ECE 3318
Applied Electricity and Magnetism

Spring 2023

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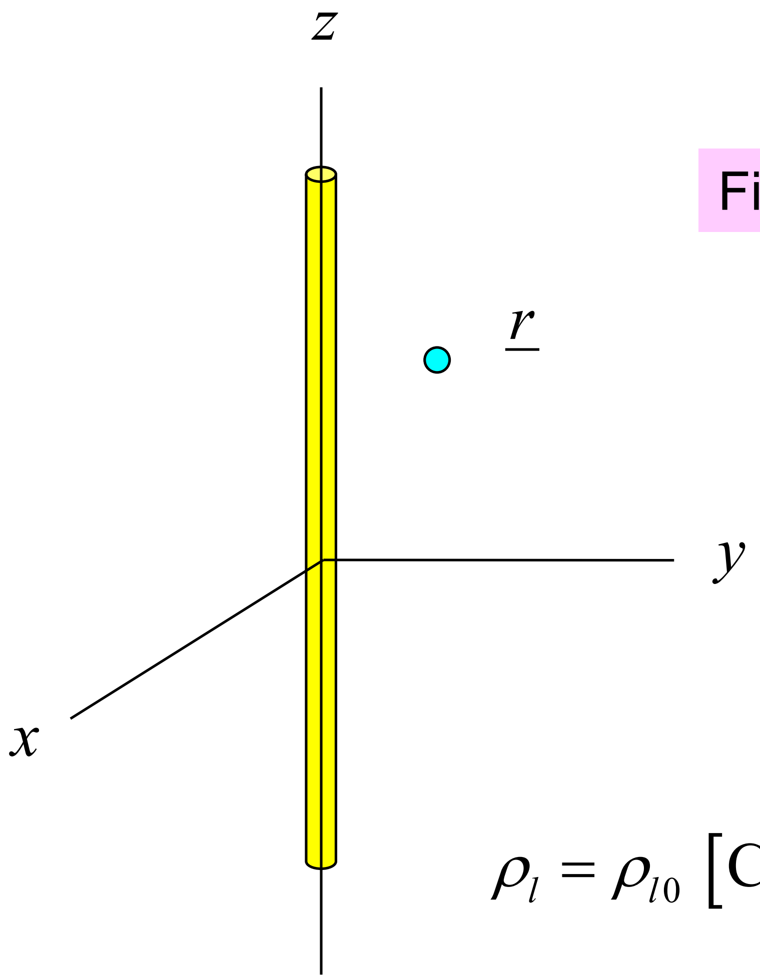
Dept. of ECE

