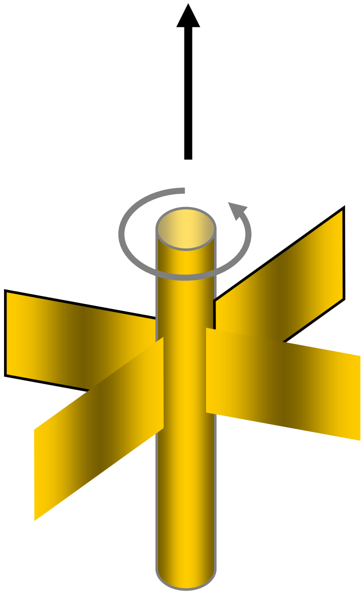
ECE 3317
Applied Electromagnetic Waves

Prof. David R. Jackson

Fall 2023



Notes 3
Review of Vector Calculus

Adapted from notes by Prof. Stuart A. Long

Overview

Here we present a brief overview of vector calculus. A much more thorough discussion of vector calculus may be found in the class notes for ECE 3318:

http://courses.egr.uh.edu/ECE/ECE3318

Notes 13: Divergence
Notes 17: Curl
Notes 19: Gradient and Laplacian

Please also see the textbooks and the following supplementary books (on reserve in the Library):

- H. M. Schey, *Div, Grad, Curl, and All That: an Informal Text on Vector Calculus*, 2nd Ed., W. W. Norton and Company, 1992.
- M. R. Spiegel, *Schaum's Outline on Vector Analysis*, McGraw-Hill, 1959.

"Del" Operator

$$\nabla \equiv \underline{\hat{x}}\frac{\partial}{\partial x} + \underline{\hat{y}}\frac{\partial}{\partial y} + \underline{\hat{z}}\frac{\partial}{\partial z}$$

This is an "operator"*.

Gradient

$$\nabla \phi = \underline{\hat{x}}\frac{\partial \phi}{\partial x} + \underline{\hat{y}}\frac{\partial \phi}{\partial y} + \underline{\hat{z}}\frac{\partial \phi}{\partial z}$$

(Vector)

Laplacian

$$\nabla^2 \phi = \left(\nabla \cdot \nabla\right)\phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi$$

(Scalar)

*An operator is something that operates on a <u>function</u> to return another <u>function</u>.

"Del" Operator (cont.)

$$\nabla \equiv \underline{\hat{x}}\frac{\partial}{\partial x} + \underline{\hat{y}}\frac{\partial}{\partial y} + \underline{\hat{z}}\frac{\partial}{\partial z}$$

Vector $\underline{A}$: $\underline{A} = \underline{\hat{x}}A_x + \underline{\hat{y}}A_y + \underline{\hat{z}}A_z$

Divergence $\nabla \cdot \underline{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$ (Scalar)

Curl $\nabla \times \underline{A} = \underline{\hat{x}}\left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right] + \underline{\hat{y}}\left[\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right] + \underline{\hat{z}}\left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right]$

(Please see the next slide for more details.) (Vector)

Note:
Results for cylindrical and spherical coordinates are given in the back of your books.

"Del" Operator (cont.)

$$\nabla \equiv \underline{\hat{x}}\frac{\partial}{\partial x} + \underline{\hat{y}}\frac{\partial}{\partial y} + \underline{\hat{z}}\frac{\partial}{\partial z}$$

$$\underline{A} = \underline{\hat{x}}A_x + \underline{\hat{y}}A_y + \underline{\hat{z}}A_z$$

A few more details about calculating the divergence and the curl:

$$\nabla \cdot \underline{A} = \left(\underline{\hat{x}}\frac{\partial}{\partial x} + \underline{\hat{y}}\frac{\partial}{\partial y} + \underline{\hat{z}}\frac{\partial}{\partial z} \right) \cdot \left(\underline{\hat{x}}\,A_x + \underline{\hat{y}}\,A_y + \underline{\hat{z}}\,A_z \right) = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \underline{A} = \left(\underline{\hat{x}}\frac{\partial}{\partial x} + \underline{\hat{y}}\frac{\partial}{\partial y} + \underline{\hat{z}}\frac{\partial}{\partial z} \right) \times \left(\underline{\hat{x}}\,A_x + \underline{\hat{y}}\,A_y + \underline{\hat{z}}\,A_z \right)$$

$$= \begin{vmatrix} \underline{\hat{x}} & \underline{\hat{y}} & \underline{\hat{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$= \underline{\hat{x}}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) - \underline{\hat{y}}\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z} \right) + \underline{\hat{z}}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)$$

Vector Identities

Two fundamental "zero" identities:

$$\nabla \cdot \left(\nabla \times \underline{A} \right) = 0$$

$$\nabla \times \left(\nabla \phi \right) = \underline{0}$$

Note: It is usually easiest to prove vector identities by expanding both sides in rectangular coordinates, though any coordinate system can be used.

