ECE 3317
Applied Electromagnetic Waves

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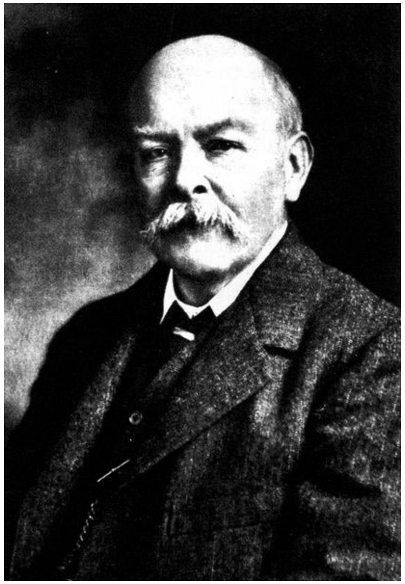
Fall 2023



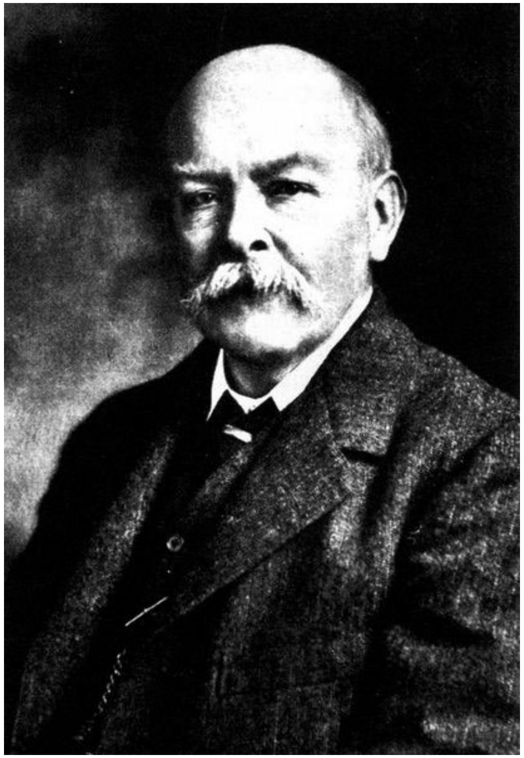
Notes 5
Poynting's Theorem

Adapted from notes by Prof. Stuart A. Long

Poynting Theorem

The Poynting theorem is one of the most important in EM theory. It tells us the power flowing in an electromagnetic field.

John Henry Poynting (1852-1914)



John Henry Poynting was an English physicist. He was a professor of physics at Mason Science College (now the University of Birmingham) from 1880 until his death.

He was the developer and eponym of the Poynting vector, which describes the direction and magnitude of electromagnetic energy flow and is used in the Poynting theorem, a statement about energy conservation for electric and magnetic fields. This work was first published in 1884. He performed a measurement of Newton's gravitational constant by innovative means during 1893. In 1903 he was the first to realize that the Sun's radiation can draw in small particles towards it. This was later coined the Poynting-Robertson effect.

In the year 1884 he analyzed the futures exchange prices of commodities using statistical mathematics.

(Wikipedia)

Poynting Theorem (cont.)

$$\nabla \times \underline{\mathscr{E}} = -\frac{\partial \underline{\mathscr{B}}}{\partial t}$$

$$\nabla \times \underline{\mathscr{H}} = \underline{\mathscr{J}} + \frac{\partial \underline{\mathscr{D}}}{\partial t}$$

From these we obtain:

$$\underline{\mathscr{H}} \cdot \left(\nabla \times \underline{\mathscr{E}} \right) = -\underline{\mathscr{H}} \cdot \frac{\partial \underline{\mathscr{B}}}{\partial t}$$

$$\underline{\mathscr{E}} \cdot \left(\nabla \times \underline{\mathscr{H}} \right) = \underline{\mathscr{J}} \cdot \underline{\mathscr{E}} + \underline{\mathscr{E}} \cdot \frac{\partial \underline{\mathscr{D}}}{\partial t}$$

Poynting Theorem (cont.)

$$\underline{\mathscr{H}} \cdot \left(\nabla \times \underline{\mathscr{E}}\right) = -\underline{\mathscr{H}} \cdot \frac{\partial \underline{\mathscr{B}}}{\partial t}$$

$$\underline{\mathscr{E}} \cdot \left(\nabla \times \underline{\mathscr{H}}\right) = \underline{\mathscr{J}} \cdot \underline{\mathscr{E}} + \underline{\mathscr{E}} \cdot \frac{\partial \underline{\mathscr{D}}}{\partial t}$$

Subtract, and use the following vector identity:

$$\underline{\mathscr{H}} \cdot \left(\nabla \times \underline{\mathscr{E}}\right) - \underline{\mathscr{E}} \cdot \left(\nabla \times \underline{\mathscr{H}}\right) = \nabla \cdot \left(\underline{\mathscr{E}} \times \underline{\mathscr{H}}\right)$$

We then have:

$$\nabla \cdot \left(\underline{\mathscr{E}} \times \underline{\mathscr{H}}\right) = -\underline{\mathscr{J}} \cdot \underline{\mathscr{E}} - \underline{\mathscr{H}} \cdot \frac{\partial \underline{\mathscr{B}}}{\partial t} - \underline{\mathscr{E}} \cdot \frac{\partial \underline{\mathscr{D}}}{\partial t}$$

Poynting Theorem (cont.)

$$\nabla \cdot \left(\underline{\mathscr{E}} \times \underline{\mathscr{H}} \right) = -\underline{\mathscr{J}} \cdot \underline{\mathscr{E}} - \underline{\mathscr{H}} \cdot \frac{\partial \underline{\mathscr{B}}}{\partial t} - \underline{\mathscr{E}} \cdot \frac{\partial \underline{\mathscr{D}}}{\partial t}$$

