ECE 3318
Applied Electricity and Magnetism

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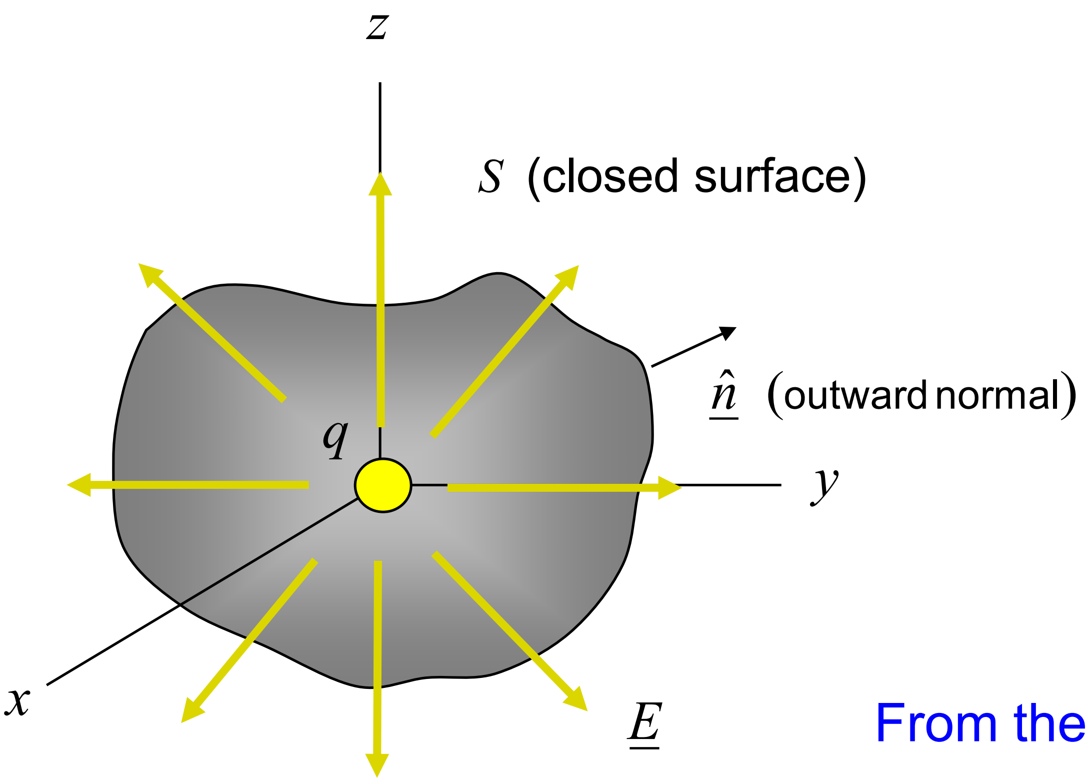
Dept. of ECE

Notes 10
Gauss's Law I

Notes prepared by the EM Group
University of Houston

Gauss's Law

A charge q is inside a closed surface.



Assume q produces N_f flux lines

$$\psi \equiv \oint_S \underline{D} \cdot \underline{\hat{n}} \, dS$$

$$\psi = \left(\frac{q}{N_f} \right) N_S$$

$N_S \equiv$ # flux lines that go through S

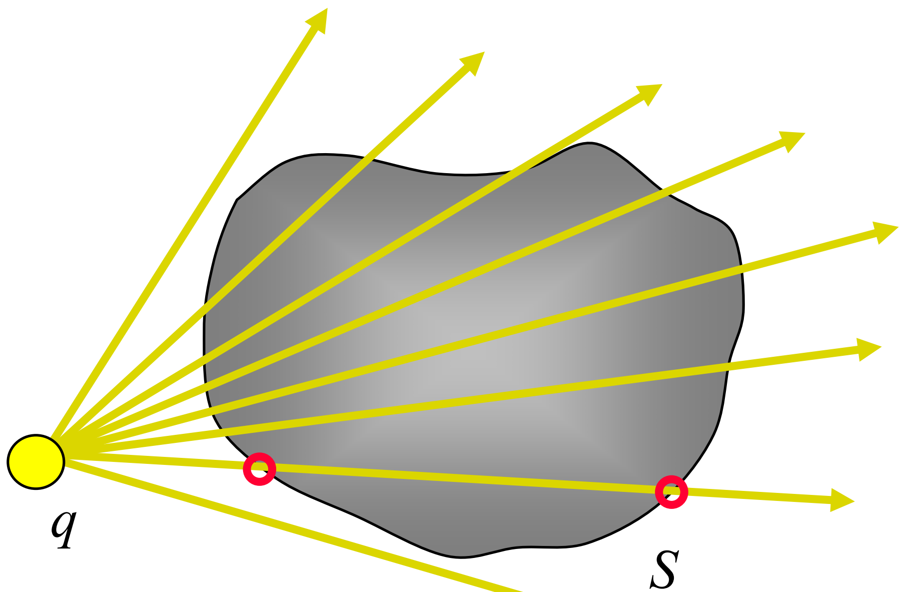
From the picture:

$N_S = N_f$ (all flux lines go through S)

Hence $\psi = q$

Gauss's Law (cont.)

The charge q is now outside the surface



$$N_S = 0$$

(All flux lines that enter the surface must leave the surface.)

