Maxwell’s Equations
Here we present an overview of Maxwell’s equations. A much more thorough discussion of Maxwell’s equations may be found in the class notes for ECE 3318:

http://courses.egr.uh.edu/ECE/ECE3318

Notes 10: Electric Gauss’s law
Notes 18: Faraday’s law
Notes 28: Ampere’s law
Notes 28: Magnetic Gauss law

**Electromagnetic Fields**

### Four vector quantities

- $\mathbf{E}$ electric field strength  
  - [Volt/meter]
- $\mathbf{D}$ electric flux density  
  - [Coulomb/meter$^2$]
- $\mathbf{H}$ magnetic field strength  
  - [Amp/meter]
- $\mathbf{B}$ magnetic flux density  
  - [Weber/meter$^2$] or [Tesla]

Each are functions of space and time  
  e.g. $\mathbf{E}(x,y,z,t)$

- $\mathbf{J}$ electric current density  
  - [Amp/meter$^2$]
- $\rho_v$ electric charge density  
  - [Coulomb/meter$^3$]
Some common prefixes and the power of ten each represent are listed below

- femto - f - 10^{-15}
- pico - p - 10^{-12}
- nano - n - 10^{-9}
- micro - μ - 10^{-6}
- milli - m - 10^{-3}
- centi - c - 10^{-2}
- deci - d - 10^{-1}
- deka - da - 10^{1}
- hecto - h - 10^{2}
- kilo - k - 10^{3}
- mega - M - 10^{6}
- giga - G - 10^{9}
- tera - T - 10^{12}
- peta - P - 10^{15}
Maxwell’s Equations

(Time-varying, differential form)

\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \]
\[ \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \]
\[ \nabla \cdot \mathbf{B} = 0 \]
\[ \nabla \cdot \mathbf{D} = \rho_v \]
James Clerk Maxwell (1831–1879)

James Clerk Maxwell was a Scottish mathematician and theoretical physicist. His most significant achievement was the development of the classical electromagnetic theory, synthesizing all previous unrelated observations, experiments and equations of electricity, magnetism and even optics into a consistent theory. His set of equations—Maxwell's equations—demonstrated that electricity, magnetism and even light are all manifestations of the same phenomenon: the electromagnetic field. From that moment on, all other classical laws or equations of these disciplines became simplified cases of Maxwell's equations. Maxwell's work in electromagnetism has been called the "second great unification in physics", after the first one carried out by Isaac Newton.

Maxwell demonstrated that electric and magnetic fields travel through space in the form of waves, and at the constant speed of light. Finally, in 1864 Maxwell wrote A Dynamical Theory of the Electromagnetic Field where he first proposed that light was in fact undulations in the same medium that is the cause of electric and magnetic phenomena. His work in producing a unified model of electromagnetism is considered to be one of the greatest advances in physics.

(Wikipedia)
Maxwell’s Equations (cont.)

\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \text{Faraday’s law} \]

\[ \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \quad \text{Ampere’s law} \]

\[ \nabla \cdot \mathbf{B} = 0 \quad \text{Magnetic Gauss law} \]

\[ \nabla \cdot \mathbf{D} = \rho_v \quad \text{Electric Gauss law} \]
$\vec{J} = \sigma \vec{E}$

Ohm’s law

$\Delta I = |\vec{J}| \Delta S$
Ohm’s law

\[ \mathbf{J} = \sigma \mathbf{E} \]

\[ \Delta I = \left( \mathbf{J} \cdot \mathbf{n} \right) \Delta S \]
$\Delta I = \left( \mathbf{J} \cdot \hat{n} \right) \Delta S$

$I = \int_S \mathbf{J} \cdot \hat{n} \, dS$

**Note:**
The direction of the unit normal vector determines whether the current is measured going in or out.
Law of Conservation of Electric Charge (Continuity Equation)

\[ \nabla \times \mathcal{H} = \mathcal{J} + \frac{\partial \mathcal{D}}{\partial t} \]

\[ \nabla \cdot (\nabla \times \mathcal{H}) = \nabla \cdot \mathcal{J} + \nabla \cdot \left( \frac{\partial \mathcal{D}}{\partial t} \right) \]

\[ 0 = \nabla \cdot \mathcal{J} + \frac{\partial}{\partial t}(\nabla \cdot \mathcal{D}) \]