Next, assume that Ohm's law applies for the electric current:

$$\underline{\mathscr{J}} = \sigma \underline{\mathscr{E}}$$

$$\Rightarrow \quad \nabla \cdot \left(\underline{\mathscr{E}} \times \underline{\mathscr{H}} \right) = -\sigma \left(\underline{\mathscr{E}} \cdot \underline{\mathscr{E}} \right) - \underline{\mathscr{H}} \cdot \frac{\partial \underline{\mathscr{B}}}{\partial t} - \underline{\mathscr{E}} \cdot \frac{\partial \underline{\mathscr{D}}}{\partial t}$$

or

$$\nabla \cdot \left(\underline{\mathscr{E}} \times \underline{\mathscr{H}} \right) = -\sigma \left| \underline{\mathscr{E}} \right|^2 - \underline{\mathscr{H}} \cdot \frac{\partial \underline{\mathscr{B}}}{\partial t} - \underline{\mathscr{E}} \cdot \frac{\partial \underline{\mathscr{D}}}{\partial t}$$

Poynting Theorem (cont.)

$$\nabla \cdot \left(\underline{\mathcal{E}} \times \underline{\mathcal{H}} \right) = -\sigma \left| \underline{\mathcal{E}} \right|^2 - \underline{\mathcal{H}} \cdot \frac{\partial \underline{\mathcal{B}}}{\partial t} - \underline{\mathcal{E}} \cdot \frac{\partial \underline{\mathcal{D}}}{\partial t}$$

From calculus (chain rule), we have that

Note : $\frac{\partial}{\partial t}\left(\underline{A} \cdot \underline{B} \right) = \underline{A} \cdot \frac{\partial \underline{B}}{\partial t} + \frac{\partial \underline{A}}{\partial t} \cdot \underline{B}$

$\Rightarrow \frac{\partial}{\partial t}\left(\underline{A} \cdot \underline{A} \right) = 2 \underline{A} \cdot \frac{\partial \underline{A}}{\partial t}$

$$\underline{\mathcal{E}} \cdot \frac{\partial \underline{\mathcal{D}}}{\partial t} = \varepsilon \left(\underline{\mathcal{E}} \cdot \frac{\partial \underline{\mathcal{E}}}{\partial t} \right) = \varepsilon \frac{1}{2} \frac{\partial}{\partial t} \left(\underline{\mathcal{E}} \cdot \underline{\mathcal{E}} \right)$$

$$\underline{\mathcal{H}} \cdot \frac{\partial \underline{\mathcal{B}}}{\partial t} = \mu \left(\underline{\mathcal{H}} \cdot \frac{\partial \underline{\mathcal{H}}}{\partial t} \right) = \mu \frac{1}{2} \frac{\partial}{\partial t} \left(\underline{\mathcal{H}} \cdot \underline{\mathcal{H}} \right)$$

Hence, we have

$$\nabla \cdot \left(\underline{\mathcal{E}} \times \underline{\mathcal{H}} \right) = -\sigma \left| \underline{\mathcal{E}} \right|^2 - \mu \frac{1}{2} \frac{\partial}{\partial t} \left(\underline{\mathcal{H}} \cdot \underline{\mathcal{H}} \right) - \varepsilon \frac{1}{2} \frac{\partial}{\partial t} \left(\underline{\mathcal{E}} \cdot \underline{\mathcal{E}} \right)$$

Poynting Theorem (cont.)

$$\nabla \cdot \left(\underline{\mathscr{E}} \times \underline{\mathscr{H}} \right) = -\sigma \left| \underline{\mathscr{E}} \right|^2 - \mu \frac{1}{2} \frac{\partial}{\partial t} \left(\underline{\mathscr{H}} \cdot \underline{\mathscr{H}} \right) - \varepsilon \frac{1}{2} \frac{\partial}{\partial t} \left(\underline{\mathscr{E}} \cdot \underline{\mathscr{E}} \right)$$

This may be written as

$$\nabla \cdot \left(\underline{\mathscr{E}} \times \underline{\mathscr{H}} \right) = -\sigma \left| \underline{\mathscr{E}} \right|^2 - \frac{\partial}{\partial t} \left(\frac{1}{2} \mu \left| \underline{\mathscr{H}} \right|^2 \right) - \frac{\partial}{\partial t} \left(\frac{1}{2} \varepsilon \left| \underline{\mathscr{E}} \right|^2 \right)$$

Poynting Theorem (cont.)

Final differential (point) form of Poynting's theorem:

$$\nabla \cdot \left(\underline{\mathscr{E}} \times \underline{\mathscr{H}} \right) = -\sigma \left| \underline{\mathscr{E}} \right|^2 - \frac{\partial}{\partial t}\left(\frac{1}{2}\mu \left| \underline{\mathscr{H}} \right|^2 \right) - \frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon \left| \underline{\mathscr{E}} \right|^2 \right)$$

Poynting Theorem (cont.)

Volume (integral) form

Integrate both sides over a volume and then apply the divergence theorem:

$$\nabla \cdot \left(\underline{\mathcal{E}} \times \underline{\mathcal{H}} \right) = -\sigma \left| \underline{\mathcal{E}} \right|^2 - \frac{\partial}{\partial t}\left(\frac{1}{2}\mu \left| \underline{\mathcal{H}} \right|^2 \right) - \frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon \left| \underline{\mathcal{E}} \right|^2 \right)$$

$$\Downarrow$$

$$\boxed{\int_V \nabla \cdot \left(\underline{\mathcal{E}} \times \underline{\mathcal{H}} \right) dV} = -\int_V \sigma \left| \underline{\mathcal{E}} \right|^2 dV - \int_V \frac{\partial}{\partial t}\left(\frac{1}{2}\mu \left| \underline{\mathcal{H}} \right|^2 \right) dV - \int_V \frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon \left| \underline{\mathcal{E}} \right|^2 \right) dV$$

Divergence theorem

$$\Downarrow$$

$$\boxed{\oint_S \left(\underline{\mathcal{E}} \times \underline{\mathcal{H}} \right) \cdot \underline{\hat{n}}\, dS} = -\int_V \sigma \left| \underline{\mathcal{E}} \right|^2 dV - \int_V \frac{\partial}{\partial t}\left(\frac{1}{2}\mu \left| \underline{\mathcal{H}} \right|^2 \right) dV - \int_V \frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon \left| \underline{\mathcal{E}} \right|^2 \right) dV$$

Poynting Theorem (cont.)