Hence

$$\psi = \oint_S \underline{D} \cdot \underline{\hat{n}}\, dS = 0$$

Gauss's Law (cont.)

To summarize both cases, we have:

$$\oint_S \underline{D} \cdot \underline{\hat{n}}\, dS = Q_{encl}$$

We have proved that this is true for a point charge.

By superposition, this law must be true for *arbitrary charges*.

This is then called Gauss's law.

Gauss's Law (cont.)

Gauss's law:

$$\oint_S \underline{D} \cdot \underline{\hat{n}} \, dS = Q_{encl}$$

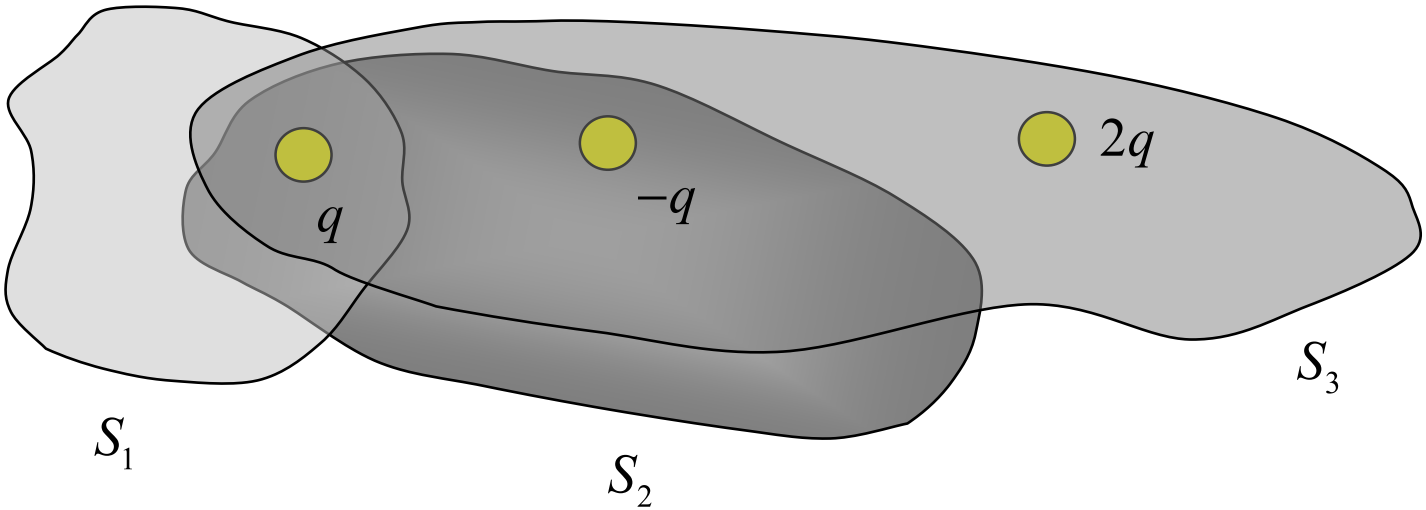
$\underline{\hat{n}}$ = outward normal

This surface S is called the "Gaussian surface".

Carl Friedrich Gauss

(his signature)

Example



$$\oint_{S_1} \underline{D} \cdot \underline{\hat{n}}\, dS = q \qquad \oint_{S_2} \underline{D} \cdot \underline{\hat{n}}\, dS = 0 \qquad \oint_{S_3} \underline{D} \cdot \underline{\hat{n}}\, dS = 2q$$

Note: $\underline{E} \neq 0$ on S_2 !

Note: All of the charges contribute to the electric field in space.

Using Gauss's Law

Gauss's law can be used to obtain the electric field from charges in a simple way.

The problems must be *highly symmetrical.*

The problem must reduce to *one unknown field component* (in one of the three coordinate systems).

Note:
When Gauss's law works, it is usually easier to use than Coulomb's law.

Choice of Gaussian Surface

$$\oint_S \underline{D} \cdot \underline{\hat{n}} \, dS = Q_{encl}$$

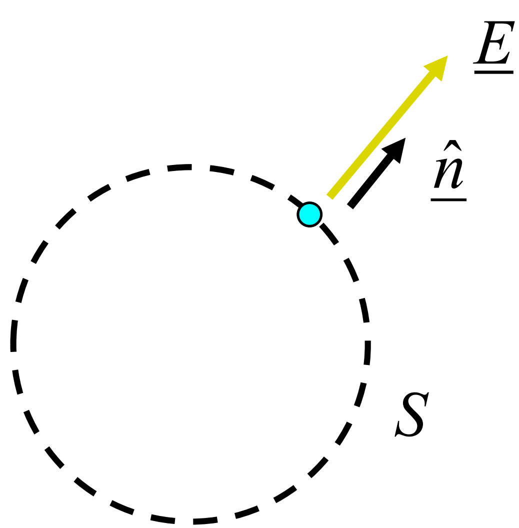
Rule 1: S must be a closed surface.

Rule 2: S should go through the observation point (usually called $\underline{r}$).

