X_{in} = 0$.

Wire Antenna (cont.)

The radiation resistance is now evaluated.

$$R_{rad} = \frac{2P_{rad}}{\left|I_0\right|^2}$$

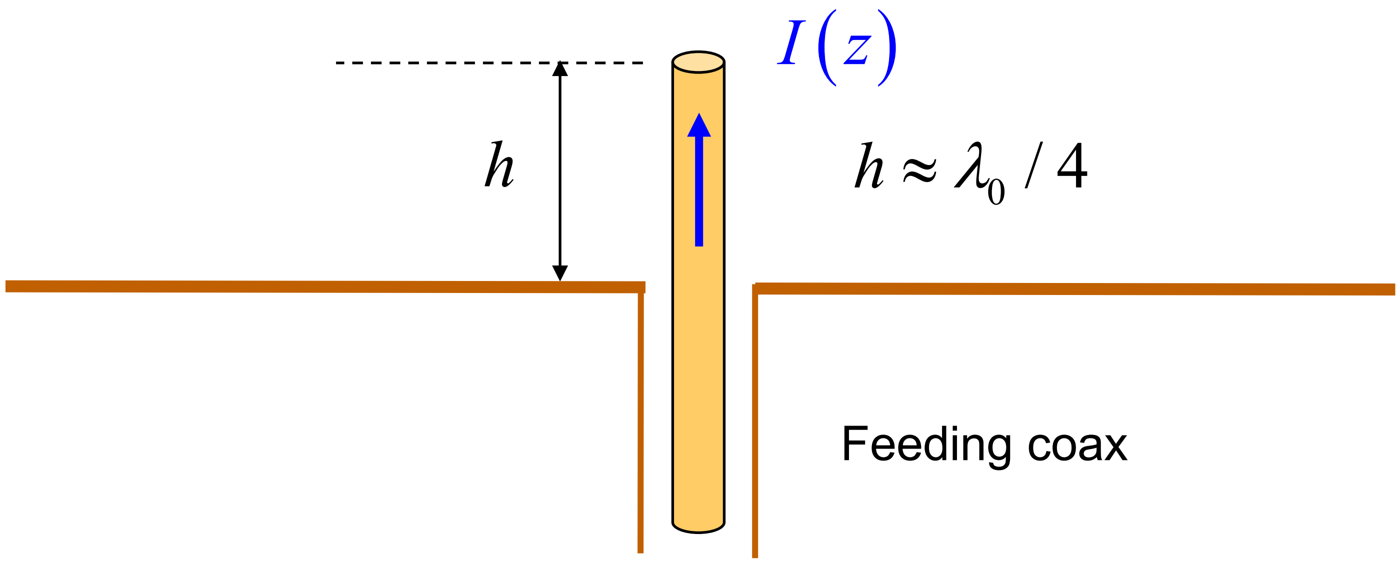
Using the previous formula for P_{rad}, we have:

$$R_{rad} = \left(\frac{\eta_0}{2\pi}\right)\left(\frac{1}{\sin(k_0 h)}\right)^2 \int_0^{\pi} \frac{\left[\cos(k_0 h\cos\theta) - \cos(k_0 h)\right]^2}{\sin\theta}\, d\theta$$

Resonant $\lambda_0/2$ Dipole: $h = \frac{\lambda_0}{4},\; k_0 h = \frac{\pi}{2}$ → $R_{rad} \approx 73\ [\Omega]$

Wire Antenna (cont.)

The result can be extended to the case of a monopole antenna



$$Z_{in}^{\text{monopole}} = \frac{1}{2} Z_{in}^{\text{dipole}}$$

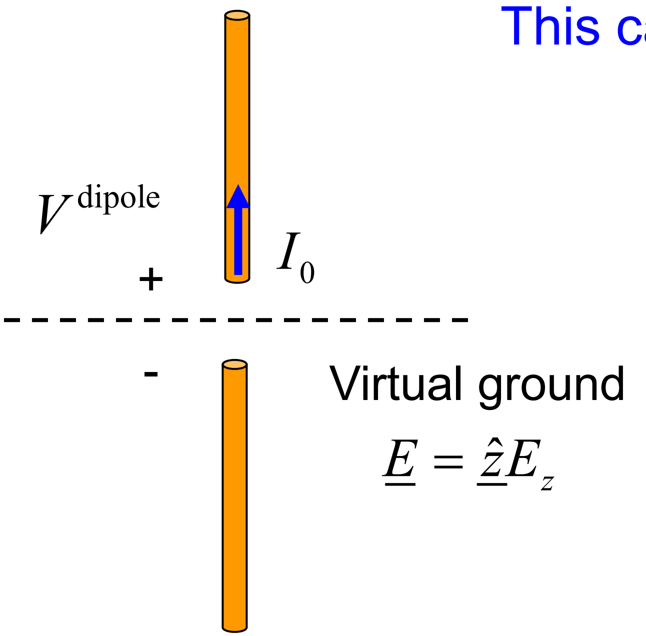
(see the next slide)

$$R_{rad} \approx 36.5 \ [\Omega]$$

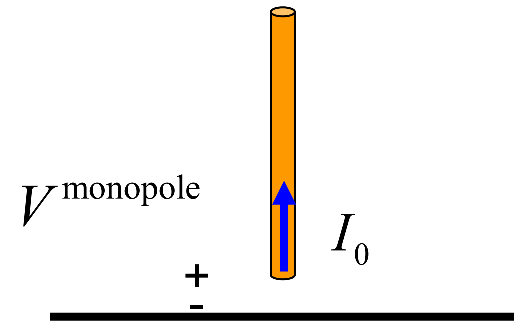
Wire Antenna (cont.)

$$Z_{in}^{\text{monopole}} = \frac{1}{2} Z_{in}^{\text{dipole}}$$

This can be justified as shown.



Dipole



Monopole

$$V^{\text{monopole}} = \frac{1}{2} V^{\text{dipole}}$$

$$I^{\text{monopole}} = I^{\text{dipole}}$$

$$\Rightarrow \quad Z_{in}^{\text{monopole}} = \frac{1}{2} Z_{in}^{\text{dipole}}$$