Final volume form of Poynting theorem in the most general case:

$$\oint_S \left(\underline{\mathscr{E}} \times \underline{\mathscr{H}}\right) \cdot \underline{\hat{n}}\, dS = -\int_V \sigma \left|\underline{\mathscr{E}}\right|^2 dV - \int_V \frac{\partial}{\partial t}\left(\frac{1}{2}\mu\left|\underline{\mathscr{H}}\right|^2\right) dV - \int_V \frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon\left|\underline{\mathscr{E}}\right|^2\right) dV$$

stationary surface

For a stationary surface:

$$\oint_S \left(\underline{\mathscr{E}} \times \underline{\mathscr{H}}\right) \cdot \hat{n}\, dS = -\int_V \sigma \left|\underline{\mathscr{E}}\right|^2 dV - \frac{\partial}{\partial t}\int_V \left(\frac{1}{2}\mu\underline{\mathscr{H}}^2\right) dV - \frac{\partial}{\partial t}\int_V \left(\frac{1}{2}\varepsilon\left|\underline{\mathscr{E}}\right|^2\right) dV$$

Poynting Theorem (cont.)

Poynting's theorem: (We assume here that S is stationary.)

$$-\oint_S \left(\underline{\mathscr{E}} \times \underline{\mathscr{H}}\right) \cdot \underline{\hat{n}}\, dS = \int_V \sigma \left|\underline{\mathscr{E}}\right|^2 dV + \frac{\partial}{\partial t}\int_V \left(\frac{1}{2}\mu \underline{\mathscr{H}}^2\right) dV + \frac{\partial}{\partial t}\int_V \left(\frac{1}{2}\varepsilon \left|\underline{\mathscr{E}}\right|^2\right) dV$$

Power dissipation as heat (Joule's law)

Rate of change of stored magnetic energy

Rate of change of stored electric energy

Conservation of energy:

Left-hand side = power flowing <u>into</u> the region V.

Poynting Theorem (cont.)

Hence

$$-\oint_S \left(\underline{\mathscr{E}} \times \underline{\mathscr{H}}\right) \cdot \underline{\hat{n}}\, dS = \text{power flowing into the region}$$

Or, we can say that

$$\oint_S \left(\underline{\mathscr{E}} \times \underline{\mathscr{H}}\right) \cdot \underline{\hat{n}}\, dS = \text{power flowing out of the region}$$

Define the Poynting vector: $\underline{\mathscr{S}} \equiv \underline{\mathscr{E}} \times \underline{\mathscr{H}}$

$$\oint_S \underline{\mathscr{S}} \cdot \underline{\hat{n}}\, dS = \text{power flowing out of the region}$$

Poynting Theorem (cont.)

Analogy:

$$\oint_S \underline{\mathscr{S}} \cdot \underline{\hat{n}}\, dS = \text{power flowing out of the region}$$

$$\oint_S \underline{\mathscr{J}} \cdot \underline{\hat{n}}\, dS = \text{current flowing out of the region}$$

$\underline{\mathscr{J}} =$ current density vector

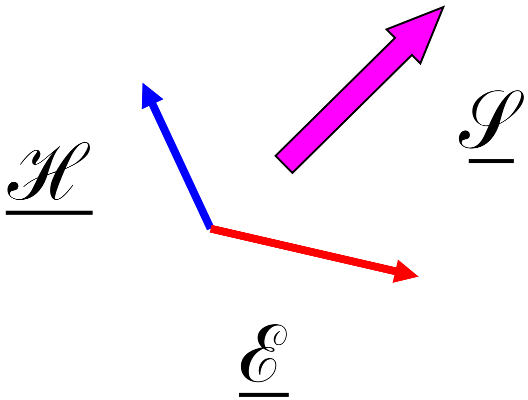
$\underline{\mathscr{S}} =$ power flow vector

Poynting Theorem (cont.)

Poynting vector

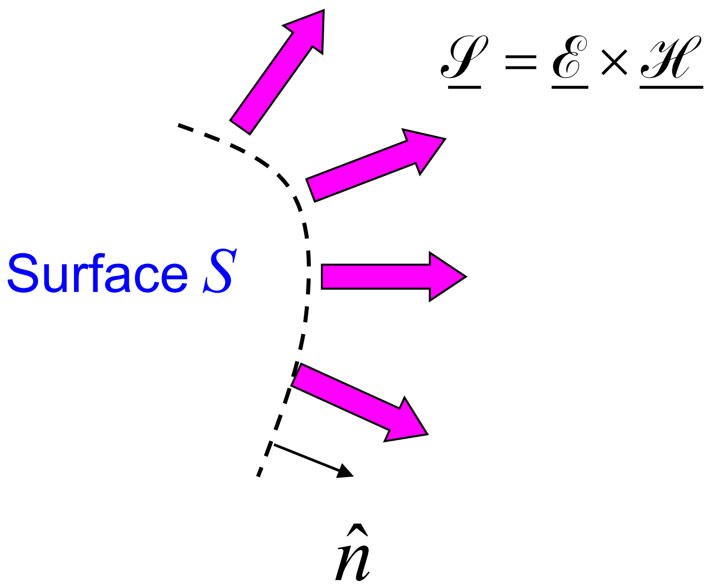
$$\underline{\mathscr{S}} = \underline{\mathscr{E}} \times \underline{\mathscr{H}}$$

Direction of power flow in $\mathrm{W/m^2}$



Note:
At high frequencies (e.g., light or x rays), it is often convenient to think of photons moving in space, carrying the power flow. *The two points of view are consistent*.