Guideline: Pick S to be $\perp$ to $\underline{E}$ as much as possible

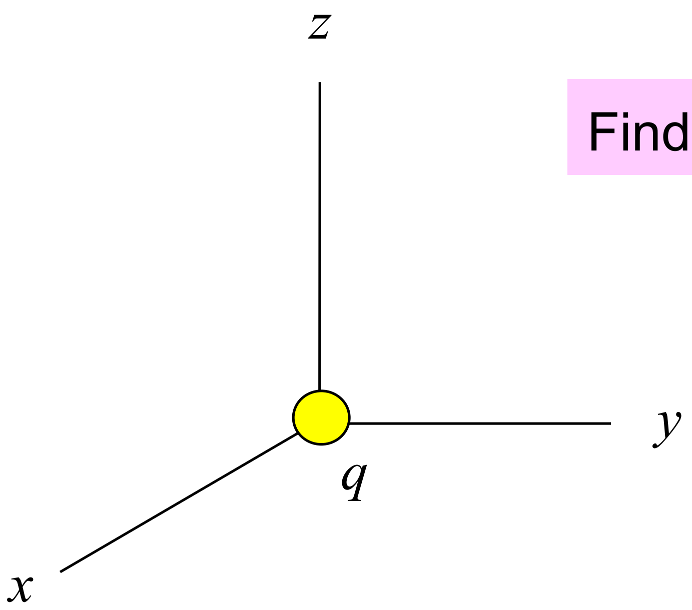
$$S \perp \underline{E} \quad \Rightarrow \quad \underline{\hat{n}} \,||\, \underline{D}$$

(This simplifies the dot product calculation.)



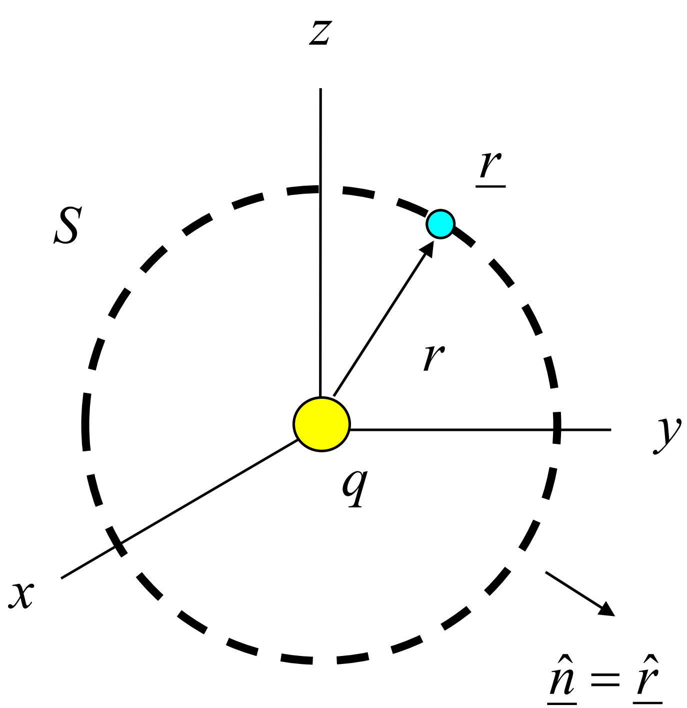
Example

Point charge



Find $\underline{E}$

Example (cont.)



$$\oint_S \underline{D} \cdot \underline{\hat{n}}\, dS = Q_{encl} = q$$

Assume $\underline{D} = \underline{\hat{r}}\, D_r$ (only an r component)

$$\oint_S \left(D_r\, \underline{\hat{r}}\right) \cdot \underline{\hat{r}}\, dS = q$$

$$\oint_S D_r\, dS = q$$

Assume $D_r = D_r\left(r\right)$ (only a function of r)

Then $D_r \oint_S dS = q$ or $D_r\left(4\pi r^2\right) = q$

Example (cont.)

We then have

$$\text{LHS} = \oint_S \underline{D} \cdot \underline{\hat{n}}\, dS = D_r \left(4\pi r^2\right)$$

$$\text{RHS} = Q_{encl} = q$$

so

$$D_r \left(4\pi r^2\right) = q$$

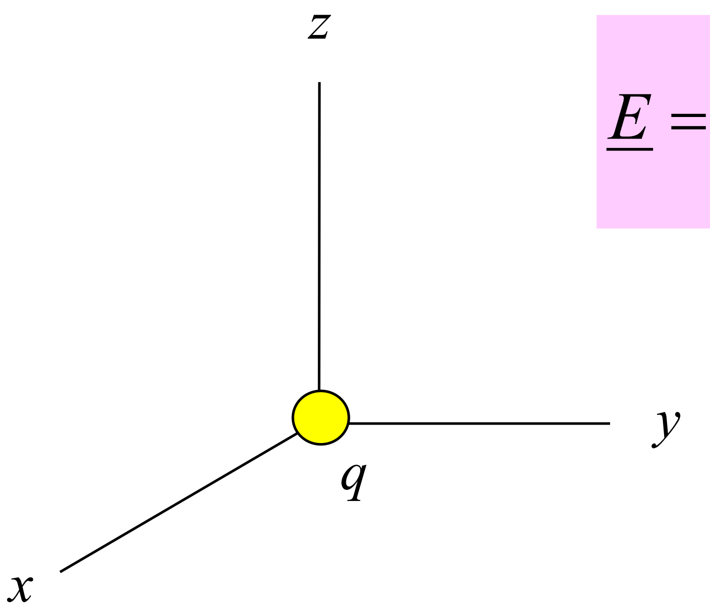
$$D_r = \frac{q}{4\pi r^2}$$

Hence

$$\underline{D} = \underline{\hat{r}} \left(\frac{q}{4\pi r^2} \right) \quad \left[\text{C/m}^2\right] \Rightarrow \quad \underline{E} = \underline{\hat{r}} \left(\frac{q}{4\pi \varepsilon_0 r^2} \right)$$

Example (cont.)