Vector Identities (cont.)

Another useful identity:

$$\nabla \cdot \left(\underline{A} \times \underline{B} \right) = \underline{B} \cdot \left(\nabla \times \underline{A} \right) - \underline{A} \cdot \left(\nabla \times \underline{B} \right)$$

This will be useful in the derivation of the <u>Poynting theorem</u>.

Vector Laplacian

The vector Laplacian of a vector function is a vector function.

$$\nabla^2 \underline{A} \equiv \nabla\left(\nabla \cdot \underline{A}\right) - \nabla \times \left(\nabla \times \underline{A}\right)$$

The vector Laplacian is very useful for deriving the vector Helmholtz equation (the fundamental differential equation that the vector electric and magnetic fields obey in free space). This will be done later.

$$\nabla^2 \underline{E} + k^2 \underline{E} = \underline{0}$$

$$\nabla^2 \underline{H} + k^2 \underline{H} = \underline{0}$$

Vector Laplacian (cont.)

In rectangular coordinates, the vector Laplacian has a very nice property:

$$\nabla^2 \underline{A} = \underline{\hat{x}}\left(\nabla^2 A_x\right) + \underline{\hat{y}}\left(\nabla^2 A_y\right) + \underline{\hat{z}}\left(\nabla^2 A_z\right)$$

This identity is a key property that will help us reduce the vector Helmholtz equation to the <u>scalar Helmholtz equation</u>, which the <u>components</u> of the fields satisfy in rectangular coordinates.

$$\nabla^2 E_x + k^2 E_x = 0$$

$$\nabla^2 E_y + k^2 E_y = 0$$

$$\nabla^2 E_z + k^2 E_z = 0$$

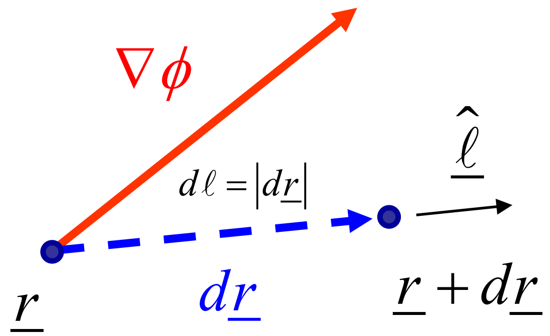
Gradient

Physical Property

$$d\phi \equiv \phi(\underline{r} + d\underline{r}) - \phi(\underline{r}) = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz$$ (from calculus)

$$= \left[\underline{\hat{x}}\left(\frac{\partial \phi}{\partial x}\right) + \underline{\hat{y}}\left(\frac{\partial \phi}{\partial y}\right) + \underline{\hat{z}}\left(\frac{\partial \phi}{\partial z}\right)\right] \cdot \left[\underline{\hat{x}}(dx) + \underline{\hat{y}}(dy) + \underline{\hat{z}}(dz)\right] = \nabla\phi \cdot d\underline{r}$$

$$\frac{d\phi}{d\ell} = \frac{\nabla\phi \cdot d\underline{r}}{d\ell} = \nabla\phi \cdot \left(\frac{d\underline{r}}{d\ell}\right)$$



$$\Rightarrow \quad \frac{d\phi}{d\ell} = \nabla\phi \cdot \left(\underline{\hat{\ell}}\right)$$

The gradient gives us the "directional derivative", which tells us the rate of change in a function as we march in a certain direction $\underline{\hat{\ell}}$.

Gradient (cont.)