Notes 11
Gauss's Law II

Notes prepared by the EM Group
University of Houston

Example

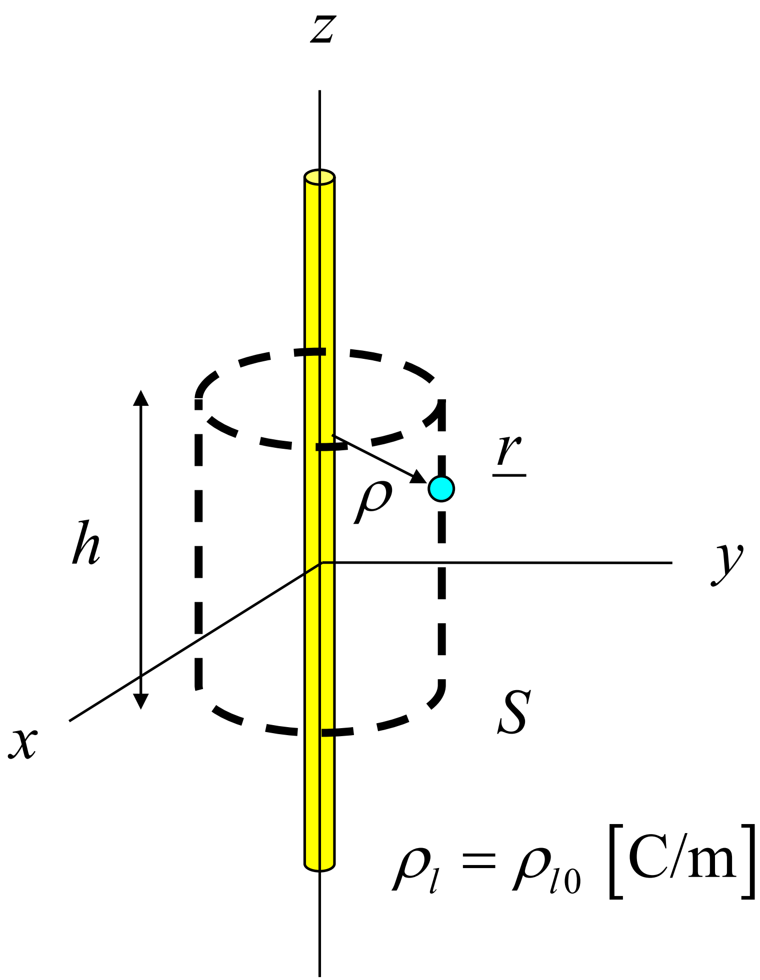
Infinite uniform line charge



Find the electric field vector

$$\rho_l = \rho_{l0}\ [\mathrm{C/m}]$$

Example (cont.)



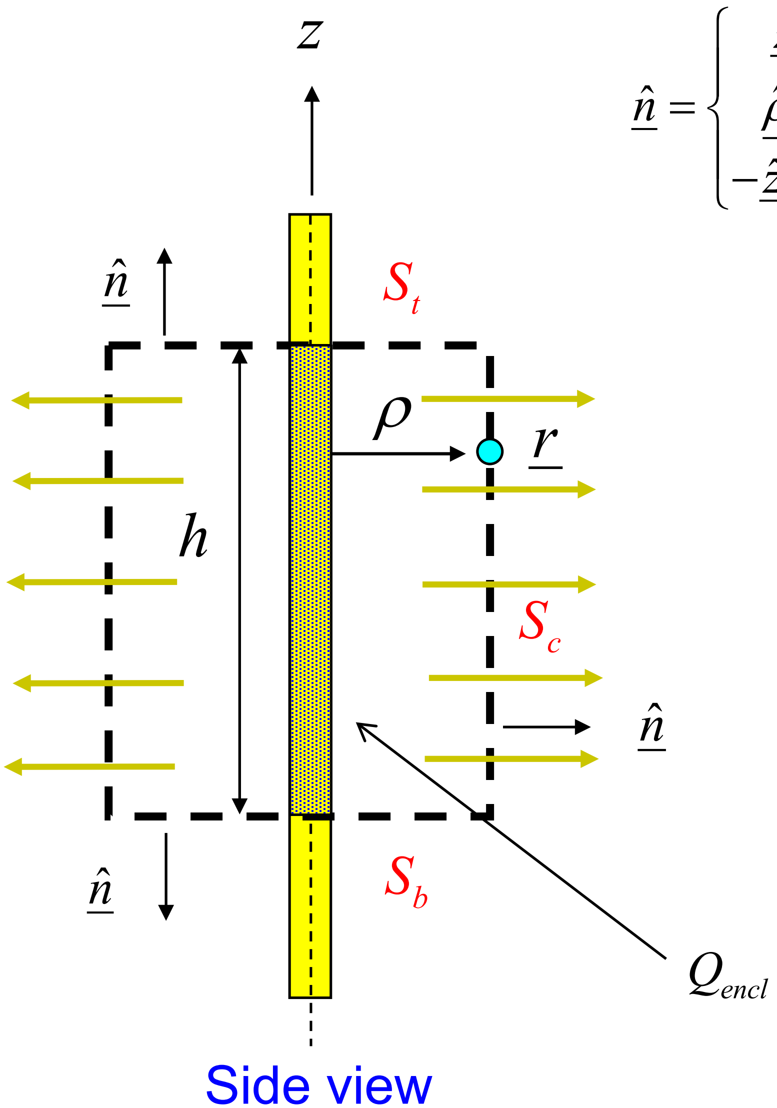
$$\rho_l = \rho_{l0} \; [\mathrm{C/m}]$$

$$\oint_S \underline{D} \cdot \underline{\hat{n}} \, dS = Q_{encl}$$

Assume $\underline{D} = \underline{\hat{\rho}} \, D_\rho(\rho)$

Note: The Gaussian surface cannot be an infinitely tall cylinder.