Summary



$$\underline{E} = \underline{\hat{r}}\left(\frac{q}{4\pi\varepsilon_0 r^2}\right) \quad [\text{V/m}]$$

Note About Spherical Coordinates

Note: In spherical coordinates, the LHS is always the same:

$$\text{LHS} = D_r\left(4\pi r^2\right)$$

← **Helpful shortcut!**

Assumption:

$$\rho_v\left(r,\theta,\phi\right) = f\left(r\right) \quad \text{(a function of } r \text{ only, not } \theta \text{ and } \phi\text{)}$$

$$\Rightarrow \quad \underline{D} = \underline{\hat{r}}\, D_r\left(r\right)$$

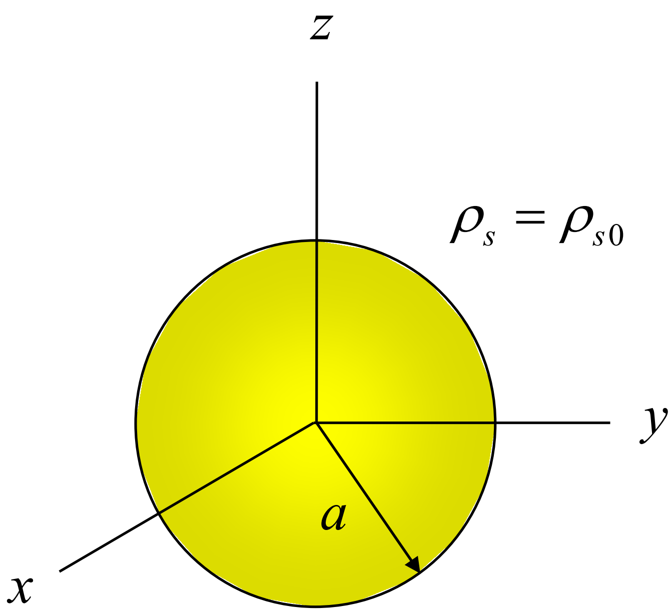
$$\Rightarrow \quad \text{LHS} = \oint_S \underline{D}\cdot\underline{\hat{n}}\, dS = \oint_S \underline{D}\cdot\underline{\hat{r}}\, dS = \oint_S D_r\, dS = D_r\left(4\pi r^2\right)$$

↑ Spherical Gaussian surface

↑ From the mathematical form of D_r

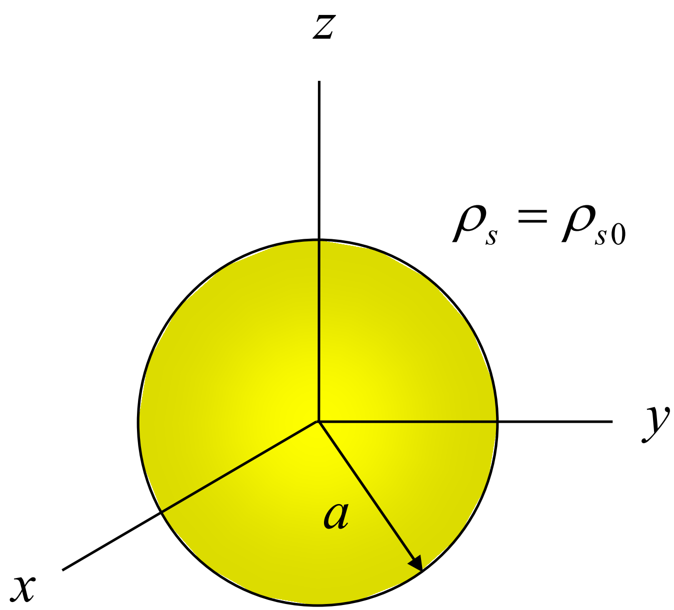
Example

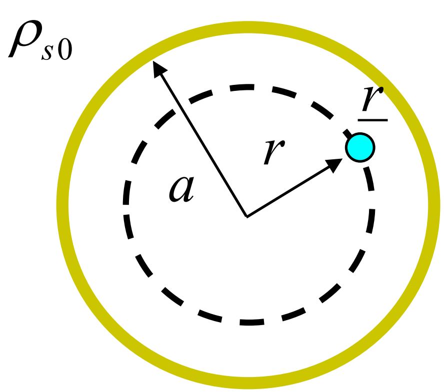
Hollow <u>shell</u> of uniform surface charge density



Find $\underline{E}$ everywhere

Example (cont.)