Rectangular

$$\nabla\Phi = \underline{\hat{x}}\,\frac{\partial\Phi}{\partial x} + \underline{\hat{y}}\,\frac{\partial\Phi}{\partial y} + \underline{\hat{z}}\,\frac{\partial\Phi}{\partial z}$$

Cylindrical

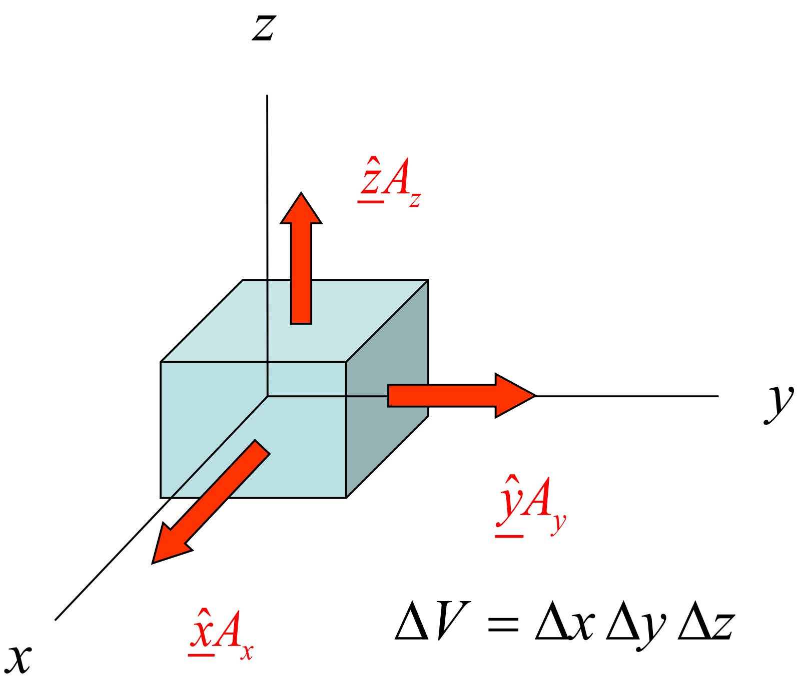
$$\nabla\Phi = \underline{\hat{\rho}}\,\frac{\partial\Phi}{\partial\rho} + \underline{\hat{\phi}}\,\frac{1}{\rho}\frac{\partial\Phi}{\partial\phi} + \underline{\hat{z}}\,\frac{\partial\Phi}{\partial z}$$

Spherical

$$\nabla\Phi = \underline{\hat{r}}\,\frac{\partial\Phi}{\partial r} + \underline{\hat{\theta}}\,\frac{1}{r}\frac{\partial\Phi}{\partial\theta} + \underline{\hat{\phi}}\,\frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}$$

