ECE 3317
Applied Electromagnetic Waves

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Notes 4
Maxwell’s Equations

Adapted from notes by Prof. Stuart A. Long
Here we present an overview of Maxwell’s equations. A much more thorough discussion of Maxwell’s equations may be found in the class notes for ECE 3318:

http://courses.egr.uh.edu/ECE/ECE3318

- Notes 10: Electric Gauss’s law
- Notes 18: Faraday’s law
- Notes 28: Ampere’s law
- Notes 28: Magnetic Gauss law

Electromagnetic Fields

Four vector quantities

- $\mathbf{E}$ electric field strength [Volt/meter]
- $\mathbf{D}$ electric flux density [Coulomb/meter$^2$]
- $\mathbf{H}$ magnetic field strength [Amp/meter]
- $\mathbf{B}$ magnetic flux density [Weber/meter$^2$] or [Tesla]

Each are functions of space and time
e.g. $\mathbf{E}(x,y,z,t)$

- $\mathbf{J}$ electric current density [Amp/meter$^2$]
- $\rho_v$ electric charge density [Coulomb/meter$^3$]
MKS units

length – meter [m]
mass – kilogram [kg]
time – second [sec]

Some common prefixes and the power of ten each represent are listed below

femto - f - 10^{-15}
centi - c - 10^{-2}
mega - M - 10^6
pico - p - 10^{-12}
deci - d - 10^{-1}
giga - G - 10^9
nano - n - 10^{-9}
deka - da - 10^1
tera - T - 10^{12}
micro - μ - 10^{-6}
hecto - h - 10^2
peta - P - 10^{15}
milli - m - 10^{-3}
kilo - k - 10^3
Maxwell’s Equations

(Time-varying, differential form)

\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \]

\[ \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \]

\[ \nabla \cdot \mathbf{B} = 0 \]

\[ \nabla \cdot \mathbf{D} = \rho_v \]
James Clerk Maxwell (1831–1879)

**James Clerk Maxwell** was a Scottish mathematician and theoretical physicist. His most significant achievement was the development of the classical electromagnetic theory, synthesizing all previous unrelated observations, experiments and equations of electricity, magnetism and even optics into a consistent theory. His set of equations—Maxwell's equations—demonstrated that electricity, magnetism and even light are all manifestations of the same phenomenon: the electromagnetic field. From that moment on, all other classical laws or equations of these disciplines became simplified cases of Maxwell's equations. Maxwell's work in electromagnetism has been called the *second great unification in physics*, after the first one carried out by Isaac Newton.

Maxwell demonstrated that electric and magnetic fields travel through space in the form of waves, and at the constant speed of light. Finally, in 1864 Maxwell wrote *A Dynamical Theory of the Electromagnetic Field* where he first proposed that light was in fact undulations in the same medium that is the cause of electric and magnetic phenomena. His work in producing a unified model of electromagnetism is considered to be one of the greatest advances in physics.

(Wikipedia)
Maxwell’s Equations (cont.)

\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \text{Faraday’s law} \]

\[ \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \quad \text{Ampere’s law} \]

\[ \nabla \cdot \mathbf{B} = 0 \quad \text{Magnetic Gauss law} \]

\[ \nabla \cdot \mathbf{D} = \rho_v \quad \text{Electric Gauss law} \]

Questions: When does a magnetic field produce an electric field? When does an electric field produce a magnetic field? When does a current flow produce a magnetic field? When does a charge density produce an electric field?
Charge Density

\[ \rho_v(x, y, z) = \lim_{{\Delta V \to 0}} \frac{\Delta Q}{\Delta V} = \frac{dQ}{dV} \]

Example: Protons are closer together as we move to the right.
Current flow is defined to be in the direction that positive charges move in.
Current Density Vector (cont.)