Example (cont.)



$$\underline{\hat{n}} = \begin{cases} \underline{\hat{z}} & \text{top} \\ \underline{\hat{\rho}} & \text{side} \\ -\underline{\hat{z}} & \text{bottom} \end{cases}$$

$$\oint_S \underline{D} \cdot \underline{\hat{n}}\, dS = Q_{encl}$$

$$\begin{aligned} \text{LHS} &= \oint_S \underline{D} \cdot \underline{\hat{n}}\, dS \\ &= \int_{S_c} \left(D_\rho\, \underline{\hat{\rho}}\right) \cdot \underline{\hat{\rho}}\, dS \\ &+ \int_{S_t} \left(D_\rho\, \underline{\hat{\rho}}\right) \cdot \underline{\hat{z}}\, dS \\ &+ \int_{S_b} \left(D_\rho\, \underline{\hat{\rho}}\right) \cdot \left(-\underline{\hat{z}}\right) dS \end{aligned}$$

Example (cont.)

$$\text{LHS} = \int_{S_c} \left(D_\rho \, \underline{\hat{\rho}} \right) \cdot \underline{\hat{\rho}} \, dS = \int_{S_c} D_\rho \, dS = D_\rho \int_{S_c} dS = D_\rho \left(2\pi\rho h \right)$$