Divergence

Physical Property



$$\Delta V = \Delta x \, \Delta y \, \Delta z$$

$\hat{\underline{n}}$ = outward normal

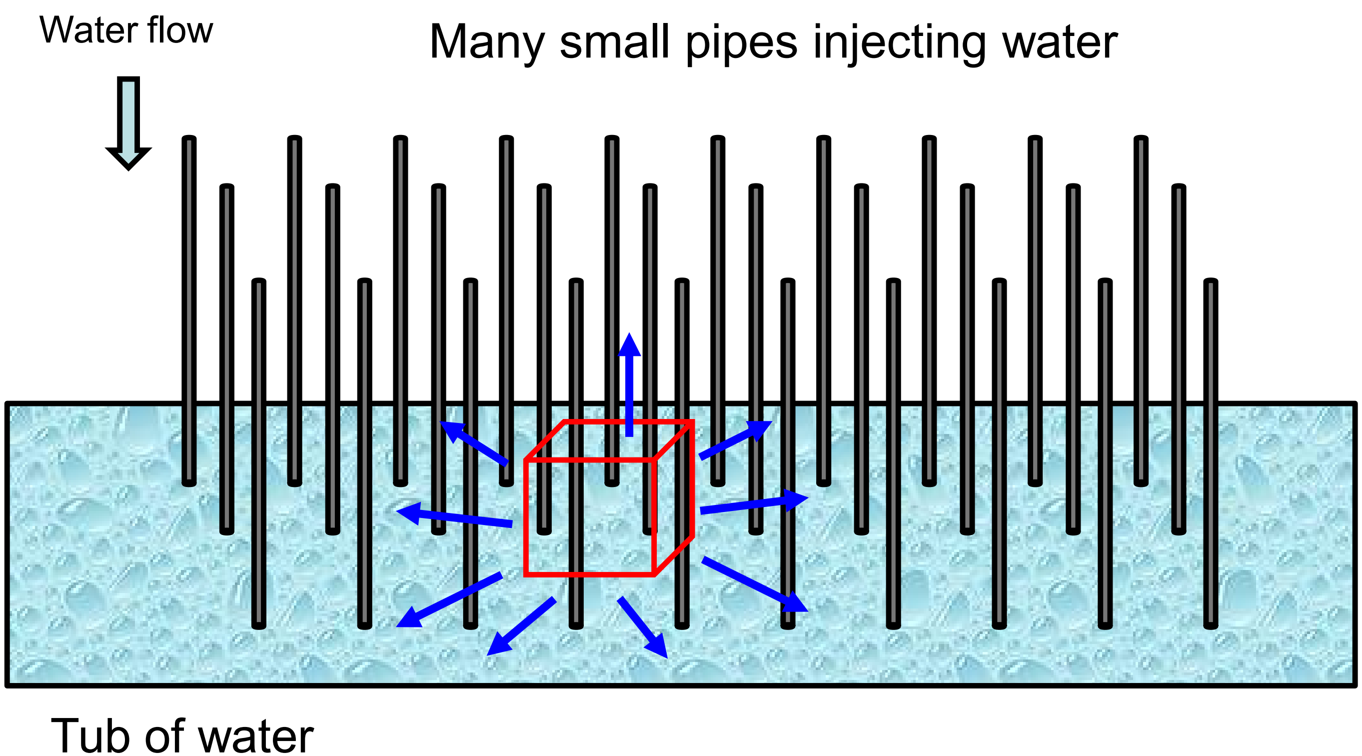
$$\nabla \cdot \underline{A} = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \oint_S \underline{A} \cdot \hat{\underline{n}} \, dS$$

The divergence measures the rate at which the "flux" of the vector function emanates from a region of space.

Divergence > 0: "source of flux"
Divergence < 0: "sink of flux"

Note: The shape of the small volume is actually arbitrary; it does not have to be a cube (please see the ECE 3318 notes).

Divergence (cont.)



$\underline{A}(x, y, z) =$ velocity vector of water inside tub

$$\nabla \cdot \underline{A} > 0 \quad (\text{inside tub})$$

Divergence (cont.)

Rectangular:

$$\nabla \cdot \underline{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

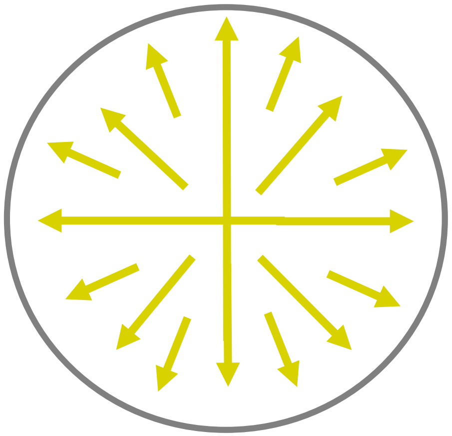
Cylindrical:

$$\nabla \cdot \underline{A} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho A_\rho\right) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

Spherical:

$$\nabla \cdot \underline{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \theta}\left(A_\theta \sin\theta\right) + \frac{1}{r \sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

Divergence (cont.)



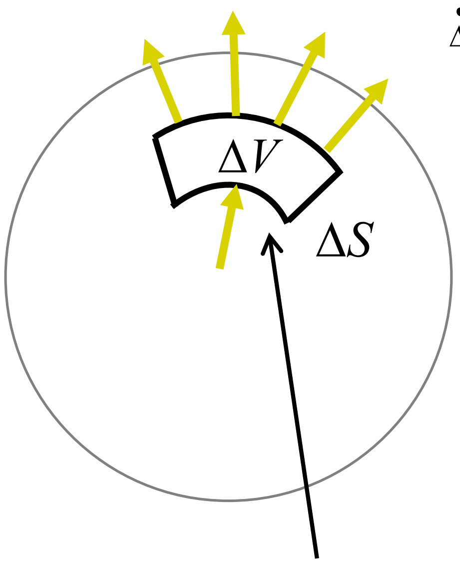
Example: $\underline{E} = \hat{\underline{r}}\left(\frac{r}{3}\right)$ (electric field vector)

$$\nabla \cdot \underline{E} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 E_r\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{r}{3}\right) = \frac{1}{3}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^3\right) = 1$$

The divergence is positive.

$$\oint_{\Delta S} \underline{E} \cdot \hat{\underline{n}}\, dS > 0$$

$\hat{\underline{n}}$ = outward normal



The net flux of $\underline{E}$ out of a small volume ΔV is positive.