Case a) $r < a$

LHS = RHS

$$D_r\left(4\pi r^2\right) = Q_{encl} = 0$$

so

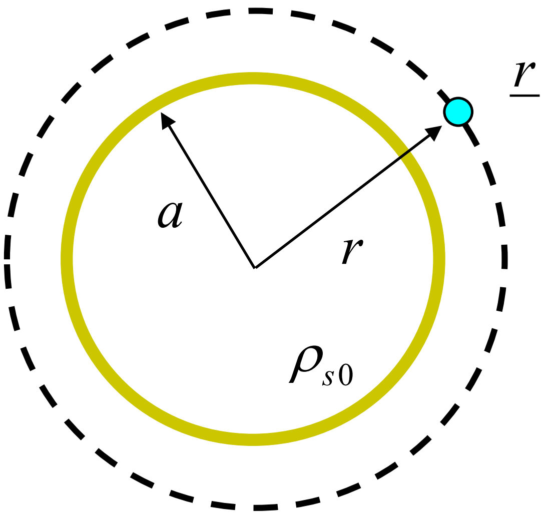
$$D_r = 0$$

Hence

$$\underline{E} = \underline{0} \;\; [\text{V/m}]$$

Example (cont.)

Case b) $r > a$



LHS = RHS

$$D_r\left(4\pi r^2\right) = Q_{encl} = \rho_{s0} 4\pi a^2$$

$$\Rightarrow D_r = \frac{4\pi a^2 \rho_{s0}}{4\pi r^2}$$

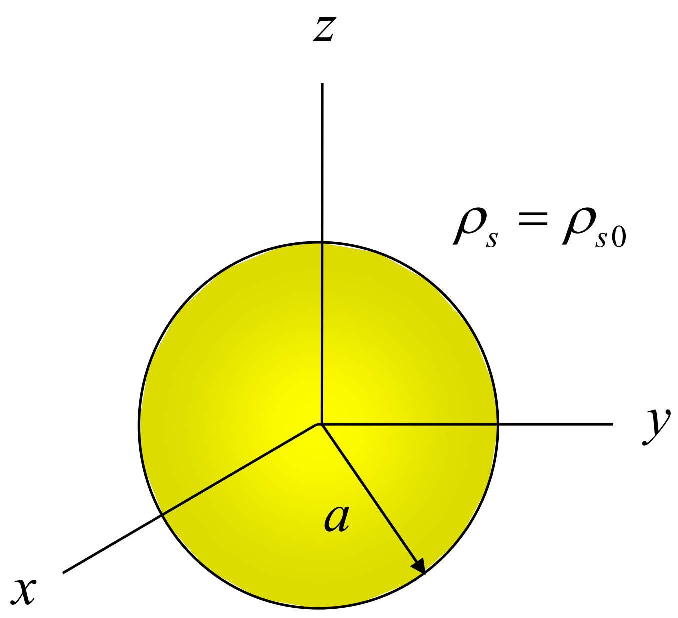
$$\Rightarrow D_r = \frac{Q}{4\pi r^2} \quad \left(Q = \rho_{s0} 4\pi a^2\right)$$

Hence

$$\underline{E} = \underline{\hat{r}} \frac{Q}{4\pi\varepsilon_0 r^2} \;\; [\text{V/m}]$$

The electric field outside a sphere of uniform surface charge density is the same as from a point charge at the origin.

Example (cont.)



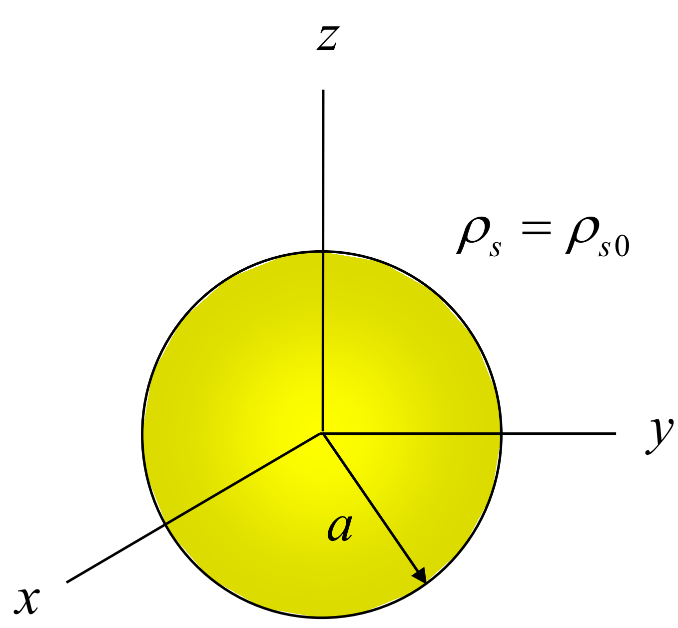
Summary

$r < a$ $\qquad \underline{E} = \underline{0}$ [V/m]

$r > a$ $\qquad \underline{E} = \underline{\hat{r}} \dfrac{Q}{4\pi\varepsilon_0 r^2}$ [V/m]