Ohm’s law

\[ \mathbf{J} = \sigma \mathbf{E} \]

<table>
<thead>
<tr>
<th>Material</th>
<th>( \sigma ) [S/m]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Silver</td>
<td>6.3 \times 10^7</td>
</tr>
<tr>
<td>Copper</td>
<td>6.0 \times 10^7</td>
</tr>
<tr>
<td>Copper (annealed)</td>
<td>5.8 \times 10^7</td>
</tr>
<tr>
<td>Gold</td>
<td>4.1 \times 10^7</td>
</tr>
<tr>
<td>Aluminum</td>
<td>3.5 \times 10^7</td>
</tr>
<tr>
<td>Zinc</td>
<td>1.7 \times 10^7</td>
</tr>
<tr>
<td>Brass</td>
<td>1.6 \times 10^7</td>
</tr>
<tr>
<td>Nickel</td>
<td>1.4 \times 10^7</td>
</tr>
<tr>
<td>Iron</td>
<td>1.0 \times 10^7</td>
</tr>
<tr>
<td>Tin</td>
<td>9.2 \times 10^6</td>
</tr>
<tr>
<td>Steel (carbon)</td>
<td>7.0 \times 10^6</td>
</tr>
<tr>
<td>Steel (stainless)</td>
<td>1.5 \times 10^6</td>
</tr>
</tbody>
</table>

http://en.wikipedia.org/wiki/Electrical_resistivity_and_conductivity
Current through a tilted surface:

\[ \Delta I = (\mathbf{J} \cdot \hat{n}) \Delta S \]
\[ \Delta I = \left( \mathbf{J} \cdot \hat{n} \right) \Delta S \]

Note:
The direction of the unit normal vector determines whether the current is measured going in or out.

\[ I = \int_{S} \mathbf{J} \cdot \hat{n} \, dS \]
Law of Conservation of Electric Charge (Continuity Equation)

\[ \nabla \times \mathcal{H} = \mathcal{J} + \frac{\partial \mathcal{D}}{\partial t} \]

\[ \nabla \cdot (\nabla \times \mathcal{H}) = \nabla \cdot \mathcal{J} + \nabla \cdot \left( \frac{\partial \mathcal{D}}{\partial t} \right) \]

\[ 0 = \nabla \cdot \mathcal{J} + \frac{\partial}{\partial t} \left( \nabla \cdot \mathcal{D} \right) \]

Flow of electric current out of volume (per unit volume)

\[ \nabla \cdot \mathcal{J} = - \frac{\partial \rho_v}{\partial t} \]

Rate of decrease of electric charge (per unit volume)
Continuity Equation (cont.)

\[ \nabla \cdot \mathcal{J} = -\frac{\partial \rho_v}{\partial t} \]

Integrate both sides over an arbitrary volume \( V \):

\[ \int_V \nabla \cdot \mathcal{J} \, dV = \int_V -\frac{\partial \rho_v}{\partial t} \, dV \]

Apply the divergence theorem:

\[ \int_V \nabla \cdot \mathcal{J} \, dV = \oint_S \mathcal{J} \cdot \hat{n} \, dS \]

Hence:

\[ \oint_S \mathcal{J} \cdot \hat{n} \, dV = \int_V -\frac{\partial \rho_v}{\partial t} \, dV \]
Continuity Equation (cont.)

Physical interpretation:

\[ \oint_S \mathbf{J} \cdot \hat{n} \, dS = \int_V -\frac{\partial \rho_v}{\partial t} \, dV \]

\[ i_{out} = \int_V -\frac{\partial \rho_v}{\partial t} \, dV = -\frac{\partial}{\partial t} \int_V \rho_v \, dV \]

(This assumes that the surface is stationary.)

\[ i_{out} = -\frac{\partial Q_{encl}}{\partial t} \quad \text{or} \quad i_{in} = \frac{\partial Q_{encl}}{\partial t} \]
Continuity Equation (cont.)

\[ i_{in} = \frac{\partial Q_{encl}}{\partial t} \]

This implies that charge is never created or destroyed. It only moves from one place to another!
Maxwell’s Equations (cont.)