Flow of electric current out of volume (per unit volume)

\[ \nabla \cdot \mathcal{J} = -\frac{\partial \rho_v}{\partial t} \]

Rate of decrease of electric charge (per unit volume)
Integrate both sides over an arbitrary volume $V$:

$$
\int_V \nabla \cdot \underline{\mathcal{I}} = \int_V -\frac{\partial \rho_v}{\partial t} \, dV
$$

Apply the divergence theorem:

$$
\int_V \nabla \cdot \underline{\mathcal{I}} = \oint_S \underline{\mathcal{I}} \cdot \hat{n}
$$

Hence:

$$
\oint_S \underline{\mathcal{I}} \cdot \hat{n} = \int_V -\frac{\partial \rho_v}{\partial t} \, dV
$$
Continuity Equation (cont.)

\[ \int_S \mathbf{J} \cdot \hat{n} = \int_V -\frac{\partial \rho_v}{\partial t} dV \]

Physical interpretation:

\[ i_{out} = \int_V -\frac{\partial \rho_v}{\partial t} dV = -\frac{\partial}{\partial t} \int_V \rho_v dV \]

(This assumes that the surface is stationary.)

\[ i_{out} = -\frac{\partial Q_{encl}}{\partial t} \quad \text{or} \quad i_{in} = \frac{\partial Q_{encl}}{\partial t} \]
Continuity Equation (cont.)

This implies that charge is never created or destroyed. It only moves from one place to another!
Maxwell’s Equations (cont.)

Time - Dependent

\[ \nabla \times E = -\frac{\partial B}{\partial t} \quad \nabla \times H = J + \frac{\partial D}{\partial t} \quad \nabla \cdot B = 0 \quad \nabla \cdot D = \rho_v \]

Time - Independent (Statics)

\[ \nabla \times E = 0 \quad \nabla \cdot D = \rho_v \quad \nabla \times H = J \quad \nabla \cdot B = 0 \]

Decouples \( E \) and \( H \) \( \Rightarrow \) \( E \) comes from \( \rho_v \) and \( H \) comes from \( J \)

**Note:** Regular (not script) font is used for statics, just as it is for phasors.
Maxwell’s Equations (cont.)

Time-harmonic (phasor) domain

\[
\nabla \times \mathbf{E} = -j\omega \mathbf{B} \\
\nabla \times \mathbf{H} = \mathbf{J} + j\omega \mathbf{D} \\
\n\nabla \cdot \mathbf{B} = 0 \\
\n\n\nabla \cdot \mathbf{D} = \rho_v
\]
The characteristics of the media relate $\mathbf{D}$ to $\mathbf{E}$ and $\mathbf{H}$ to $\mathbf{B}$.

**Free Space**

$$
\mathbf{D} = \varepsilon_0 \mathbf{E} \quad (\varepsilon_0 = \text{permittivity})
$$

$$
\mathbf{B} = \mu_0 \mathbf{H} \quad (\mu_0 = \text{permeability})
$$

$$
\varepsilon_0 \doteq 8.8541878 \times 10^{-12} \, \text{[F/m]}
$$

$$
\mu_0 = 4\pi \times 10^{-7} \, \text{[H/m]} \quad \text{(exact)}
$$

$$
c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}
$$

$$
c \equiv 2.99792458 \times 10^8 \, \text{[m/s]} \quad \text{(exact value that is defined)}
$$
Definition of the Amp:

Two infinite wires carrying DC currents

From ECE 3318:

\[
F_{x2} = \frac{I^2 \mu_0}{2\pi d}
\]

\[
\mu_0 = 4\pi \times 10^{-7} \quad [\text{A/m}]
\]

Definition of \( I = 1 \) Amp:

\[
F_{x2} = 2 \times 10^{-7} \quad [\text{N/m}] \quad \text{when} \quad d = 1 \quad [\text{m}]
\]
Free space, in the **phasor domain**:

\[
D = \varepsilon_0 E \quad (\varepsilon_0 = \text{permittivity})
\]

\[
B = \mu_0 H \quad (\mu_0 = \text{permeability})
\]

This follows from the fact that

\[
a'V(t) \Leftrightarrow aV
\]

