ECE 3317
Applied Electromagnetic Waves

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Notes 9
Transmission Lines (Frequency Domain)
Why is the frequency domain important?

- Most communication systems use a sinusoidal signal (which may be modulated).

(Some systems, like Ethernet, communicate in “baseband”, meaning that there is no carrier.)

Examples:
100BASET, 10GBASET
Why is the frequency domain important?

- A solution in the frequency-domain allows to solve for an **arbitrary** time-varying signal on a lossy line (by using the Fourier transform method).

**Fourier-transform pair**

\[
\tilde{v}(\omega) = \int_{-\infty}^{\infty} v(t) e^{-j\omega t} dt
\]

\[
v(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{v}(\omega) e^{j\omega t} d\omega
\]

For a physically-realizable (real-valued) signal, we can write

\[
v(t) = \frac{1}{\pi} \int_{0}^{\infty} \text{Re}\{\tilde{v}(\omega) e^{j\omega t}\} d\omega
\]

Jean-Baptiste-Joseph Fourier
A pulse is resolved into a collection (spectrum) of infinite sine waves.

\[ v(t) = \int_0^\infty \text{Re} \left\{ \left( \frac{1}{\pi} \tilde{v}(\omega) d\omega \right) e^{j\omega t} \right\} \]

A collection of phasor-domain signals!
Example: rectangular pulse

\[ v(t) = 1.0 \quad \text{for} \quad -W/2 < t < W/2 \]

Frequency Domain (cont.)

\[ \tilde{v}(\omega) = \int_{-\infty}^{\infty} v(t) e^{-j\omega t} \, dt \]

\[ \tilde{v}(\omega) = W \, \text{sinc} \left( \frac{W \omega}{2} \right) \]

\[ \text{sinc}(x) \equiv \frac{\sin(x)}{x} \]
Phasor domain:

\[ V_{in}(\omega) \rightarrow H(\omega) \rightarrow V_{out}(\omega) \]

In the frequency domain, the system has a transfer function \( H(\omega) \):

\[ V_{out}(\omega) = H(\omega)V_{in}(\omega) \]

Time domain:

\[ v_{in}(t) \rightarrow H(\omega) \rightarrow v_{out}(t) \]

The time-domain response of the system to an input signal is:

\[ v_{out}(t) = \frac{1}{\pi} \int_{0}^{\infty} \text{Re}\{H(\omega)\tilde{v}_{in}(\omega) e^{j\omega t}\} d\omega \]
If we can solve the system in the phasor domain (i.e., get the transfer function \( H(\omega) \)), we can get the output for any time-varying input signal.

This is one reason why the phasor domain is so important!

This applies for transmission lines also!
Telegrapher’s Equations

\[
\frac{\partial^2 v}{\partial z^2} - (RG)v - (RC + LG) \frac{\partial v}{\partial t} - LC \left( \frac{\partial^2 v}{\partial t^2} \right) = 0
\]
To convert to the phasor domain, we use: \[
\frac{\partial}{\partial t} \rightarrow j\omega
\]

\[
\frac{\partial^2 v}{\partial z^2} - (RG)v - (RC + LG)\frac{\partial v}{\partial t} - LC\left(\frac{\partial^2 v}{\partial t^2}\right) = 0
\]

or

\[
\frac{d^2 V}{dz^2} = (RG) V + j\omega(RC + LG)V - (\omega^2 LC)V
\]
\[
\frac{d^2V}{dz^2} = \left[ (RG) + j\omega(RC + LG) - (\omega^2 LC) \right]V
\]

Note that

\[
RG + j\omega(RC + LG) - \omega^2 LC = (R + j\omega L)(G + j\omega C)
\]

\[
Z = R + j\omega L = \text{series impedance/length}
\]

\[
Y = G + j\omega C = \text{parallel admittance/length}
\]

Then we can write:

\[
\frac{d^2V}{dz^2} = (ZY)V
\]
Frequency Domain (cont.)

\[ \frac{d^2V}{dz^2} = (ZY)V \]

Define

\[ \gamma^2 \equiv ZY = (R + j\omega L)(G + j\omega C) \]

Then

\[ \frac{d^2V}{dz^2} = (\gamma^2)V \]

Solution:

\[ V(z) = Ae^{-\gamma z} + Be^{+\gamma z} \]

Note: We have an exact solution, even for a lossy line!
Convention: we choose the (complex) square root to be the principle branch:

\[ \gamma \equiv \sqrt{(R + j\omega L)(G + j\omega C)} \]  
(lossy case)

\( \gamma \) is called the propagation constant, with units of [1/m]