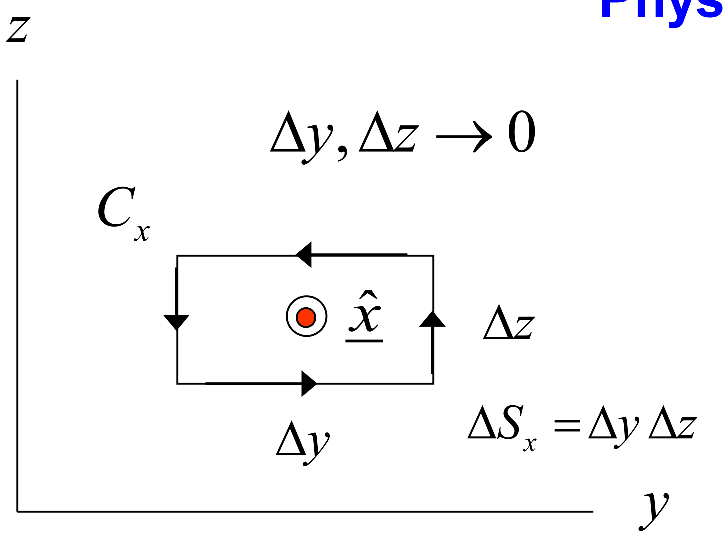
Small "curvilinear cube"

Practical note: This corresponds to a region of charge density.

$$\nabla \cdot \underline{E} = \rho_v / \varepsilon_0$$

Curl

Physical Property



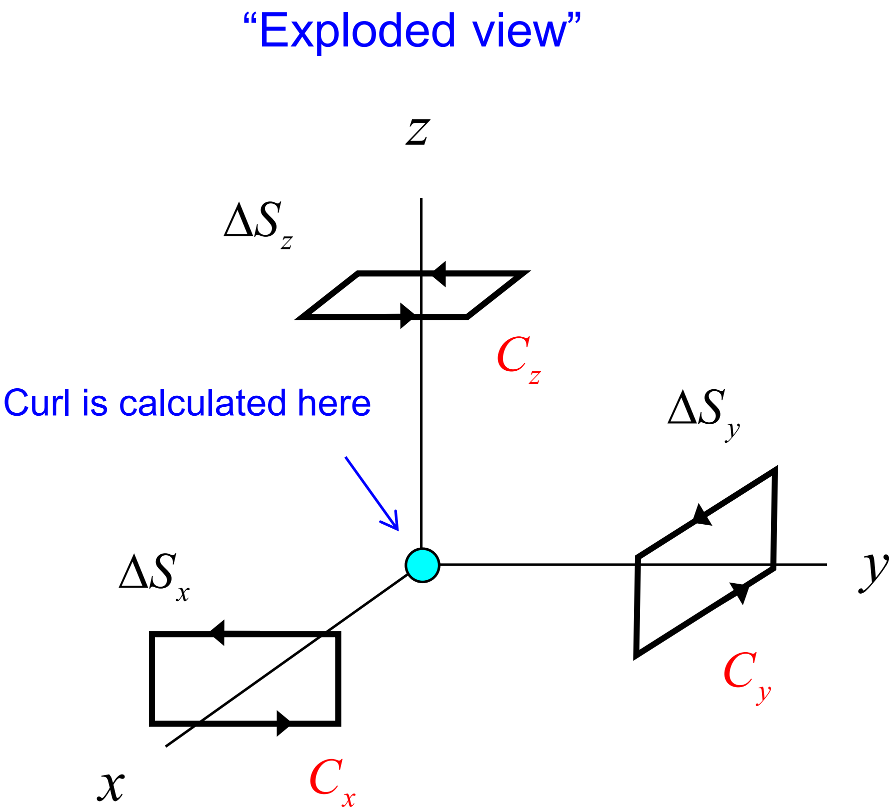
"right-hand rule for C"

$$\left(\nabla \times \underline{A}\right) \cdot \underline{\hat{x}} = \lim_{\Delta S_x \to 0} \frac{1}{\Delta S_x} \oint_{C_x} \underline{A} \cdot d\underline{r}$$

Note:
The x component of the curl measures the tendency of the vector field to "curl" or "rotate" about the x axis.