Power Flow



The power $\mathscr{P}$ flowing through the surface S (from left to right) is:

$$\mathscr{P}(t) = \int_S \underline{\mathscr{S}} \cdot \underline{\hat{n}}\, dS$$

Time-Average Poynting Vector

Assume sinusoidal (time-harmonic) fields:

$$\underline{\mathscr{E}}(x,y,z,t) = \mathrm{Re}\left\{\underline{E}(x,y,z)\,e^{j\omega t}\right\}$$

$$\underline{\mathscr{H}}(x,y,z,t) = \mathrm{Re}\left\{\underline{H}(x,y,z)\,e^{j\omega t}\right\}$$

From our previous discussion (Notes 2) about time averages, we know that

$$\left\langle \underline{\mathscr{S}}(t) \right\rangle = \left\langle \underline{\mathscr{E}}(t) \times \underline{\mathscr{H}}(t) \right\rangle = \frac{1}{2}\mathrm{Re}\left(\underline{E} \times \underline{H}^{*}\right)$$

Complex Poynting Vector

Define the complex Poynting vector:

$$\underline{S} \equiv \frac{1}{2}\left(\underline{E} \times \underline{H}^*\right)$$

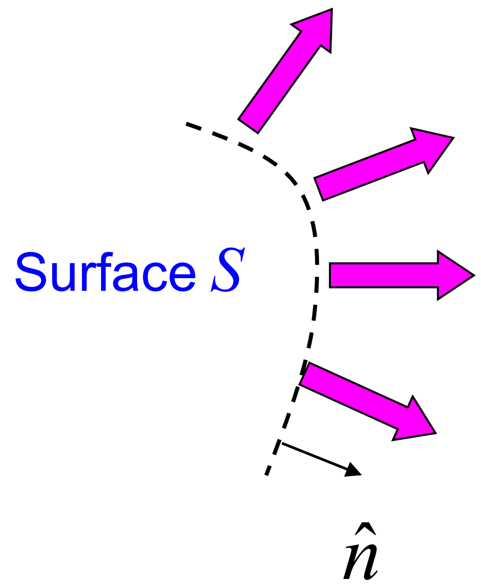
The units of $\underline{S}$ are [VA/m^2].

We then have that

$$\left\langle \underline{\mathscr{S}}(x,y,z,t) \right\rangle = \mathrm{Re}\left(\underline{S}(x,y,z)\right)$$

Note:
The imaginary part of the complex Poynting vector corresponds to the Vars/m^2 flowing in space.

Complex Poynting Vector (cont.)



$$\underline{S} = \frac{1}{2}\underline{E} \times \underline{H}^*$$

Note:
The direction of the unit normal determines whether the power flow is form left to right or from right to left.

The complex power P flowing through the surface S (from left to right in the above figure) is:

$$P = \int_S \underline{S} \cdot \underline{\hat{n}}\, dS$$

$\mathrm{Re}(\mathrm{P}) =$ Watts

$\mathrm{Im}(\mathrm{P}) =$ Vars

Complex Poynting Vector (cont.)

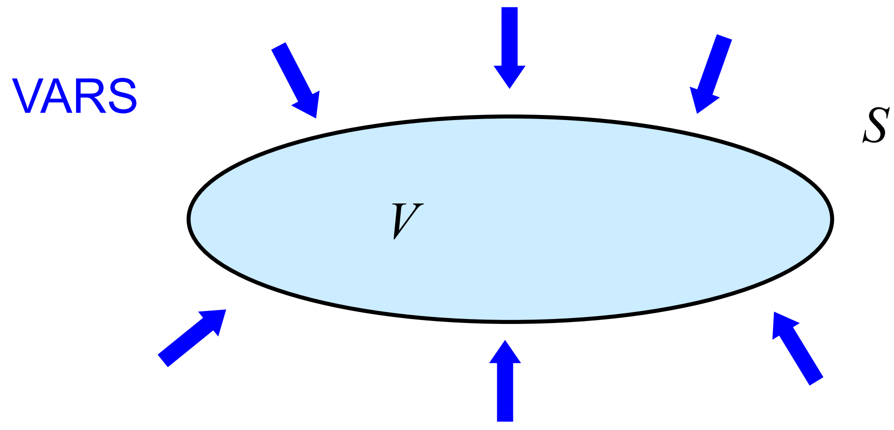
What does Vars flowing in space mean?

Equation for Vars flowing into a region (derivation omitted):

$$\text{Vars} = 2\omega \int_V \left(\frac{1}{4}\mu \left|\underline{H}\right|^2 - \frac{1}{4}\varepsilon \left|\underline{E}\right|^2 \right) dV$$

$$= 2\omega \left(\langle \mathscr{W}_m \rangle - \langle \mathscr{W}_e \rangle \right)$$

The Vars flowing into the region V is equal to the difference in the time-average magnetic and electric stored energies inside the region (times a factor of 2ω).



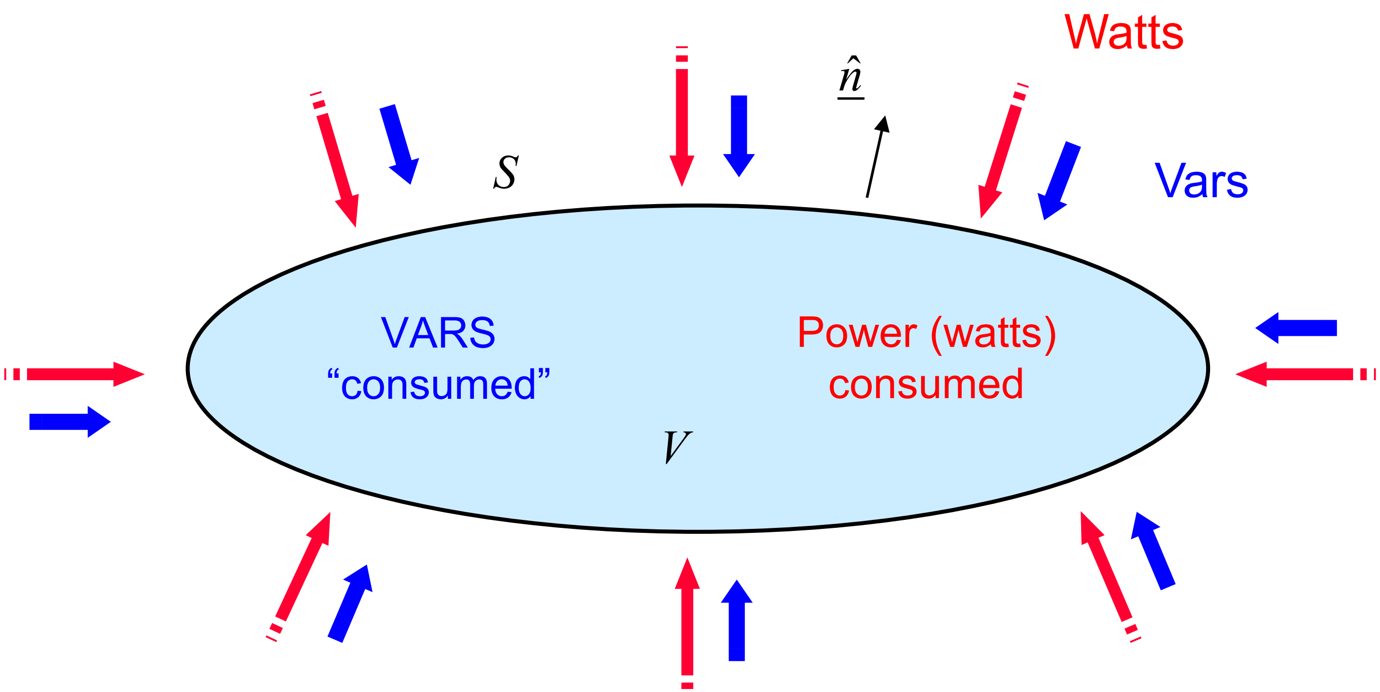
Question:
does an inductor absorb positive or negative Vars?