Time-Dependent

\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \quad \nabla \cdot \mathbf{B} = 0 \quad \nabla \cdot \mathbf{D} = \rho_v \]

Time-Independent (Statics)

\[ \nabla \times \mathbf{E} = 0 \quad \nabla \cdot \mathbf{D} = \rho_v \quad \nabla \times \mathbf{H} = \mathbf{J} \quad \nabla \cdot \mathbf{B} = 0 \]

Decouples \( \mathbf{E} \) and \( \mathbf{H} \)  \( \Rightarrow \)  \( \mathbf{E} \) comes from \( \rho_v \) and \( \mathbf{H} \) comes from \( \mathbf{J} \)

Note: Regular (not script) font is used for statics, just as it is for phasors.
Maxwell’s Equations (cont.)

Time-harmonic (phasor) domain

\[
\nabla \times \underline{E} = -j\omega \underline{B} \\
\nabla \times \underline{H} = \underline{J} + j\omega \underline{D} \\
\n\nabla \cdot \underline{B} = 0 \\
\n\n\nabla \cdot \underline{D} = \rho_v
\]
Constitutive Relations

The characteristics of the media relate \( \mathbf{D} \) to \( \mathbf{E} \) and \( \mathbf{H} \) to \( \mathbf{B} \).

**Free Space**

\[
\mathbf{D} = \varepsilon_0 \mathbf{E} \\
\mathbf{B} = \mu_0 \mathbf{H}
\]

\( \varepsilon_0 \approx 8.8541878 \times 10^{-12} \) [F/m]

\( \mu_0 = 4\pi \times 10^{-7} \) [H/m] (exact*)

\( c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \)

\( c \equiv 2.99792458 \times 10^8 \) [m/s] (exact value that is defined)

*Prior to 2019
Definition of the Amp*: *Prior to 2019

Two infinite wires carrying DC currents

From ECE 3318:

\[ F_{x2} = \frac{I^2 \mu_0}{2\pi d} \]

\[ \mu_0 = 4\pi \times 10^{-7} \text{ [A/m]} \]

Definition of \( I = 1 \) Amp:

\[ F_{x2} = 2 \times 10^{-7} \text{ [N/m]} \text{ when } d = 1 \text{ [m]} \]
Free space, in the **phasor** domain:

\[
\begin{align*}
D &= \varepsilon_0 E \\
B &= \mu_0 H
\end{align*}
\]

\(\varepsilon_0 = \text{permittivity} \)  \\
\(\mu_0 = \text{permeability} \)

This follows from the fact that

\[ a'V(t) \iff aV \]

(\(a\) is a real number)
Given the following electric field in free space:

\[ \mathcal{E}(t) = \hat{\theta}\left( E_0 \cos(\omega t - k_0 r + \phi_0) \right) \frac{1}{r} \sin \theta \]

\[ k_0 = \omega \sqrt{\mu_0 \epsilon_0} \]

Find the magnetic field.

In the phasor domain:

\[ \nabla \times \mathcal{E} = -j\omega \mathcal{B} \]

\[ \nabla \times \mathcal{H} = \mathcal{J} + j\omega \mathcal{D} \]

\[ \nabla \cdot \mathcal{B} = 0 \]

\[ \nabla \cdot \mathcal{D} = \rho_v \]

So

\[ \mathcal{H} = \frac{1}{-j\omega \mu_0} \nabla \times \mathcal{E} \]

\[ \nabla \times \mathcal{E} = \frac{1}{r} \frac{\partial}{\partial \phi} \left[ \frac{\partial \mathcal{E}_\phi}{\partial \phi} \sin \theta \right] - \frac{1}{r} \frac{\partial}{\partial \rho} \left[ \frac{\partial \mathcal{E}_\rho}{\partial \rho} \right] + \frac{1}{r} \frac{\partial}{\partial \theta} \left[ \frac{\partial \mathcal{E}_\theta}{\partial \theta} \right] \]

(no \ \phi \ \text{variation})
Example (cont.)