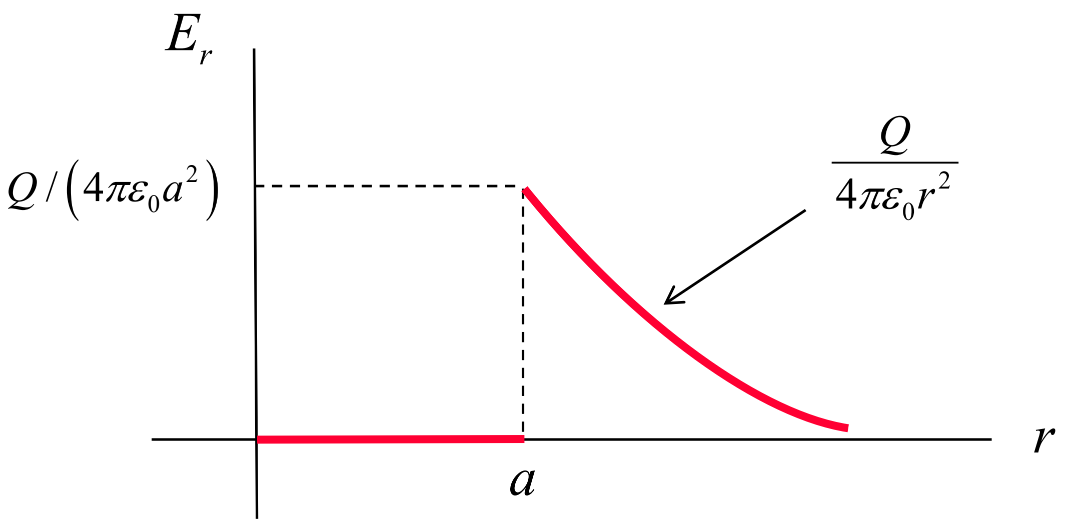
Note:
A similar result holds for the force due to gravity from a shell of material mass.

Example (cont.)



Important Point:

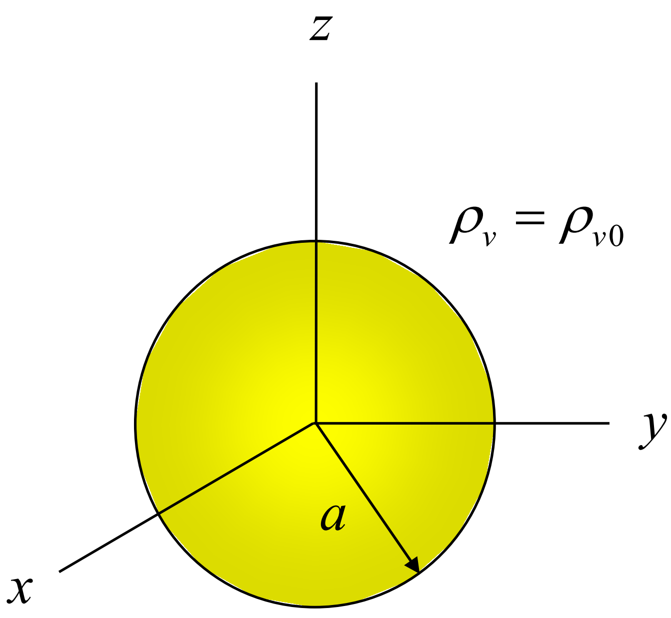
The electric field is <u>discontinuous</u> as we cross the boundary of a <u>surface charge density</u>.



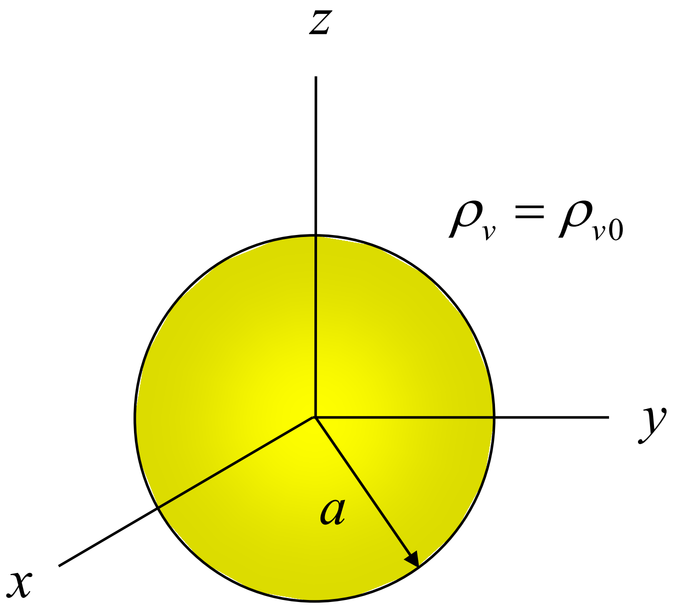
Example

Solid sphere of uniform volume charge density

Find $\underline{E}\,(r)$ everywhere



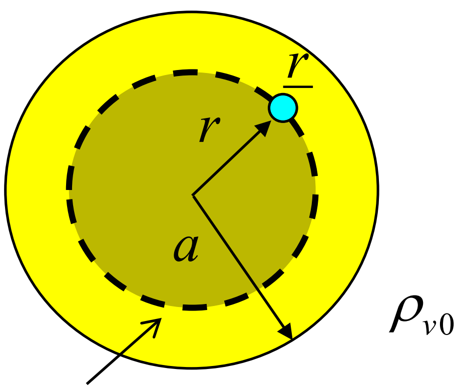
Example (cont.)