Note: The shape of the small area is actually arbitrary; it does not have to be a rectangle (please see the ECE 3318 notes).

Curl (cont.)



$$\left(\nabla \times \underline{A}\right) \cdot \underline{\hat{x}} = \lim_{\Delta S_x \to 0} \frac{1}{\Delta S_x} \oint_{C_x} \underline{A} \cdot d\underline{r}$$

$$\left(\nabla \times \underline{A}\right) \cdot \underline{\hat{y}} = \lim_{\Delta S_x \to 0} \frac{1}{\Delta S_y} \oint_{C_y} \underline{A} \cdot d\underline{r}$$

$$\left(\nabla \times \underline{A}\right) \cdot \underline{\hat{z}} = \lim_{\Delta S_x \to 0} \frac{1}{\Delta S_z} \oint_{C_z} \underline{A} \cdot d\underline{r}$$

Note:
The paths are defined according to the "right-hand rule."
(Your thumb is in the direction of the unit normal.)

Note:
The paths are all centered at the point of interest (shown as the origin for simplicity).
(A separation between the paths is shown in the exploded view for clarity.)

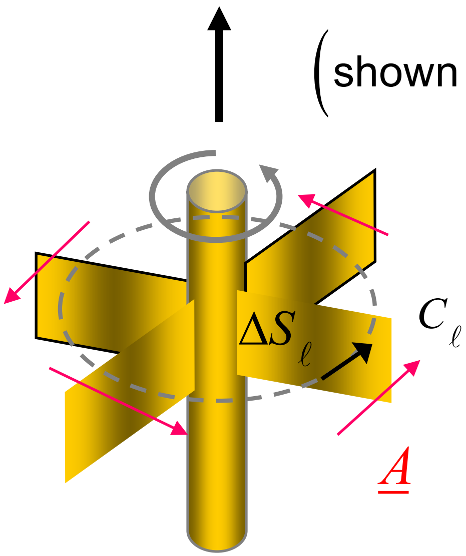
Curl (cont.)

"Curl meter"

T_l = torque when pointed in $\hat{\underline{\ell}}$ direction

$\hat{\underline{\ell}} = \underline{\hat{x}}, \ \underline{\hat{y}}$, or $\underline{\hat{z}}$

(shown for $\hat{\underline{\ell}} = \underline{\hat{z}}$)



Assume that $\underline{A}$ represents the velocity of a fluid.

$$\hat{\underline{\ell}} \cdot \left(\nabla \times \underline{A} \right) \equiv \lim_{\Delta s_\ell \to 0} \frac{1}{\Delta S_\ell} \oint_{C_\ell} \underline{A} \cdot \underline{dr}$$

The term $\underline{A} \cdot \underline{dr}$ measures the force on the paddles.

(going counterclockwise)

Hence,

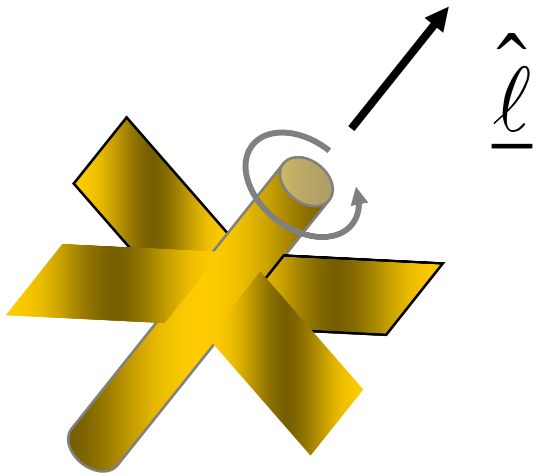
$\hat{\underline{\ell}} \cdot \left(\nabla \times \underline{A} \right)$ is a measure of T_l, the component of torque in the $\hat{\underline{\ell}}$ direction.

(If this component is positive, the paddle wheel will spin counterclockwise.)

Curl (cont.)