$$\text{RHS} = Q_{encl} = \rho_{l0} h$$

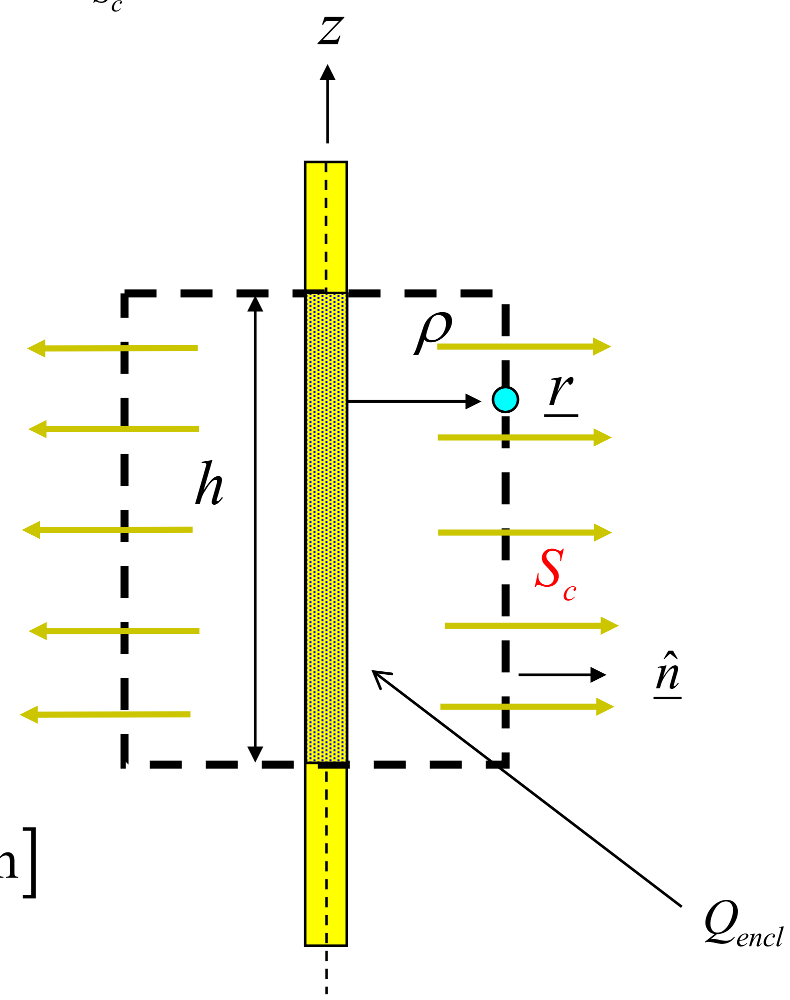
Hence,

$$D_\rho \left(2\pi\rho h \right) = \rho_{l0} h$$

$$\Rightarrow D_\rho = \frac{\rho_{l0}}{2\pi\rho}$$

We then have

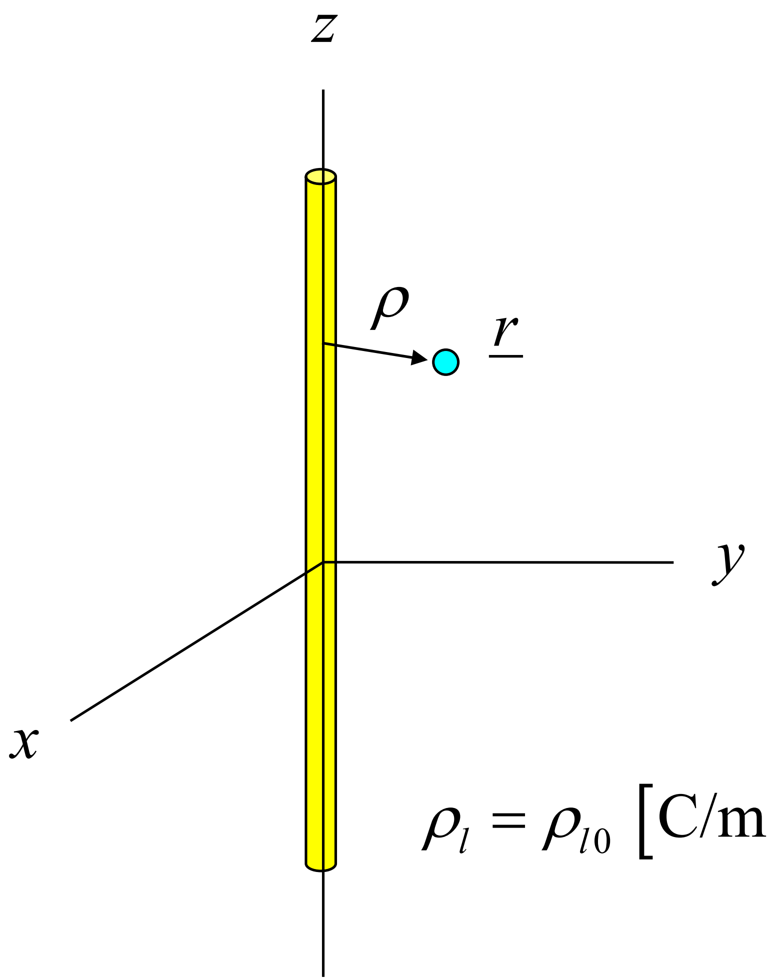
$$\underline{E} = \underline{\hat{\rho}} \left(\frac{\rho_{l0}}{2\pi\varepsilon_0 \rho} \right) \quad [\text{V/m}]$$



Side view

Example (cont.)

Summary



$$\underline{E} = \underline{\hat{\rho}}\left(\frac{\rho_{l0}}{2\pi\varepsilon_0\rho}\right) \quad [\text{V/m}]$$

$$\rho_l = \rho_{l0} \; [\text{C/m}]$$

Note About Cylindrical Coordinates

Note: In cylindrical coordinates, the LHS is always:

$$\text{LHS} = D_\rho \left(2\pi\rho h\right)$$

← **Helpful shortcut!**

Assumption:

$$\rho_v\left(\rho,\phi,z\right) = f\left(\rho\right) \quad \text{(a function of } \rho \text{ only, not } \phi \text{ and } z\text{)}$$