Principle branch of square root:

\[ c = |c| e^{j\phi} \]
\[ \sqrt{c} = \left( |c| e^{j\phi} \right)^{1/2} \]
\[ \sqrt{|c|} e^{j(\phi/2)} \]
\[ -\pi < \phi \leq \pi \]

Note:
\[ -\pi/2 < \phi/2 \leq \pi/2 \]
\[ \text{Re}\sqrt{c} \geq 0 \]
\[
\gamma = \sqrt{(R + j\omega L)(G + j\omega C)}
\]

Denote: \( \gamma = \alpha + j\beta \)

\( \gamma = \) propagation constant [1/m]
\( \alpha = \) attenuation constant [nepers/m]
\( \beta = \) phase constant [radians/m]

Choosing the principle branch means that

\[ \text{Re}\, \gamma \geq 0 \quad \Rightarrow \quad \alpha \geq 0 \]
For a lossless line, we consider this as the limit of a lossy line, in the limit as the loss tends to zero:

\[ \gamma = \sqrt{(R + j\omega L)(G + j\omega C)} = (\omega \sqrt{LC}) \sqrt{-1} \]

Hence

\[ \gamma = j\omega \sqrt{LC} \]

(lossless case)

Hence we have that

\[ \alpha = 0 \]
\[ \beta = \omega \sqrt{LC} \]

Note: \( \alpha = 0 \) for a lossless line.
Physical interpretation of waves:

\[ V^+(z) = A e^{-\gamma z} \]  
\[ V^-(z) = B e^{+\gamma z} \]

(forward traveling wave)  
(backward traveling wave)

(This interpretation will be shown shortly.)

\[ \gamma = \alpha + j \beta \]

Forward traveling wave:  
\[ V^+(z) = A e^{-az} e^{-j\beta z} \]
Propagation Wavenumber

Alternative notations:

\[ \gamma = \alpha + j \beta \]  
\quad \text{(propagation constant)}

\[ k_z = -j \gamma = \beta - j \alpha \]  
\quad \text{(propagation wavenumber)}

Note: \[ \gamma = j k_z \]

\[ V^+(z) = A e^{-\gamma z} = A e^{-jk_z z} = A e^{-\alpha z} e^{-j \beta z} \]
Forward traveling wave:

\[ V^+(z) = A e^{-az} e^{-j\beta z} \]

Denote \( A = |A| e^{j\phi} \)

Then \( V^+(z) = |A| e^{j\phi} e^{-az} e^{-j\beta z} \)

In the time domain we have:

\[ v^+(z,t) = \text{Re}\left\{V^+(z) e^{j\omega t}\right\} \]

Hence we have

\[ v^+(z,t) = |A| e^{-\alpha z} \cos(\omega t - \beta z + \phi) \]
\[ v^+(z, t) = |A| e^{-\alpha z} \cos(\omega t - \beta z + \phi) \]

Snapshot of Waveform:

The distance \( \lambda \) is the distance it takes for the waveform to “repeat” itself in meters.

\[ \lambda = \text{wavelength} \]
Wavelength

The wave “repeats” (except for the amplitude decay) when:

$$\beta \lambda = 2\pi$$

Hence:

$$\beta = \frac{2\pi}{\lambda}$$
The attenuation constant controls how fast the wave decays.

\[ v^+(z, t) = |A| e^{-\alpha z} \cos(\omega t - \beta z + \phi) \]

envelope = \(|A| e^{-\alpha z}\)
The forward-traveling wave is moving in the positive $z$ direction.

Consider a lossless transmission line for simplicity ($\alpha = 0$):

$$v^+ (z, t) = |A| \cos(\omega t - \beta z + \phi)$$

Crest of wave: $\omega t - \beta z + \phi = 0$
The phase velocity \( v_p \) is the velocity of a point on the wave, such as the crest.

Set \( \omega t - \beta z = -\phi = \text{constant} \)

Take the derivative with respect to time: \( \omega - \beta \frac{dz}{dt} = 0 \)

Hence

\[
\frac{dz}{dt} = \frac{\omega}{\beta}
\]

We thus have

\[
v_p = \frac{\omega}{\beta}
\]

Note: This result holds also for a lossy line.
Let’s calculate the phase velocity for a lossless line:

\[ v_p = \frac{\omega}{\beta} = \frac{\omega}{\omega \sqrt{LC}} = \frac{1}{\sqrt{LC}} \]

Also, we know that \( LC = \mu \varepsilon = \frac{1}{c_d^2} \)

Hence \( v_p = c_d \) (lossless line)

Recall: \( c_d = \frac{1}{\sqrt{\mu \varepsilon}} = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = \frac{1}{\sqrt{\mu_r \varepsilon_r}} = \frac{c}{\sqrt{\mu_r \varepsilon_r}} \)
Let’s now consider the backward-traveling wave (lossless, for simplicity):

\[
v^-(z, t) = |A| \cos(\omega t + \beta z + \phi)
\]
The group velocity $v_g$ is the velocity of a pulse.