This property actually holds for <u>any</u> direction, not just $\hat{\underline{x}}, \hat{\underline{y}}, \hat{\underline{z}}$

$\hat{\underline{\ell}} \cdot \left(\nabla \times \underline{A} \right)$ is a measure of T_l = torque when pointed in $\hat{\underline{\ell}}$ direction



(A proof of this is given in the ECE 3318 notes.)

Curl (cont.)

Rectangular

$$\nabla \times \underline{A} = \underline{\hat{x}}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \underline{\hat{y}}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \underline{\hat{z}}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

Cylindrical

$$\nabla \times \underline{A} = \underline{\hat{\rho}}\left(\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right) + \underline{\hat{\phi}}\left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right) + \underline{\hat{z}}\,\frac{1}{\rho}\left(\frac{\partial\left(\rho A_\phi\right)}{\partial \rho} - \frac{\partial A_\rho}{\partial \phi}\right)$$

Spherical

$$\nabla \times \underline{A} = \underline{\hat{r}}\,\frac{1}{r\sin\theta}\left[\frac{\partial\left(A_\phi \sin\theta\right)}{\partial \theta} - \frac{\partial A_\theta}{\partial \phi}\right] + \underline{\hat{\theta}}\,\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial \phi} - \frac{\partial\left(rA_\phi\right)}{\partial r}\right] + \underline{\hat{\phi}}\,\frac{1}{r}\left[\frac{\partial\left(rA_\theta\right)}{\partial r} - \frac{\partial A_r}{\partial \theta}\right]$$

Curl (cont.)

Determinant Forms

Rectangular

$$\nabla \times \underline{A} = \begin{vmatrix} \underline{\hat{x}} & \underline{\hat{y}} & \underline{\hat{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

Cylindrical

$$\nabla \times \underline{A} = \frac{1}{\rho} \begin{vmatrix} \underline{\hat{\rho}} & \rho\,\underline{\hat{\phi}} & \underline{\hat{z}} \\ \dfrac{\partial}{\partial \rho} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix}$$

Spherical

$$\nabla \times \underline{A} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \underline{\hat{r}} & r\underline{\hat{\theta}} & r\sin\theta\,\underline{\hat{\phi}} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin\theta\, A_\phi \end{vmatrix}$$

Curl (cont.)

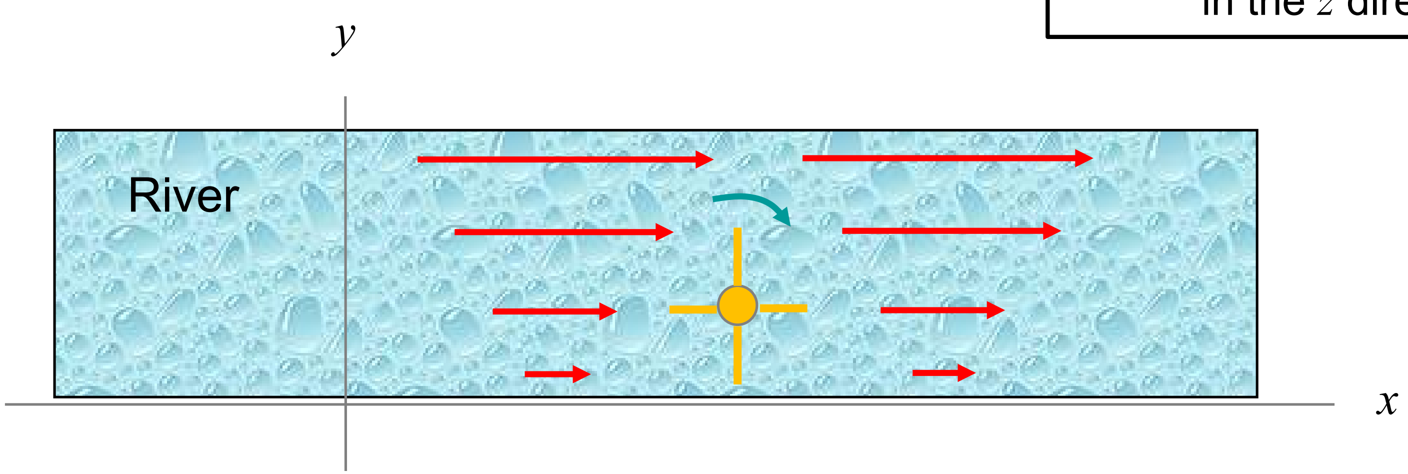
Example: $\underline{V} = \underline{\hat{x}}\,(y)$ $\quad V_x = y,\ V_y = 0,\ V_z = 0$ (water velocity vector)

$$\nabla \times \underline{V} = \underline{\hat{x}}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) + \underline{\hat{y}}\left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right) + \underline{\hat{z}}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)$$