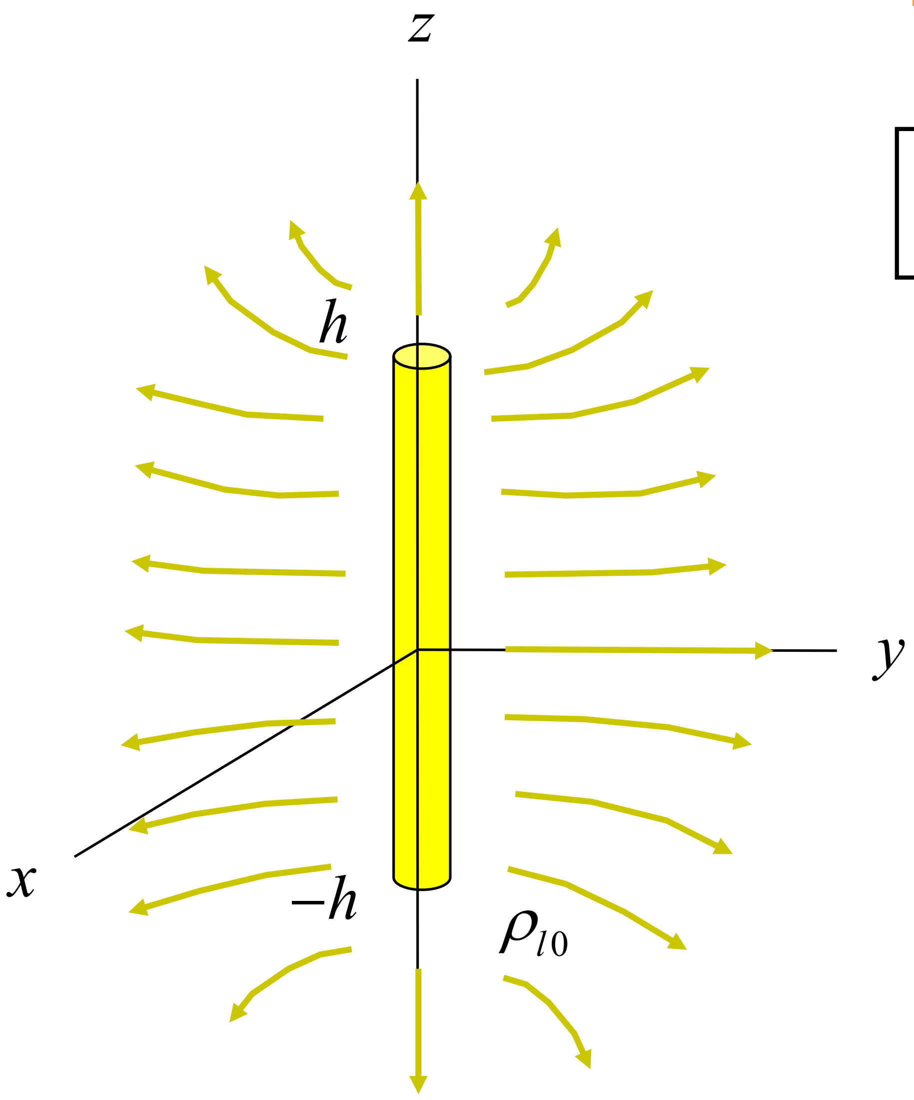
$$\Rightarrow \quad \underline{D} = \underline{\hat{\rho}}\, D_\rho\left(\rho\right)$$

$$\Rightarrow \quad \text{LHS} = \oint_S \underline{D} \cdot \underline{\hat{n}}\, dS = \oint_S \underline{D} \cdot \underline{\hat{\rho}}\, dS = \oint_S D_\rho\, dS = D_\rho\left(2\pi\rho h\right)$$

↑ Cylindrical Gaussian surface

↑ Since D_ρ is constant on cylinder

Example



Finite uniform line charge

This example illustrates when Gauss's Law is <u>not</u> useful.

$$\oint_S \underline{D} \cdot \underline{\hat{n}}\, dS = Q_{encl}$$

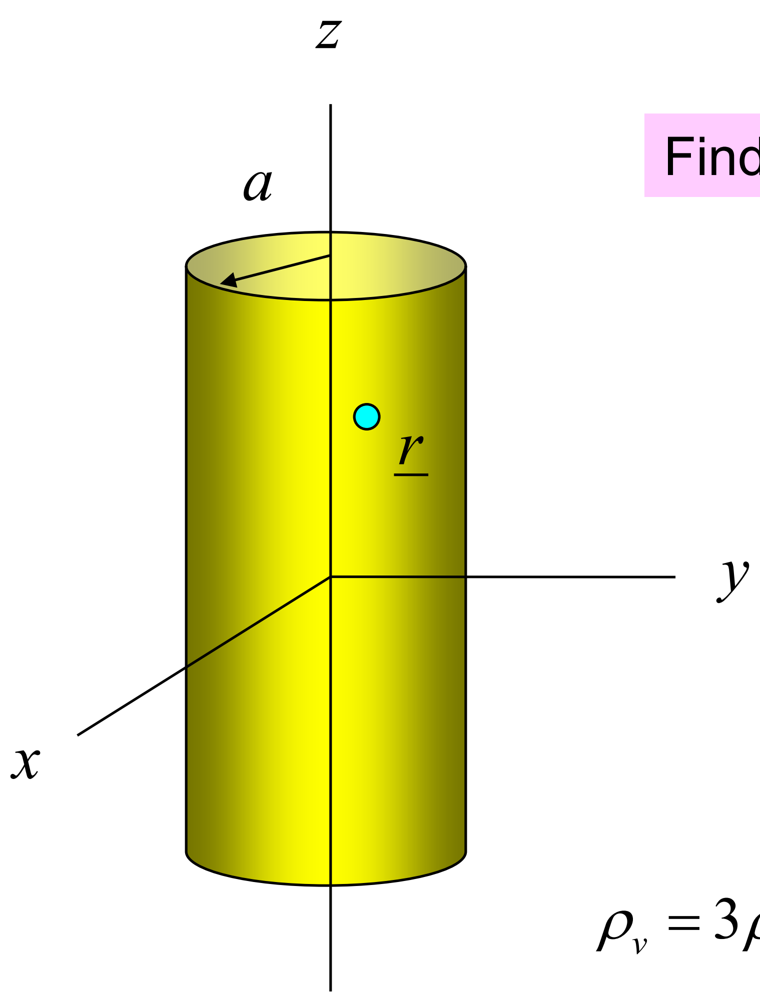
but $\underline{D} \neq \underline{\hat{\rho}}\, D_\rho$