(where \(a\) is a real number)
In a material medium:

\[ D = \varepsilon E \quad (\varepsilon = \text{permittivity}) \]

\[ B = \mu H \quad (\mu = \text{permeability}) \]

\[ \varepsilon = \varepsilon_0 \varepsilon_r \quad \varepsilon_r = \text{relative permittivity} \]

\[ \mu = \mu_0 \mu_r \quad \mu_r = \text{relative permittivity} \]
Where does permittivity come from?

$$D \equiv \varepsilon_0 E + P$$

$$P \equiv \frac{1}{\Delta V} \sum_{\Delta V} p_i$$

$$p_i = p_{\hat{l}_i}$$

$$p = qd$$
Constitutive Relations (cont.)

\[ D = \varepsilon_0 E + P \]

Linear material:

\[ P = \varepsilon_0 \chi_e E \]

The term \( \chi_e \) is called the “electric susceptibility.”

Note: \( \chi_e > 0 \) for most materials

so

\[ D = \varepsilon_0 E + \varepsilon_0 \chi_e E = \varepsilon_0 (1 + \chi_e) E \]

Define: \( \varepsilon_r \equiv 1 + \chi_e \)

Then

\[ D = \varepsilon_0 \varepsilon_r E \]
Constitutive Relations (cont.)

Teflon \( \varepsilon_r = 2.2 \)

Water \( \varepsilon_r = 81 \) (a very polar molecule, fairly free to rotate)

Styrofoam \( \varepsilon_r = 1.03 \)

Quartz \( \varepsilon_r = 5 \)

Note: \( \varepsilon_r > 1 \) for most materials: \( \varepsilon_r \equiv 1 + \chi_e, \ \chi_e > 0 \)
Where does permeability come from?

Because of electron spin, atoms tend to act as little current loops, and hence as electromagnetics, or bar magnets. When a magnetic field is applied, the little atomic magnets tend to line up.

\[
\mathcal{M} \equiv \frac{1}{\mu_0} \mathbf{B} - \mathcal{M}
\]

\[
\mathcal{M} \equiv \frac{1}{\Delta V} \sum_{\Delta V} m_i
\]

\[
m_i = \hat{n}_i (iA)
\]
Constitutive Relations (cont.)

\[ \mathbf{B} = \mu_0 \mathbf{H} + \mu_0 \mathbf{M} \]

Linear material:

\[ \mathbf{M} = \chi_m \mathbf{H} \]

The term \( \chi_m \) is called the "magnetic susceptibility."

Note: \( \chi_m > 0 \) for most materials

So

\[ \mathbf{B} = \mu_0 \mathbf{H} + \mu_0 \chi_m \mathbf{H} \]

\[ = \mu_0 (1 + \chi_m) \mathbf{H} \]

Define: \( \mu_r = (1 + \chi_m) \)

Then

\[ \mathbf{B} = \mu_0 \mu_r \mathbf{H} \]
Constitutive Relations (cont.)

<table>
<thead>
<tr>
<th>Material</th>
<th>Relative Permeability $\mu_r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vacuum</td>
<td>1</td>
</tr>
<tr>
<td>Air</td>
<td>1.0000004</td>
</tr>
<tr>
<td>Water</td>
<td>0.999992</td>
</tr>
<tr>
<td>Copper</td>
<td>0.999994</td>
</tr>
<tr>
<td>Aluminum</td>
<td>1.00002</td>
</tr>
<tr>
<td>Silver</td>
<td>0.99998</td>
</tr>
<tr>
<td>Nickel</td>
<td>600</td>
</tr>
<tr>
<td>Iron</td>
<td>5000</td>
</tr>
<tr>
<td>Carbon Steel</td>
<td>100</td>
</tr>
<tr>
<td>Transformer Steel</td>
<td>2000</td>
</tr>
<tr>
<td>Mumetal</td>
<td>50,000</td>
</tr>
<tr>
<td>Supermalloy</td>
<td>1,000,000</td>
</tr>
</tbody>
</table>

Note: Values can often vary depending on purity and processing.

