ECE 3317
Fall 2023

Homework #2

Assigned: Thursday, Aug. 31
Due: Thursday, Sept. 7

Assigned Problems: 1–2, 4–7, 9–10 (Probs. 3 and 8 are only for your own benefit.)

1) If $\underline{A}(x,y,z) = xy^2\underline{\hat{x}} + xy^2z^3\underline{\hat{y}} + x^2yz^2\underline{\hat{z}}$, find $\nabla \times \underline{A}$ and $\nabla \cdot \underline{A}$.

2) Prove that for an arbitrary vector $\underline{A}$ (x, y, z) and an arbitrary scalar function $\phi(x, y, z)$, that

$$\nabla \cdot \left(\nabla \times \underline{A}\right) = 0$$

$$\nabla \times \left(\nabla \phi\right) = \underline{0}.$$

Hint: It is easiest if you use rectangular coordinates to prove these identities. (But keep in mind that if you prove them in any coordinate system, the identities must be true in all coordinate systems.)

3) Recall that the vector Laplacian of a vector function $\underline{A}$ (x, y, z) is defined as

$$\nabla^2 \underline{A} \equiv \nabla\left(\nabla \cdot \underline{A}\right) - \nabla \times \left(\nabla \times \underline{A}\right).$$

Prove that in rectangular coordinates the following identity holds:

$$\nabla^2 \underline{A} = \underline{\hat{x}}\nabla^2 A_x + \underline{\hat{y}}\nabla^2 A_y + \underline{\hat{z}}\nabla^2 A_z .$$

In words, this identity says that the rectangular components of the vector Laplacian are equal to the scalar Laplacian of the corresponding components of the vector function. This identity only holds in rectangular coordinates, not in cylindrical or spherical coordinates.

(The vector Laplacian and the identity that you are proving above are both very useful when deriving the scalar Helmholtz equation for the components of the electric and magnetic fields, something we will be doing later in the course.)

4) If $\phi(x,y,z) = x^3y^2z$, find $\nabla\phi$ and $\nabla \cdot \left(\nabla\phi\right)$ (which is the same as $\nabla^2\phi$).

5) Given that $\underline{\mathscr{E}}(x,y,z,t) = 2x\underline{\hat{x}} + y\underline{\hat{y}} + 3z\underline{\hat{z}}$ is a field that exists in space, but does not vary with time, does $\underline{\mathscr{B}}(x,y,z,t)$ vary with time? Justify your answer by using Maxwell's equations.

Repeat for $\underline{\mathscr{E}}(x,y,z,t) = 2y\underline{\hat{x}} + z\underline{\hat{y}} + 3x\underline{\hat{z}}$.

6) Given that $\underline{\mathscr{D}}(x,y,z,t) = 2xy^2\underline{\hat{x}} + 3x\underline{\hat{y}} - x^2 y\underline{\hat{z}}$ in some region, what is the charge density in that region?

7) For the following sinusoidal electric fields in free space, find the magnetic field $\underline{\mathscr{H}}(x, y, z, t)$. Do this by using Maxwell's equations in the phasor domain. (In free space the permittivity is ε_0 and the permeability is μ_0, and there is no charge or current present.)

(a) $\underline{\mathscr{E}}(x,y,z,t) = E_0\left(x\underline{\hat{x}} + \cos(2\pi z)\,\underline{\hat{y}} + 2z\underline{\hat{z}}\right)\sin\omega t$

(b) $\underline{\mathscr{E}}(x,y,z,t) = E_0\underline{\hat{z}}\sin\left[2\pi\times10^6 t + 2\pi(2x+3y)\right]$

(c) $\underline{\mathscr{E}}(x,y,z,t) = \underline{\hat{y}}\,E_0 x^2\cos\omega t$.

8) Show that in the time-harmonic (sinusoidal) steady state, there is no volume charge density ($\rho_v = 0$) in a region of space that is homogeneous (i.e., the material parameters are constants, and do not vary with position).

Do this by taking the divergence of both sides of Ampere's law (one of Maxwell's equations) in the phasor domain. Then recall that we have two zero identities in vector calculus. Can you use one of them? Also assume that we have a linear medium for which Ohm's law applies, so that in the phasor domain $\underline{J} = \sigma\underline{E}$. This should allow you to first prove that the divergence of the electric field in the homogeneous region must be zero. What does this then tell you about the volume charge density in the homogeneous region?

9) The phasor-domain fields in a region of space are given by:

$$\underline{E} = \left(\underline{\hat{x}} - j\underline{\hat{y}}\right)e^{-j(2\pi)z}$$
$$\underline{H} = 0.002654\left(\underline{\hat{x}} - j\underline{\hat{y}}\right)e^{-j(2\pi)z}.$$

Find the complex Poynting vector $\underline{S}$ and the time average Poynting vector $\left\langle\underline{\mathscr{S}}(t)\right\rangle$. Then find the watts and VARs flowing upward through the inside of a circle of radius a = 2 meters that lies in the z = 0 plane and is centered at the origin.

10) Consider a lossless coaxial cable transmission line of inner radius a and outer radius b, filled with a lossless nonmagnetic (μ_r = 1) dielectric material. The phasor-domain fields inside of

the cable ($a < \rho < b$) for a time-harmonic (i.e., sinusoidally varying) wave traveling down the cable in the z direction are given in cylindrical coordinates as

$$\underline{E} = \underline{\hat{\rho}}\left(\frac{1}{\rho}\right)e^{-jkz}$$

$$\underline{H} = \underline{\hat{\phi}}\left(\frac{1}{\rho}\right)\frac{1}{\eta_0\sqrt{\varepsilon_r}}e^{-jkz}\,.$$

Calculate the total time-average power (in Watts) flowing in the z direction down the cable. Also calculate the amount of VARS flowing down the coaxial cable.

Note: The real-valued wavenumber k is the wavenumber of the medium filling the coax, given by $k = \omega\sqrt{\mu\varepsilon} = \omega\sqrt{\mu_0\varepsilon_0}\sqrt{\varepsilon_r}$, but its value should not affect your answer. Also, the intrinsic impedance of free space is $\eta_0 = \sqrt{\mu_0 / \varepsilon_0} \approx 376.7303\,[\Omega]$.