\[ E = \hat{\theta} \left( E_0 e^{j\phi_0} \right) \frac{1}{r} e^{-jk_0r} \sin \theta \]

\[ \nabla \times E = \hat{\phi} \frac{1}{r} \left[ \frac{\partial (rE_{\theta})}{\partial r} \right] \]

\[ = \hat{\phi} \frac{1}{r} \left[ \frac{\partial \left( r \left( E_0 e^{j\phi_0} \frac{1}{r} e^{-jk_0r} \right) \sin \theta \right)}{\partial r} \right] \]

\[ = \hat{\phi} \frac{1}{r} e^{j\phi_0} E_0 \sin \theta \left[ \frac{\partial \left( e^{-jk_0r} \right)}{\partial r} \right] \]

\[ = \hat{\phi} \frac{1}{r} e^{j\phi_0} E_0 \sin \theta (-jk_0) e^{-jk_0r} \]

\[ H = \frac{1}{-j\omega\mu_0} \nabla \times E \]

\[ H = \frac{1}{-j\omega\mu_0} \hat{\phi} e^{j\phi_0} E_0 \sin \theta (-jk_0) e^{-jk_0r} \]

\[ H = \hat{\phi} \left[ E_0 \left( \frac{k_0}{\omega\mu_0} \right) e^{j\phi_0} \right] \frac{1}{r} \sin \theta e^{-jk_0r} \]

\[ \mathcal{H} = \hat{\phi} \left[ E_0 \left( \frac{k_0}{\omega\mu_0} \right) \right] \frac{1}{r} \sin \theta \cos (\omega t - k_0 r + \phi) \]
Alternative approach (in the time domain directly):

\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \]

\[ \frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} \]

\[ \nabla \times \mathbf{E} = \hat{\phi} \frac{1}{r} \left[ \frac{\partial (r E_\theta)}{\partial r} \right] \]

\[ = \hat{\phi} \frac{1}{r} \left[ \frac{\partial \left( r \left( E_0 \frac{1}{r} \cos(\omega t - k_0 r + \phi) \right) \sin \theta \right)}{\partial r} \right] \]

\[ = \hat{\phi} \frac{1}{r} E_0 \sin \theta \left[ \frac{\partial \left( \cos(\omega t - k_0 r + \phi) \right)}{\partial r} \right] \]

\[ = \hat{\phi} \frac{1}{r} E_0 \sin \theta (-k_0) \left( -\sin (\omega t - k_0 r + \phi) \right) \]
\[
\begin{align*}
\frac{\partial \mathcal{B}}{\partial t} &= -\nabla \times \mathcal{E} \\
\nabla \times \mathcal{E} &= \hat{\phi} \frac{1}{r} E_0 \sin \theta (k_0) \left( \sin (\omega t - k_0 r + \phi) \right)
\end{align*}
\]

So

\[
\frac{\partial \mathcal{B}}{\partial t} = -\hat{\phi} k_0 E_0 \frac{1}{r} \sin \theta \left( \sin (\omega t - k_0 r + \phi) \right)
\]

\[
\mathcal{B} = -\hat{\phi} k_0 E_0 \frac{1}{r} \sin \theta \frac{1}{\omega} \left( -\cos (\omega t - k_0 r + \phi) \right) + C(r, \theta, \phi)
\]

\[
\mathcal{H} = \hat{\phi} \left[ E_0 \frac{k_0}{\omega \mu_0} \right] \frac{1}{r} \sin \theta \left( \cos (\omega t - k_0 r + \phi) \right)
\]

All fields must be pure sinusoidal waves in the time-harmonic steady state.
This describes the far-field radiation from a small vertical dipole antenna.
In a **material medium**:

\[
\begin{align*}
D &= \varepsilon E \quad (\varepsilon = \text{permittivity}) \\
B &= \mu H \quad (\mu = \text{permeability})
\end{align*}
\]

\[
\begin{align*}
\varepsilon &= \varepsilon_0 \varepsilon_r \quad \varepsilon_r = \text{relative permittivity} \\
\mu &= \mu_0 \mu_r \quad \mu_r = \text{relative permeability}
\end{align*}
\]
Where does permittivity come from?

\[ D \equiv \varepsilon_0 E + P \]

\[ P \equiv \frac{1}{\Delta V} \sum_{\Delta V} p_i \]

\[ p_i = p_{\hat{l}_i} \]

\[ p = qd \]
Material Properties (cont.)

\[ D \equiv \varepsilon_0 E + P \]

Linear material:

\[ P = \varepsilon_0 \chi_e E \]

Note: \( \chi_e > 0 \) for most materials

The term \( \chi_e \) is called the “electric susceptibility.”