We have (derivation omitted):

$$v_g = \frac{d\omega}{d\beta}$$

Note: This result holds also for a lossy line.
$v^+(z, t) = |A| e^{-az} \cos(\omega t - \beta z + \phi)$

Gain in dB: $\text{dB} = 20 \log_{10} \left| \frac{V^+(z)}{V^+(0)} \right| = 20 \log_{10} (e^{-az})$

Use the following logarithm identity: $\log_{10} x = \frac{\ln x}{\ln 10}$

Therefore, the “gain” is: $\text{dB} = 20 \frac{\ln (e^{-az})}{\ln 10} = 20 \frac{(-\alpha z)}{\ln 10}$

Hence we have: $\text{Attenuation} = \left( \frac{20}{\ln 10} \right) \alpha \quad [\text{dB/m}]$
Final attenuation formulas:

\[ \text{Attenuation} = \left( \frac{20}{\ln 10} \right) \alpha \quad [\text{dB/m}] \]

\[ \text{Attenuation} = (8.686) \alpha \quad [\text{dB/m}] \]
Example: Coaxial Cable

\[ C = \frac{2\pi \varepsilon_0 \varepsilon_r}{\ln \left( \frac{b}{a} \right)} \quad [\text{F/m}] \]

\[ L = \frac{\mu_0}{2\pi} \ln \left( \frac{b}{a} \right) \quad [\text{H/m}] \]

\[ G = \frac{2\pi \sigma_d}{\ln \left( \frac{b}{a} \right)} \quad [\text{S/m}] \]

\[ R = \left( \frac{1}{2\pi a \sigma_{ma} \delta_{ma}} + \frac{1}{2\pi b \sigma_{mb} \delta_{mb}} \right) \quad [\Omega/\text{m}] \]

\[ \delta_m = \sqrt{\frac{2}{\omega \mu_m \sigma_m}} \quad \text{(skin depth)} \]

Copper conductors
(nonmagnetic: \( \mu_m = \mu_0 \))

\[ a = 0.5 \quad [\text{mm}] \]
\[ b = 3.2 \quad [\text{mm}] \]
\[ \varepsilon_r = 2.2 \]
\[ \tan \delta_d = 0.001 \]
\[ \sigma_{ma} = \sigma_{mb} = 5.8 \times 10^7 \quad [\text{S/m}] \]
\[ f = 500 \quad [\text{MHz}] \quad \text{(UHF)} \]

Note:
The “loss tangent” of the dielectric is called \( \tan \delta_d \).
Dielectric conductivity is often specified in terms of the loss tangent:

$$\tan \delta_d \equiv \frac{\sigma_d}{\omega \varepsilon} = \frac{\sigma_d}{\omega \varepsilon_0 \varepsilon_r}$$

$$\sigma_d = \text{effective conductivity of the dielectric material}$$

**Note:**
The loss tangent of practical insulating materials (e.g., Teflon) is approximately constant over a wide range of frequencies.
Example: Coaxial Cable (cont.)

Relation between $G$ and $C$

$$C = \frac{2\pi \varepsilon_0 \varepsilon_r}{\ln\left(\frac{b}{a}\right)} \quad [\text{F/m}]$$

$$G = \frac{2\pi \sigma_d}{\ln\left(\frac{b}{a}\right)} \quad [\text{S/m}]$$

$$G = C \left(\frac{\sigma_d}{\varepsilon_0 \varepsilon_r}\right)$$

$$\Rightarrow \frac{G}{\omega C} = \frac{\sigma_d}{\omega \varepsilon_0 \varepsilon_r}$$

$$\tan \delta_d \equiv \frac{\sigma_d}{\omega \varepsilon} = \frac{\sigma_d}{\omega \varepsilon_0 \varepsilon_r}$$

Hence

$$\frac{G}{\omega C} = \tan \delta_d$$

This relationship holds for any type of transmission line.
Characteristic impedance (ignore $R$ and $G$):

$$Z_0 = \sqrt{\frac{L}{C}} = \frac{\eta_0}{2\pi\sqrt{\varepsilon_r}} \ln\left(\frac{b}{a}\right)$$

$$Z_0 = 75 \text{ [}\Omega\text{]}$$

Skin depth:

$$\delta_m = \sqrt{\frac{2}{\omega \mu \sigma_m}}$$

$$\delta_m = 2.955 \times 10^{-6} \text{ [m]}$$

$$\sigma_d = (\omega \varepsilon_0 \varepsilon_r) \tan \delta_d$$

$$\sigma_d = 6.12 \times 10^{-5} \text{ [S/m]}$$

Example: Coaxial Cable (cont.)