Complex Poynting Vector (cont.)

$$P = \int_S \underline{S} \cdot (-\hat{n})\, dS = \text{complex power flowing in} = \text{Watts} + j\,(\text{Vars})$$

$$\text{Watts} = \left\langle \int_V \sigma \left|\underline{\mathscr{E}}\right|^2 dV \right\rangle = \frac{1}{2} \int_V \sigma \left|\underline{E}\right|^2 dV$$ (heat dissipation inside the region)

$$\text{Vars} = 2\,\omega \left(\langle \mathscr{W}_m \rangle - \langle \mathscr{W}_e \rangle \right)$$



Photons

At high frequencies (e.g., light, x-rays), physicists like to think in terms of photons.

$$E_p = hf$$ (energy of photon)

$$h = 6.626068 \times 10^{-34} \ [\mathrm{J\,s}]$$
(Planck's constant)

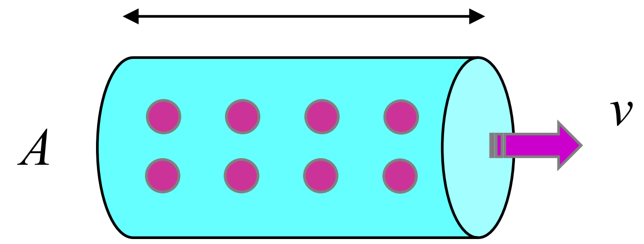
$$\langle \mathscr{S}_z \rangle = \frac{\text{Energy}}{A\,\Delta t} = \frac{E_p\big(A(c\,\Delta t)\big)N_p}{A\,\Delta t}$$

$$= E_p c\, N_p$$

$$\langle \mathscr{S}_z \rangle = c\, E_p\, N_p$$

Photons moving:

$\Delta\ell = c\Delta t$



$\underline{\mathscr{S}} = \mathscr{S}_z\, \underline{\hat{z}}$

N_p photons per m^3

Photons (cont.)

Example

A cell phone base station antenna transmits at 2.1 [GHz] with 100 [W] of power. At 1 [km] away the power density is 10^{-4} [W/m^2]. What is the photon density?

$$\langle \mathscr{S}_z \rangle = 10^{-4} = c\, E_p\, N_p = \left(2.99792458 \times 10^{8}\right)\left(6.626068 \times 10^{-34} \cdot 2.1 \times 10^{9}\right) N_p$$

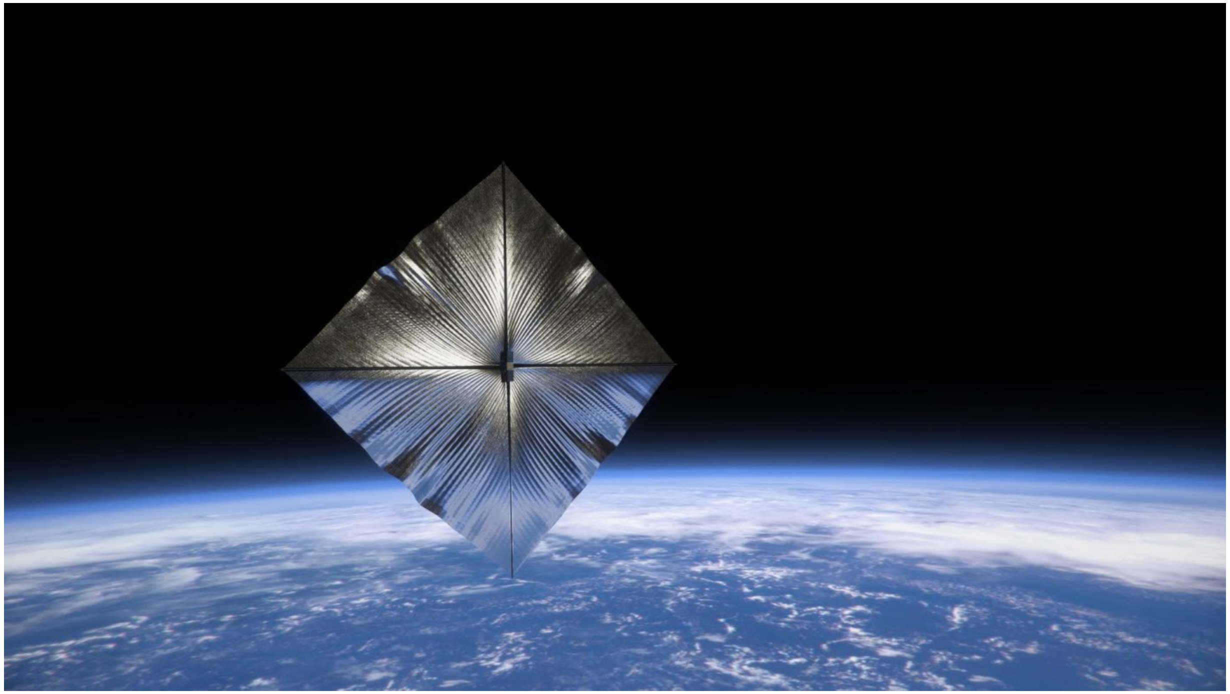
$$N_p = 2.40 \times 10^{11} \left[\text{photons/m}^3\right]$$

Photons (cont.)