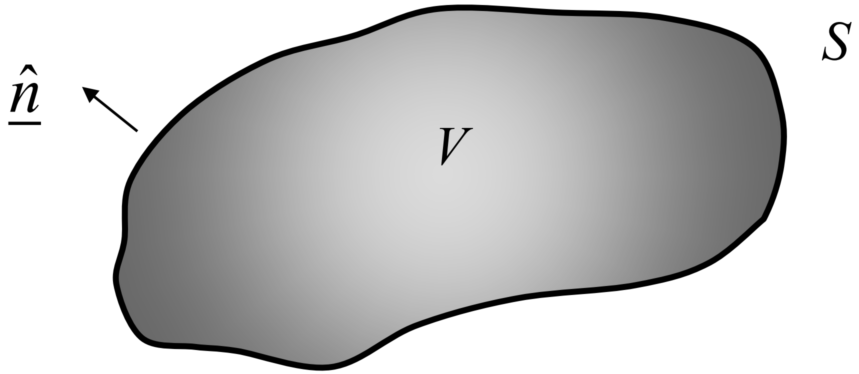
$$\nabla \times \underline{V} = \underline{\hat{z}}\,(-1)$$

$$\Rightarrow \quad \left(\nabla \times \underline{V}\right) \cdot \underline{\hat{z}} = -1 < 0$$

The paddle wheel thus spins opposite to the fingers of the right hand, if the thumb points in the z direction.



Divergence Theorem

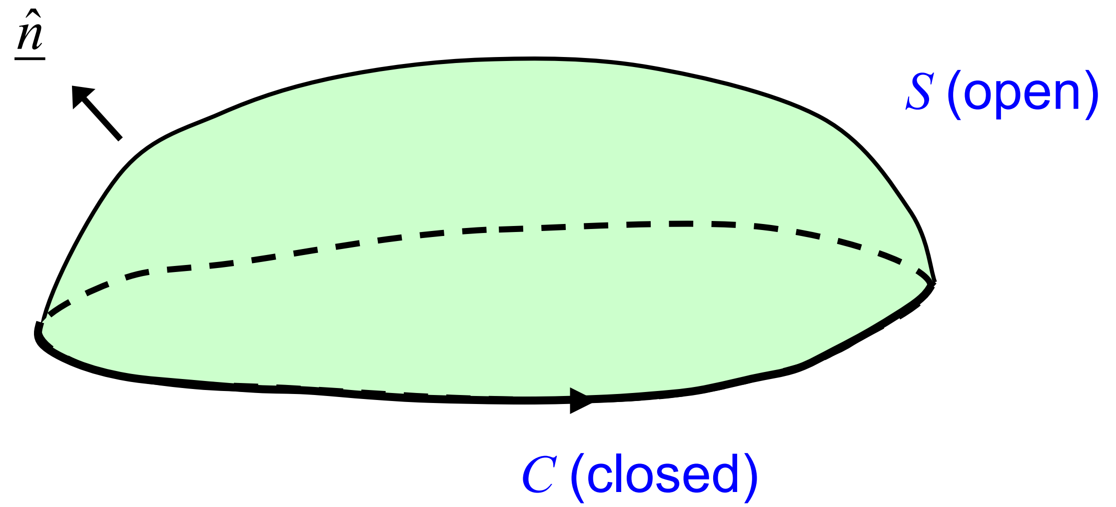


$\underline{\hat{n}}$ = outward normal

$$\int_V \nabla \cdot \underline{A} \, dV = \oint_S \underline{A} \cdot \underline{\hat{n}} \, dS$$

$\underline{A}$ = arbitrary vector function

Stokes's Theorem



The unit normal is chosen from a "right-hand rule" according to the direction along C.
(An outward normal corresponds to a counterclockwise path.)

$$\int_S \left(\nabla \times \underline{A}\right) \cdot \hat{\underline{n}}\, dS = \oint_C \underline{A} \cdot \underline{dr}$$