So

\[ D = \varepsilon_0 E + \varepsilon_0 \chi_e E \]

\[ = \varepsilon_0 (1 + \chi_e) E \]

Define: \( \varepsilon_r \equiv 1 + \chi_e \)

Then

\[ D = \varepsilon_0 \varepsilon_r E \]
Material Properties (cont.)

Teflon \( \varepsilon_r = 2.2 \)

Water \( \varepsilon_r = 81 \) (a very polar molecule, fairly free to rotate)

Styrofoam \( \varepsilon_r = 1.03 \)

Quartz \( \varepsilon_r = 5 \)

Note: \( \varepsilon_r > 1 \) for most materials: \( \varepsilon_r \equiv 1 + \chi_e, \quad \chi_e > 0 \)
Where does permeability come from?

Because of *electron spin*, atoms tend to acts as little current loops, and hence as electromagnetics, or bar magnets. When a magnetic field is applied, the little atomic magnets tend to line up.

\[
\mathcal{H} \equiv \frac{1}{\mu_0} \mathcal{B} - \mathcal{M}
\]

\[
\mathcal{M} \equiv \frac{1}{\Delta V} \sum_{\Delta V} m_i \\
 m_i = \hat{n}_i (iA)
\]

**Electron:**

\[
A = \pi a^2
\]
\[ \mathbf{B} = \mu_0 \mathbf{H} + \mu_0 \mathbf{M} \]

Linear material: \[ \mathbf{M} = \chi_m \mathbf{H} \]

The term \( \chi_m \) is called the "magnetic susceptibility."

Note: \( \chi_m > 0 \) for most materials

So

\[ \mathbf{B} = \mu_0 \mathbf{H} + \mu_0 \chi_m \mathbf{H} \]

\[ = \mu_0 (1 + \chi_m) \mathbf{H} \]

Define: \( \mu_r = (1 + \chi_m) \)

Then

\[ \mathbf{B} = \mu_0 \mu_r \mathbf{H} \]
Material Properties (cont.)

<table>
<thead>
<tr>
<th>Material</th>
<th>Relative Permeability $\mu_r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vacuum</td>
<td>1</td>
</tr>
<tr>
<td>Air</td>
<td>1.0000004</td>
</tr>
<tr>
<td>Water</td>
<td>0.999992</td>
</tr>
<tr>
<td>Copper</td>
<td>0.999994</td>
</tr>
<tr>
<td>Aluminum</td>
<td>1.00002</td>
</tr>
<tr>
<td>Silver</td>
<td>0.99998</td>
</tr>
<tr>
<td>Nickel</td>
<td>600</td>
</tr>
<tr>
<td>Iron</td>
<td>5000</td>
</tr>
<tr>
<td>Carbon Steel</td>
<td>100</td>
</tr>
<tr>
<td>Transformer Steel</td>
<td>2000</td>
</tr>
<tr>
<td>Mumetal</td>
<td>50,000</td>
</tr>
<tr>
<td>Supermalloy</td>
<td>1,000,000</td>
</tr>
</tbody>
</table>

**Note:** Values can often vary depending on purity and processing.

http://en.wikipedia.org/wiki/Permeability_(electromagnetism)
The fields $\mathbf{E}$ and $\mathbf{B}$ are the two physical fields, since they exert a force on a particle (the Lorenz force law). The $\mathbf{D}$ and $\mathbf{H}$ fields are the defined fields.