Photons also carry momentum, and can therefore exert a force on objects.

$$p_p = \frac{E_p}{c}$$ (momentum of photon)

Solar Sail



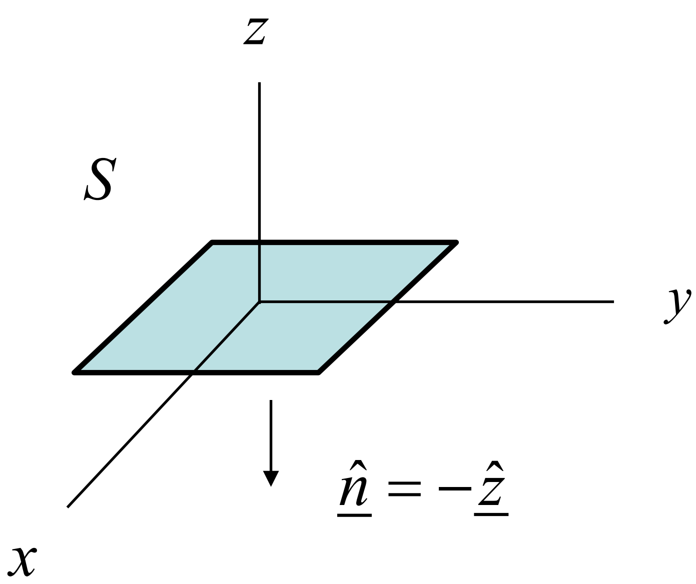
Example

At $z = 0$:

$$\underline{\mathscr{E}} = \underline{\hat{x}} \cos(\omega t)$$

$$\underline{\mathscr{H}} = \underline{\hat{y}} \cos(\omega t - \pi / 4) + \underline{\hat{z}} \sin(\omega t + 3\pi / 4)$$

Find the Watts and Vars crossing the plane $z = 0$ (downward) through a 1 m^2 area S.



Note :

$\sin x = \cos(x - \pi / 2)$

so

$\sin(\omega t + 3\pi / 4)$

$= \cos(\omega t + 3\pi / 4 - \pi / 2)$

$= \cos(\omega t + \pi / 4)$

Example (Cont.)

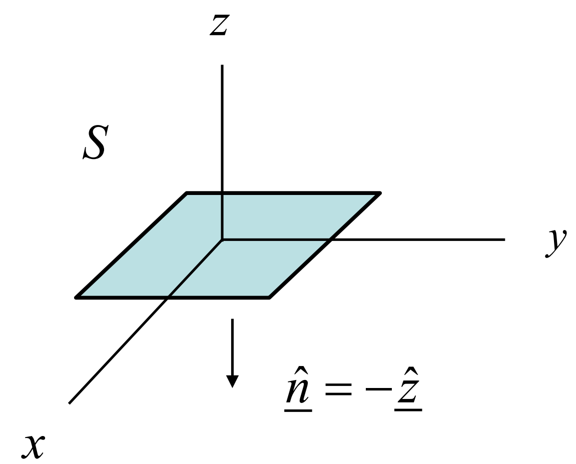
At $z = 0$:

$$\underline{E} = \underline{\hat{x}}$$

$$\underline{H} = \underline{\hat{y}}\, e^{-j\pi/4} + \underline{\hat{z}}\, e^{+j\pi/4}$$

(phasor domain)

$$\underline{S} = \frac{1}{2}\underline{E} \times \underline{H}^* = \left(\underline{\hat{z}}\frac{1}{2}e^{+j\pi/4} - \underline{\hat{y}}\frac{1}{2}e^{-j\pi/4} \right) \; \left[\mathrm{VA/m^2}\right]$$



$$P = \int_S \underline{S} \cdot \underline{\hat{n}}\, dS = \int_S \underline{S} \cdot \left(-\underline{\hat{z}}\right) dS = -\int_S S_z\, dS$$

$$= -\int_S \frac{1}{2} e^{+j\pi/4}\, dS = -\frac{1}{2} e^{+j\pi/4} (1) \; [\mathrm{VA}]$$

$$P_r = \mathrm{Re}(P) = -\frac{1}{2}\cos\left(\frac{\pi}{4}\right)$$

$$P_i = \mathrm{Im}(P) = -\frac{1}{2}\sin\left(\frac{\pi}{4}\right)$$

Hence

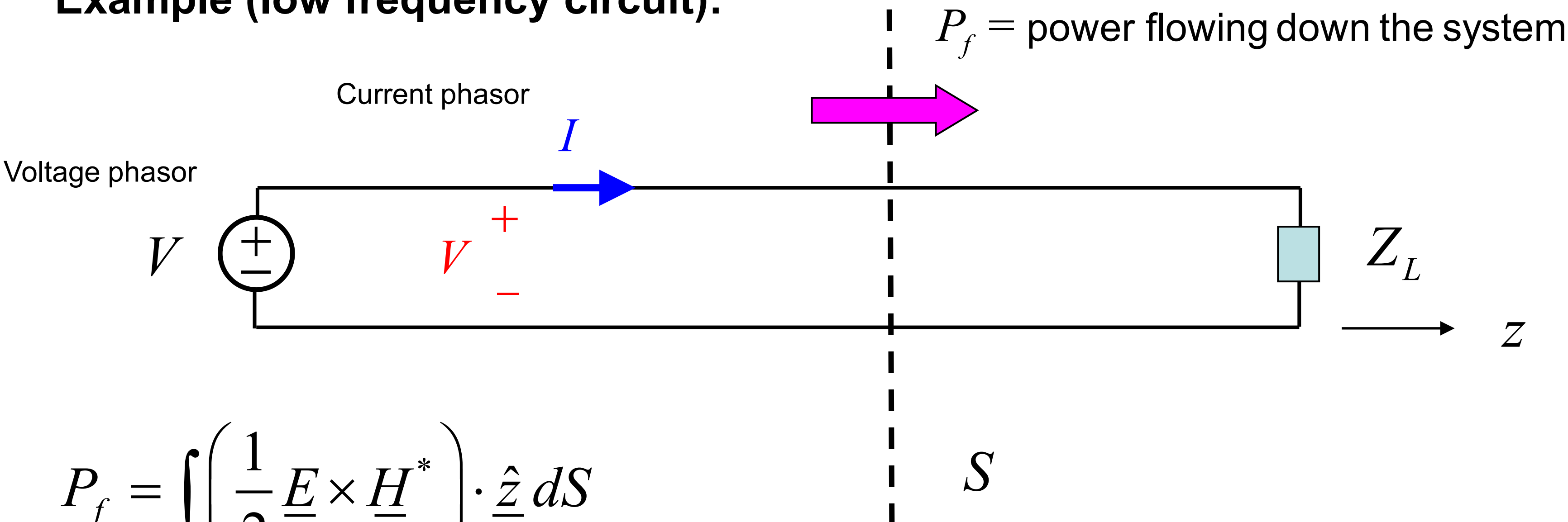
$$P_r = -\frac{1}{2\sqrt{2}} \; [\mathrm{W}]$$