- $\underline{E}$ has more than one component!
- $\underline{E}$ is not a function of only ρ!

Note:
Although Gauss's law is still <u>valid</u>, it is not <u>useful</u> in helping us to solve the problem. We must use Coulomb's law.

Example

Infinite cylinder of <u>non-uniform</u> volume charge density

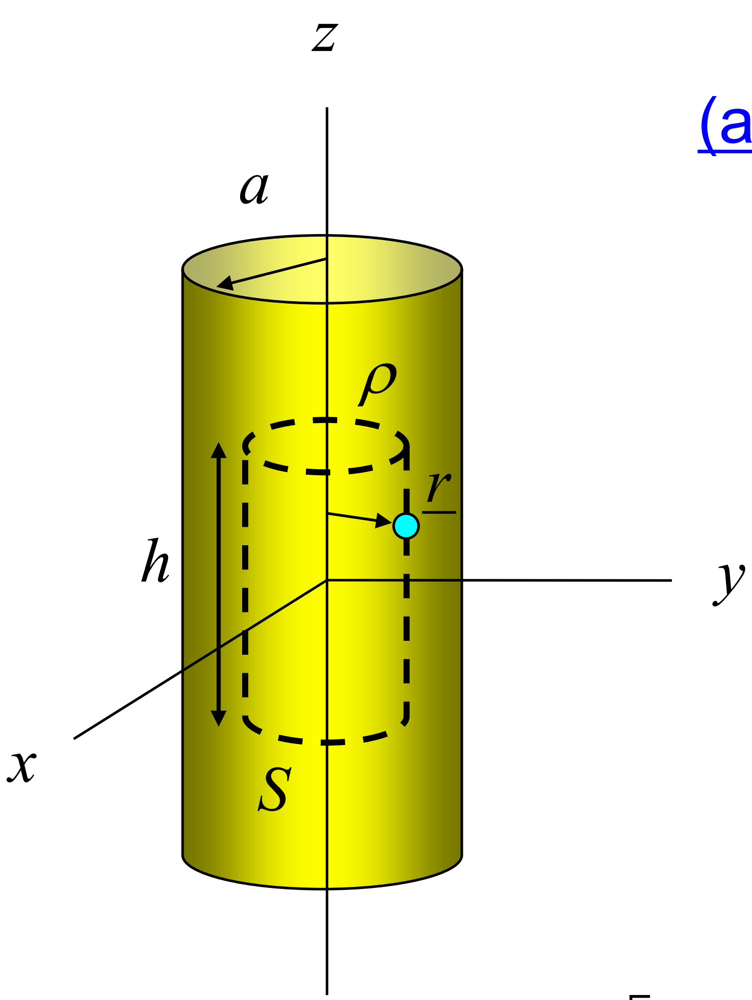


Find the electric field vector everywhere

Note:
This problem would be very difficult to solve using Coulomb's law!

$$\rho_v = 3\rho^2 \left[\mathrm{C/m^3}\right], \;\; \rho < a$$

Example (cont.)