Lorenz force law:

$$\mathbf{F} = q \left( \mathbf{E} + \mathbf{v} \times \mathbf{B} \right)$$

This experimental law gives us the force on a particle with charge $q$ moving with a velocity vector $\mathbf{v}$. 
## Terminology

### Properties of $\varepsilon$ or $\mu$

<table>
<thead>
<tr>
<th>Variation</th>
<th>Independent of</th>
<th>Dependent on</th>
</tr>
</thead>
<tbody>
<tr>
<td>Space</td>
<td>Homogenous</td>
<td>Inhomogeneous</td>
</tr>
<tr>
<td>Frequency</td>
<td>Non-dispersive</td>
<td>Dispersive</td>
</tr>
<tr>
<td>Time</td>
<td>Stationary</td>
<td>Time-varying</td>
</tr>
<tr>
<td>Field strength</td>
<td>Linear</td>
<td>Non-linear</td>
</tr>
<tr>
<td>Direction of $E$ or $H$</td>
<td>Isotropic</td>
<td>Anisotropic</td>
</tr>
</tbody>
</table>
Isotropic: This means that $\varepsilon$ and $\mu$ are scalar quantities, which means that $D \parallel E$ (and $B \parallel H$)

\[ D = \varepsilon E \]
\[ B = \mu H \]
Here $\varepsilon$ (or $\mu$) is a tensor (can be written as a matrix)

This results in $E$ and $D$ being **NOT** proportional to each other.

**Example:**

\[
\begin{aligned}
D_x &= \varepsilon_x E_x \\
D_y &= \varepsilon_y E_y \\
D_z &= \varepsilon_z E_z \\
\end{aligned}
\]

or

\[
D = \varepsilon \cdot E
\]
Practical example: **uniaxial** substrate material

\[
\begin{bmatrix}
D_x \\
D_y \\
D_z \\
\end{bmatrix} =
\begin{bmatrix}
\varepsilon_h & 0 & 0 \\
0 & \varepsilon_h & 0 \\
0 & 0 & \varepsilon_v \\
\end{bmatrix}
\begin{bmatrix}
E_x \\
E_y \\
E_z \\
\end{bmatrix}
\]

Teflon substrate

There are two different permittivity values, a **horizontal** one and a **vertical** one.

Fibers (horizontal)
Anisotropic Materials (cont.)

RT/duroid® 5870/5880/5880LZ High Frequency Laminates

This column indicates that $\varepsilon_r$ is being measured.

<table>
<thead>
<tr>
<th>PROPERTY</th>
<th>TYPICAL VALUE $^{[2]}$</th>
<th>DIRECTION</th>
<th>UNITS$^{[3]}$</th>
<th>CONDITION</th>
<th>TEST METHOD</th>
</tr>
</thead>
</table>
| [1] Dielectric Constant, $\varepsilon_r$  
Process                      | 2.33                   | 2.20      | Z            | C24/23/50   | 1 MHz IPC-TM-650 2.5.5.3                  |
|                              | 2.33 ± 0.02 spec.    | 2.20 ± 0.02 spec. | Z            | C24/23/50   | 10 GHz IPC-TM 2.5.5.5                    |
| [2] Dielectric Constant, $\varepsilon_r$  
Design                      | 2.33                   | 2.20      | Z            | C24/23/50   | 8 GHz - 40 GHz Differential Phase Length Method |
| Dissipation Factor, tan $\delta$            | 0.0005                | 0.0004    | Z            | C24/23/50   | 1 MHz IPC-TM-650, 2.5.5.3                 |
|                              | 0.0012                | 0.0009    | Z            | C24/23/50   | 10 GHz IPC-TM-2.5.5.5                    |
| Thermal Coefficient of $\varepsilon_r$             | -115                  | -125      | ppm/°C      | -50 - 150°C | IPC-TM-650, 2.5.5.5                      |
| Volume Resistivity             | $2 \times 10^7$       | $2 \times 10^9$  | Z            | C96/35/90   | ASTM D257                                |
| Surface Resistivity            | $2 \times 10^7$       | $3 \times 10^9$  | Z            | C96/35/90   | ASTM D257                                |