$$P_i = -\frac{1}{2\sqrt{2}} \; [\mathrm{VAR}]$$

Note on Circuit Theory

Although the Poynting vector can always be used to calculate power flow on a transmission line, circuit theory can also be used, and this is usually easier.

Example (low frequency circuit):

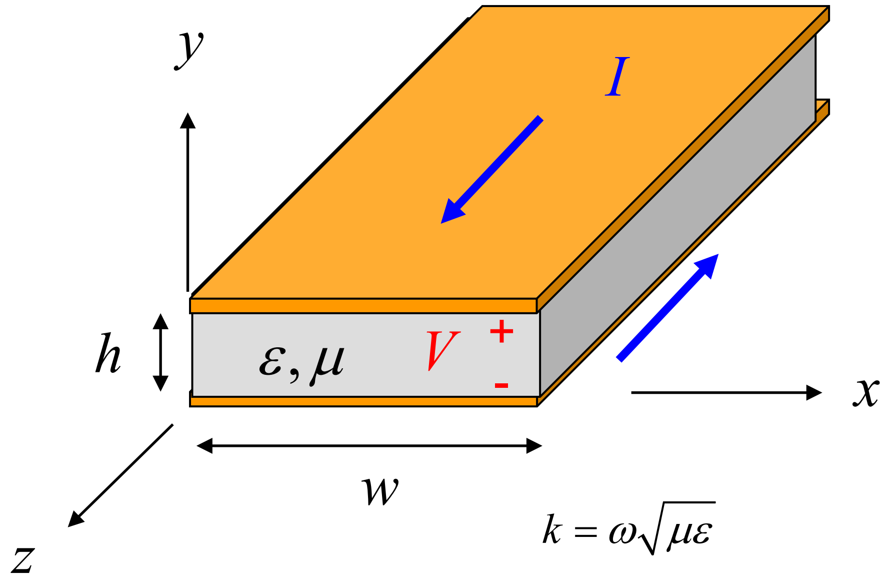


$$P_f = \int_S \left(\frac{1}{2} \underline{E} \times \underline{H}^* \right) \cdot \underline{\hat{z}} \, dS$$

$$P_f = \frac{1}{2} V I^*$$ (This is exact for a transmission line.)

The second form is much easier to calculate!

Example: Parallel-Plate Transmission Line

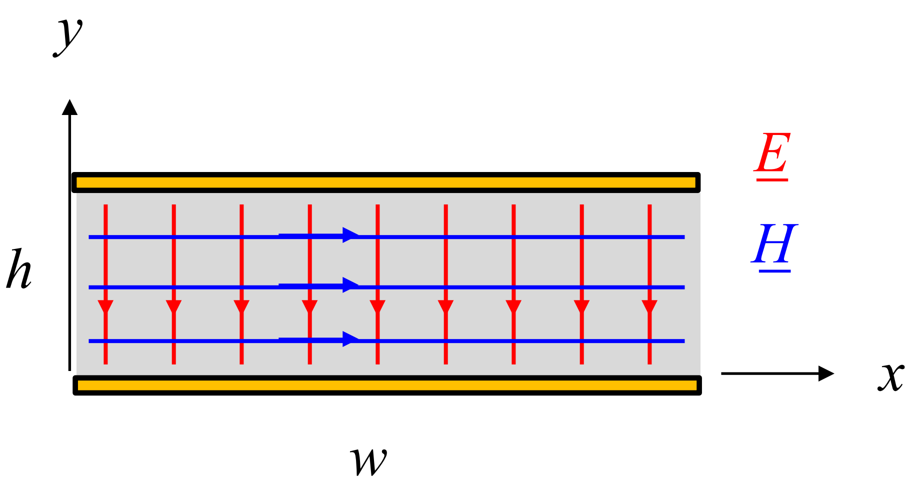


$k = \omega\sqrt{\mu\varepsilon}$

$$V(z) = V_0\, e^{-jkz}$$

$$I(z) = I_0\, e^{-jkz}$$

The voltage and current have the form of waves that travel along the line in the z direction.

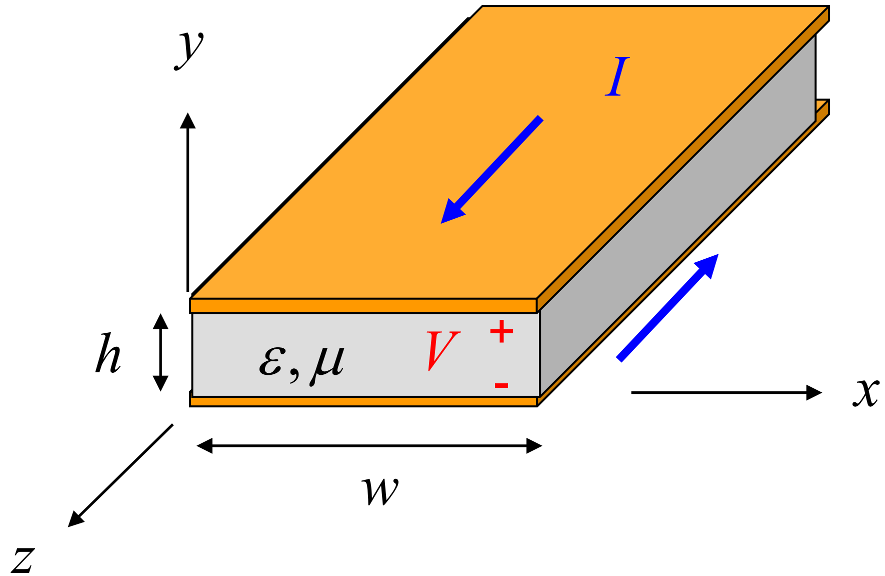


At $z = 0$:

$$V(0) = V_0$$

$$I(0) = I_0$$

Example (cont.)