http://en.wikipedia.org/wiki/Permeability_(electromagnetism)
## Terminology

### Properties of $\varepsilon$ or $\mu$

<table>
<thead>
<tr>
<th>Variation</th>
<th>Independent of</th>
<th>Dependent on</th>
</tr>
</thead>
<tbody>
<tr>
<td>Space</td>
<td>Homogenous</td>
<td>Inhomogeneous</td>
</tr>
<tr>
<td>Frequency</td>
<td>Non-dispersive</td>
<td>Dispersive</td>
</tr>
<tr>
<td>Time</td>
<td>Stationary</td>
<td>Non-stationary</td>
</tr>
<tr>
<td>Field strength</td>
<td>Linear</td>
<td>Non-linear</td>
</tr>
<tr>
<td>Direction of $E$ or $H$</td>
<td>Isotropic</td>
<td>Anisotropic</td>
</tr>
</tbody>
</table>
$\varepsilon$ (and $\mu$) are scalar quantities, which means that $D \parallel E$ (and $B \parallel H$)

\[
D = \varepsilon E \\
B = \mu H
\]
\( \varepsilon \) (or \( \mu \)) is a tensor (can be written as a matrix)

Example:

\[
\begin{align*}
D_x &= \varepsilon_x E_x \\
D_y &= \varepsilon_y E_y \\
D_z &= \varepsilon_z E_z
\end{align*}
\]

\[
\begin{bmatrix}
D_x \\
D_y \\
D_z
\end{bmatrix} =
\begin{bmatrix}
\varepsilon_x & 0 & 0 \\
0 & \varepsilon_y & 0 \\
0 & 0 & \varepsilon_z
\end{bmatrix}
\begin{bmatrix}
E_x \\
E_y \\
E_z
\end{bmatrix}
\]

or

\[
D = \varepsilon \cdot E
\]

This results in \( E \) and \( D \) being NOT proportional to each other.
Anisotropic Materials (cont.)

Practical example: **uniaxial** substrate material

\[
\begin{bmatrix}
D_x \\
D_y \\
D_z
\end{bmatrix} =
\begin{bmatrix}
\varepsilon_h & 0 & 0 \\
0 & \varepsilon_h & 0 \\
0 & 0 & \varepsilon_v
\end{bmatrix}
\begin{bmatrix}
E_x \\
E_y \\
E_z
\end{bmatrix}
\]

Teflon substrate

There are **two** different permittivity values, a **horizontal** one and a **vertical** one.

Fibers (horizontal)
RT/duroid® 5870/5880/5880LZ High Frequency Laminates

This column indicates that $\varepsilon_v$ is being measured.

<table>
<thead>
<tr>
<th>PROPERTY</th>
<th>TYPICAL VALUE [R]</th>
<th>DIRECTION</th>
<th>UNITS [S]</th>
<th>CONDITION</th>
<th>TEST METHOD</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dielectric Constant, $\varepsilon_T$</td>
<td>2.33</td>
<td>2.20</td>
<td>Z</td>
<td>C24/23/50</td>
<td>1 MHz IPC-TM-650 2.5.5.3</td>
</tr>
<tr>
<td>RT/duroid 5870</td>
<td>2.33 ± 0.02 spec.</td>
<td>Z</td>
<td>10 GHz IPC-TM-2.5.5.5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>RT/duroid 5880</td>
<td>2.20 ± 0.02 spec.</td>
<td>Z</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Design</td>
<td>2.33</td>
<td>2.20</td>
<td>Z</td>
<td>C24/23/50</td>
<td>Differential Phase Length Method</td>
</tr>
<tr>
<td>Dissipation Factor, $\tan \delta$</td>
<td>0.0005</td>
<td>0.0004</td>
<td>Z</td>
<td>C24/23/50</td>
<td>1 MHz IPC-TM-650, 2.5.5.3</td>
</tr>
<tr>
<td>RT/duroid 5870</td>
<td>0.0012</td>
<td>0.0009</td>
<td>Z</td>
<td>C24/23/50</td>
<td>10 GHz IPC-TM-2.5.5.5</td>
</tr>
<tr>
<td>RT/duroid 5880</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Thermal Coefficient of $\varepsilon_T$</td>
<td>-115</td>
<td>-125</td>
<td>ppm/°C</td>
<td>-50 - 150°C</td>
<td>IPC-TM-650, 2.5.5.5</td>
</tr>
<tr>
<td>Volume Resistivity</td>
<td>$2 \times 10^7$</td>
<td>$2 \times 10^9$</td>
<td>Z</td>
<td>C96/35/90</td>
<td>ASTM D257</td>
</tr>
<tr>
<td>Surface Resistivity</td>
<td>$2 \times 10^5$</td>
<td>$3 \times 10^7$</td>
<td>Z</td>
<td>C/96/35/90</td>
<td>ASTM D257</td>
</tr>
</tbody>
</table>