Case a) $r < a$

$$\oint_S \underline{D} \cdot \underline{\hat{n}} \, dS = Q_{encl}$$

$$\Rightarrow D_r \left(4\pi r^2\right) = Q_{encl}$$



$$Q_{encl} = \int_V \rho_v\left(\underline{r}\right) dV$$

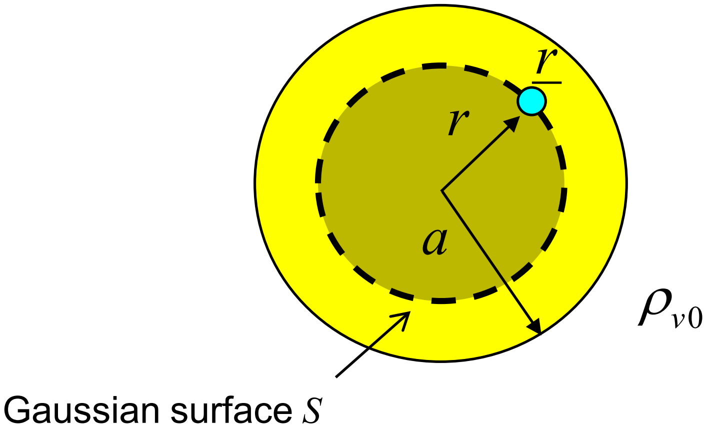
Example (cont.)

Calculate RHS:

$$Q_{encl} = \int_V \rho_{v0} \, dV$$

$$= \rho_{v0} \int_V dV$$

$$= \rho_{v0} \left(\frac{4}{3} \pi r^3 \right)$$



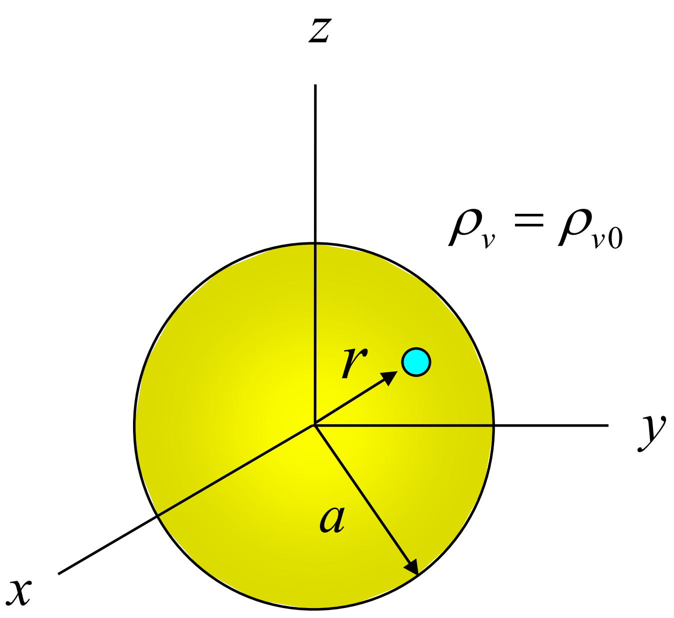
LHS = RHS

$$D_r \left(4\pi r^2 \right) = \rho_{v0} \left(\frac{4}{3} \pi r^3 \right)$$

Example (cont.)

Hence, we have

$$D_r = \rho_{v0}\left(\frac{1}{3}r\right)$$



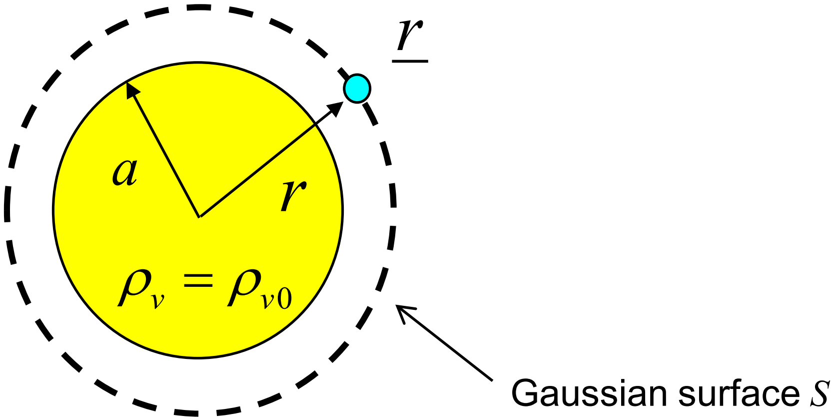
$$r < a$$

The vector electric field is then:

$$\underline{E} = \underline{\hat{r}}\,\rho_{v0}\left(\frac{r}{3\varepsilon_0}\right) \quad [\text{V/m}]$$

Example (cont.)