$$\tan \delta_d = 0.001$$

$$\sigma_{ma} = \sigma_{mb} = 5.8 \times 10^7 \text{ [S/m]}$$

$$f = 500 \text{ [MHz]}$$ (UHF)
Example: Coaxial Cable (cont.)

\[ a = 0.5 \text{ [mm]} \]

\[ b = 3.2 \text{ [mm]} \]

\[ \varepsilon_r = 2.2 \]

\[ \tan \delta_d = 0.001 \]

\[ \sigma_{ma} = \sigma_{mb} = 5.8 \times 10^7 \text{ [S/m]} \]

\[ f = 500 \text{ [MHz]} \text{ (UHF)} \]

\[ \gamma = \sqrt{(R + j\omega L)(G + j\omega C)} \]

\[ \gamma = 0.022 + j(15.543) \text{ [1/m]} \]

\[ \alpha = 0.022 \text{ [nepers/m]} \]

\[ \beta = 15.544 \text{ [rad/m]} \]

Attenuation = 0.191 [dB/m]

\[ \lambda = 0.404 \text{ [m]} \]

\[ R = 2.147 \text{ [}\Omega/\text{m]}\]

\[ L = 3.713 \times 10^{-7} \text{ [H/m]} \]

\[ G = 2.071 \times 10^{-4} \text{ [S/m]} \]

\[ C = 6.593 \times 10^{-11} \text{ [F/m]} \]
Use the first Telegrapher equation:

\[
\frac{\partial v}{\partial z} = -Ri - L \frac{\partial i}{\partial t}
\]

\[
\frac{\partial V}{\partial z} = -RI - j\omega LI
\]

Next, use

\[
V(z) = Ae^{-\gamma z} + Be^{+\gamma z}
\]

so

\[
\frac{\partial V(z)}{\partial z} = -\gamma \left[ Ae^{-\gamma z} - B e^{+\gamma z} \right]
\]
Hence we have

\[-\gamma \left[ A e^{-\gamma z} - B e^{+\gamma z} \right] = -RI - j\omega LI\]

Solving for the phasor current \( I \), we have

\[
I = \left( \frac{\gamma}{R + j\omega L} \right) \left[ A e^{-\gamma z} - B e^{+\gamma z} \right]
\]

\[
= \left( \frac{\sqrt{(R + j\omega L)(G + j\omega C)}}{R + j\omega L} \right) \left[ A e^{-\gamma z} - B e^{+\gamma z} \right]
\]

\[
= \sqrt{\frac{G + j\omega C}{R + j\omega L}} \left[ A e^{-\gamma z} - B e^{+\gamma z} \right]
\]
Define the (complex) characteristic impedance $Z_0$:

$$Z_0 \equiv \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

Then we have:

$$I(z) = \left( \frac{1}{Z_0} \right) \left[ Ae^{-\gamma z} - B e^{+\gamma z} \right]$$
Practical note: Even though $Z_0$ is always complex for a practical line (due to loss), we usually neglect this and take it to be real.

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} \approx \sqrt{\frac{L}{C}}$$
Summary of Solution

Characteristic Impedance

Lossy:

\[ Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} \]

Lossless:

\[ Z_0 = \sqrt{\frac{L}{C}} \]

Voltage and Current

\[ V(z) = Ae^{-\gamma z} + Be^{+\gamma z} \]

\[ I(z) = \left( \frac{1}{Z_0} \right) \left[ Ae^{-\gamma z} - Be^{+\gamma z} \right] \]
Appendix: Summary of Formulas

\[ Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} \]

\[ Z_0 = \sqrt{\frac{L}{C}} \]

\[ V(z) = Ae^{-\gamma z} + Be^{+\gamma z} \]

\[ I(z) = \left( \frac{1}{Z_0} \right) \left[ Ae^{-\gamma z} - Be^{+\gamma z} \right] \]

\[ \gamma \equiv \sqrt{(R + j\omega L)(G + j\omega C)} \]

\[ \tan \delta_d \equiv \frac{\sigma_d}{\omega \varepsilon} = \frac{\sigma_d}{\omega \varepsilon_0 \varepsilon_r} \]

\[ \frac{G}{\omega C} = \tan \delta_d \]

\[ \gamma = \alpha + j\beta \]

\[ \alpha = 0, \quad \beta = \omega \sqrt{LC} \]

\[ \beta = \frac{2\pi}{\lambda} \quad \nu_p = \frac{\omega}{\beta} \]

Attenuation = \((8.686)\alpha\) [dB/m]