$$\rho_v = 3\rho^2 \left[\mathrm{C/m^3}\right], \;\; \rho < a$$

(a) $\rho < a$

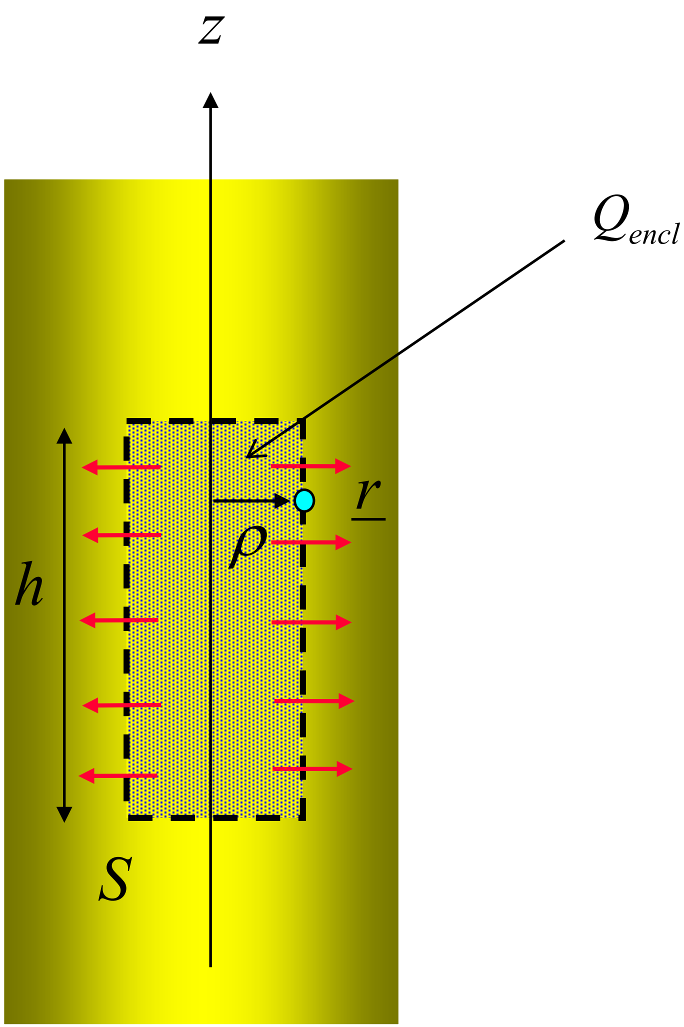
$$\oint_S \underline{D} \cdot \underline{\hat{n}}\, dS = Q_{encl}$$

$$\mathrm{LHS} = D_\rho \left(2\pi\rho h\right)$$

so

$$D_\rho \left(2\pi\rho h\right) = Q_{encl}$$

Example (cont.)



Side view

$$\text{RHS} = Q_{encl} = \int_V \rho_v \, dV$$

$$= \int_{-h/2}^{h/2} \int_0^{2\pi} \int_0^{\rho} \rho_v \, \rho \, d\rho \, d\phi \, dz$$

$$= h(2\pi) \int_0^{\rho} \rho_v \, \rho \, d\rho$$

$$= 2\pi h \int_0^{\rho} \left(3\rho^2\right) \rho \, d\rho$$

$$= 2\pi h \left(\frac{3\rho^4}{4} \right) \Bigg|_0^{\rho}$$

so

$$\text{RHS} = Q_{encl} = \frac{3}{2} \pi h \rho^4$$

Example (cont.)