Case b) $r > a$



$$\oint_S \underline{D} \cdot \underline{\hat{n}}\, dS = Q_{encl}$$

$$\Rightarrow D_r \left(4\pi r^2\right) = Q_{encl}$$

$$Q_{encl} = \rho_{v0} \left(\frac{4}{3}\pi a^3\right)$$

so

$$D_r \left(4\pi r^2\right) = \rho_{v0} \left(\frac{4}{3}\pi a^3\right)$$

$$\Rightarrow D_r = \rho_{v0} \left(\frac{a^3}{3r^2}\right)$$

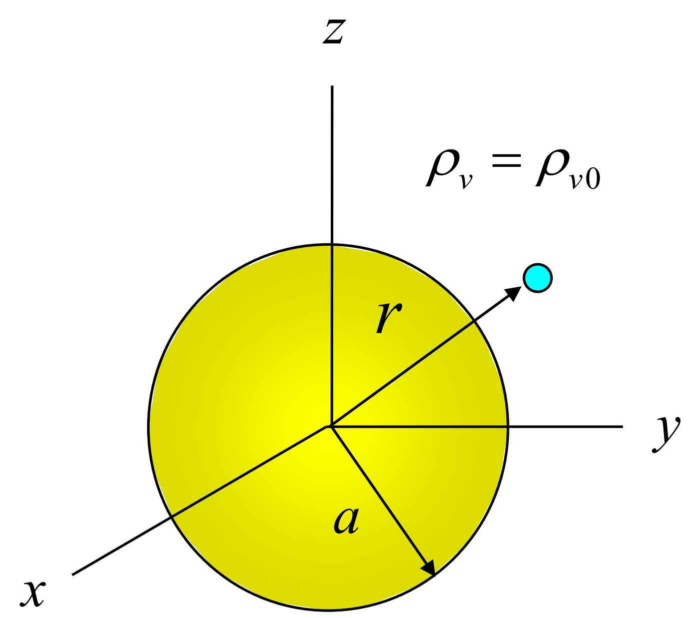
Hence, we have

$$\underline{E} = \underline{\hat{r}} \left(\frac{\rho_{v0}\, a^3}{3\varepsilon_0 r^2}\right) \quad [\mathrm{V/m}]$$

Example (cont.)

We can write this as:

$$\underline{E} = \underline{\hat{r}}\left(\frac{\rho_{v0}\, a^3}{3\varepsilon_0 r^2}\right)\left(\frac{(4/3)\pi}{(4/3)\pi}\right)$$



$r > a$

Hence

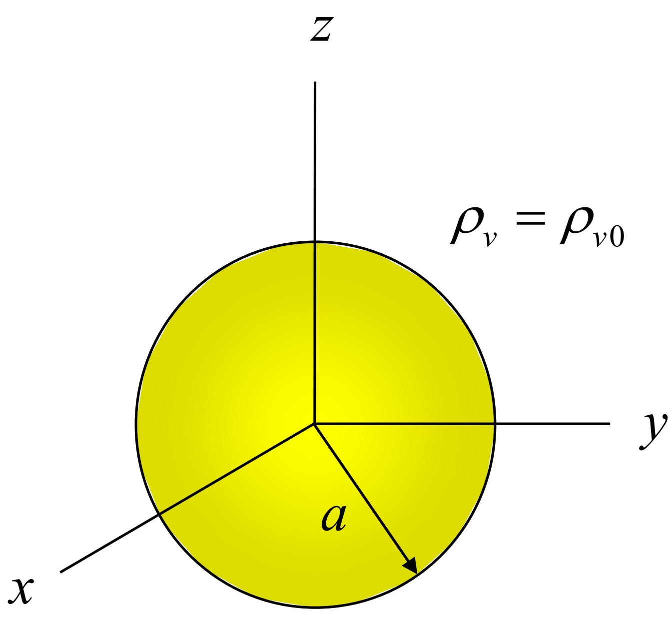
$$\underline{E} = \underline{\hat{r}}\frac{Q}{4\pi\varepsilon_0 r^2}$$

where

$$Q = \rho_{v0}\left(\frac{4}{3}\pi a^3\right)$$

The electric field <u>outside</u> a sphere of uniform volume charge density is the same as from a point charge at the origin.

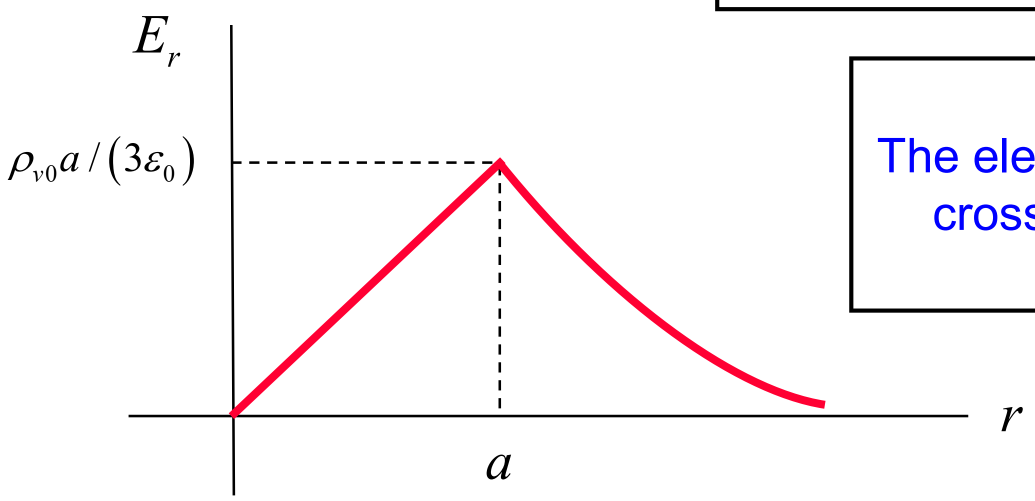
Example (cont.)



Summary

$$\underline{E} = \underline{\hat{r}}\,\rho_{v0}\left(\frac{r}{3\varepsilon_0}\right) \quad [\mathrm{V/m}] \qquad r < a$$

$$\underline{E} = \underline{\hat{r}}\,\frac{\rho_{v0}\,a^3}{3\varepsilon_0 r^2} = \underline{\hat{r}}\,\frac{Q}{4\pi\varepsilon_0 r^2} \quad [\mathrm{V/m}] \qquad r > a$$



Note:
The electric field is continuous as we cross the boundary of a volume charge density.