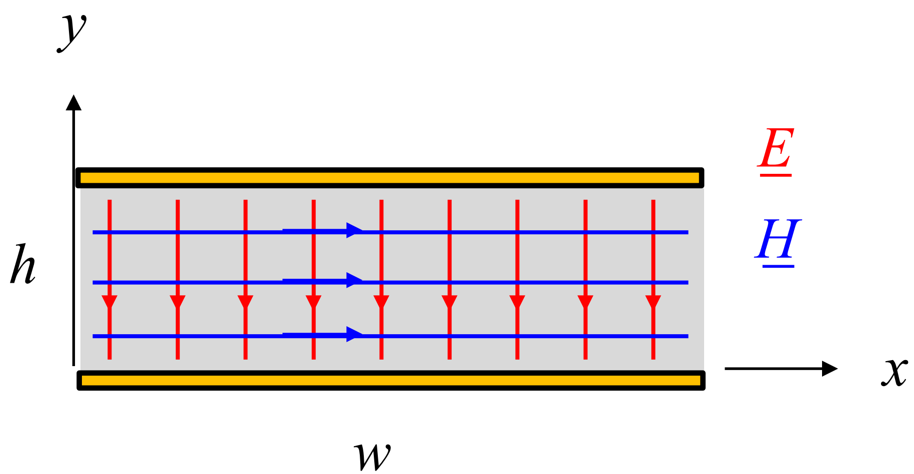


At $z = 0$:

$$\underline{E}(x, y, 0) = -\hat{y}\left(\frac{V_0}{h}\right)$$

$$\underline{H}(x, y, 0) = \hat{x}\left(\frac{I_0}{w}\right)$$

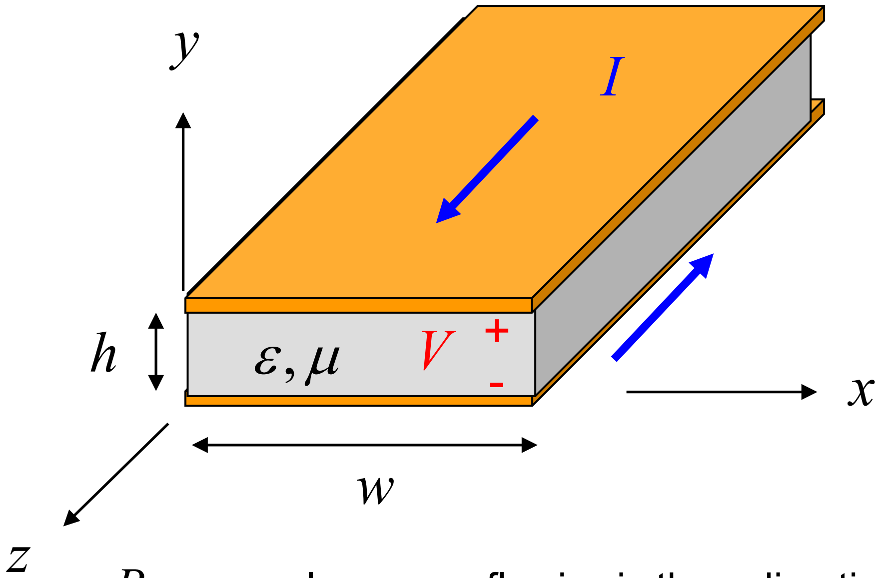
(from ECE 3318)



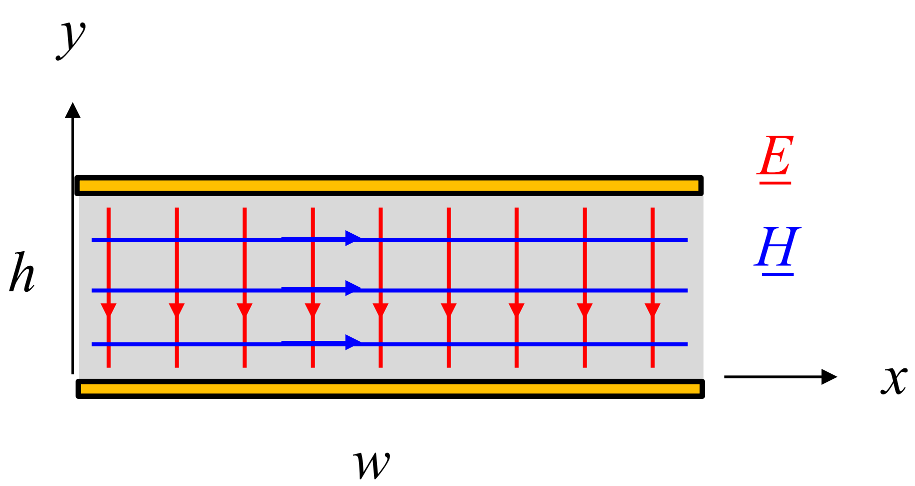
$$\underline{S} = \frac{1}{2}\left(\underline{E} \times \underline{H}^*\right)$$

$$\underline{S} = \frac{1}{2}\underline{\hat{z}}\left(\frac{V_0}{h}\right)\left(\frac{I_0}{w}\right)^*$$

Example (cont.)



P_f = complex power flowing in the z direction at $z = 0$



$$\underline{S} = \frac{1}{2}\underline{\hat{z}}\left(\frac{V_0}{h}\right)\left(\frac{I_0}{w}\right)^*$$

We then have

$$P_f = \int_0^h \int_0^w \underline{S} \cdot \underline{\hat{z}}\, dx\, dy$$

so

$$P_f = \frac{1}{2}\left(\frac{V_0}{h}\right)\left(\frac{I_0}{w}\right)^* (wh)$$

$$P_f = \frac{1}{2} V_0 I_0^*$$