$$\text{LHS} = D_\rho \left(2\pi\rho h\right)$$

$$\text{RHS} = \frac{3}{2}\pi h \rho^4$$

$$\text{LHS} = \text{RHS}$$

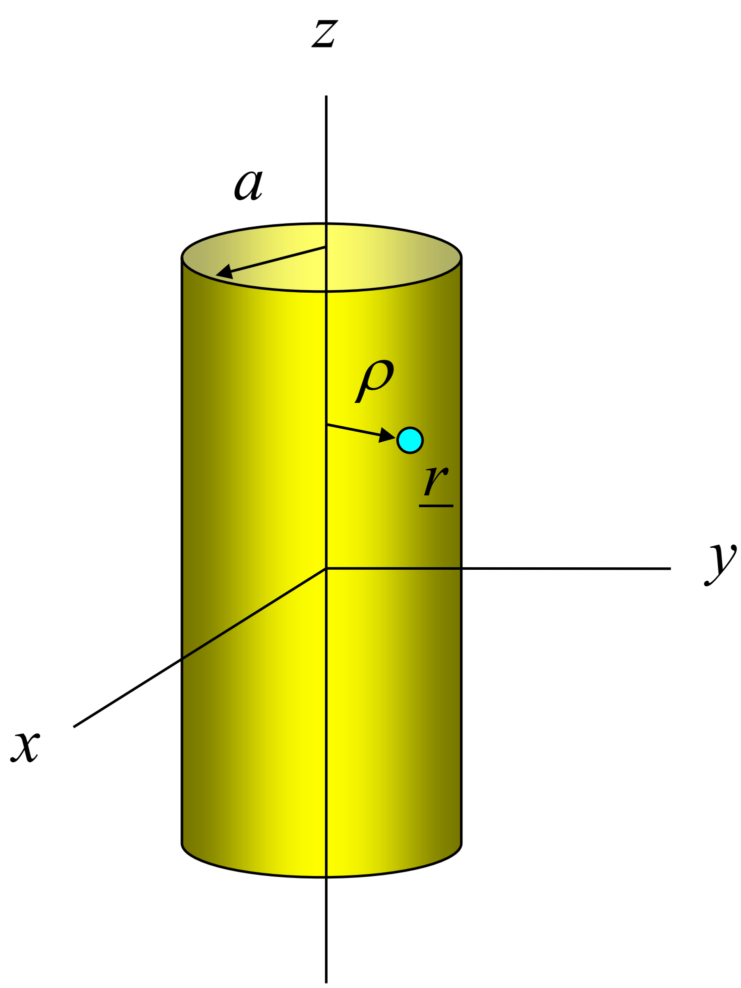
Hence

$$D_\rho \left(2\pi\rho h\right) = \frac{3}{2}\pi h \rho^4$$

$$\Rightarrow D_\rho = \frac{3}{4}\rho^3$$

so

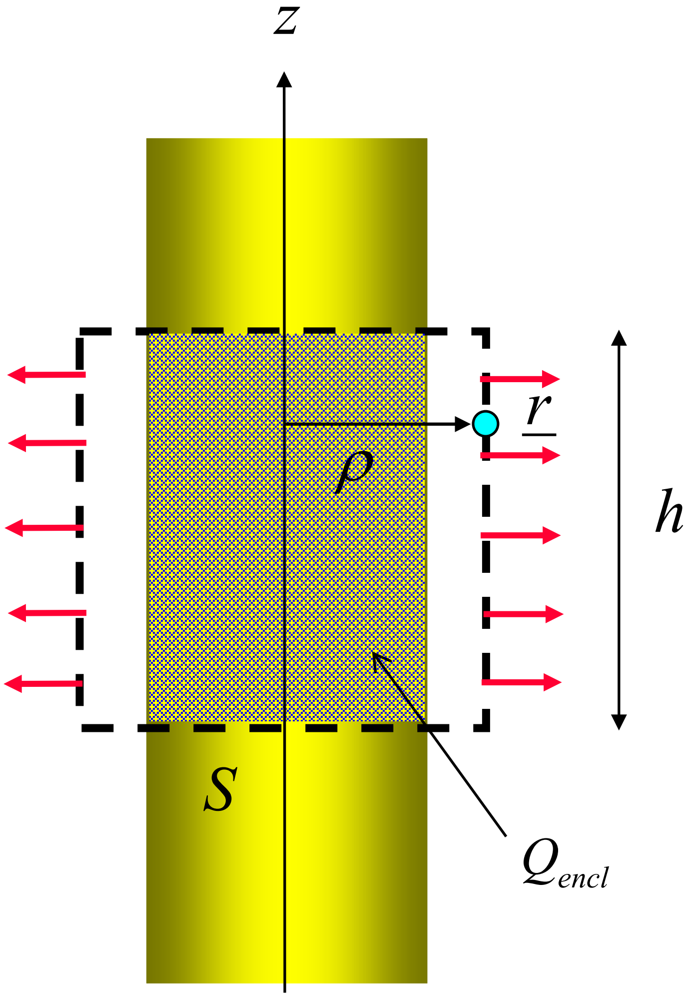
$$\underline{E} = \underline{\hat{\rho}}\left(\frac{3\rho^3}{4\varepsilon_0}\right) \quad [\text{V/m}], \quad \rho \le a$$



$\rho \le a$

Example (cont.)

(b) $\rho > a$



$$\text{LHS} = D_\rho\left(2\pi\rho h\right)$$

$$\text{RHS} = Q_{encl} = 2\pi h\left(\frac{3\rho^4}{4}\right)\Bigg|_0^a$$

so

$$Q_{encl} = \frac{3}{2}\pi h\, a^4$$

$$\text{LHS} = \text{RHS}$$

$$D_\rho\left(2\pi\rho h\right) = \frac{3}{2}\pi h a^4$$

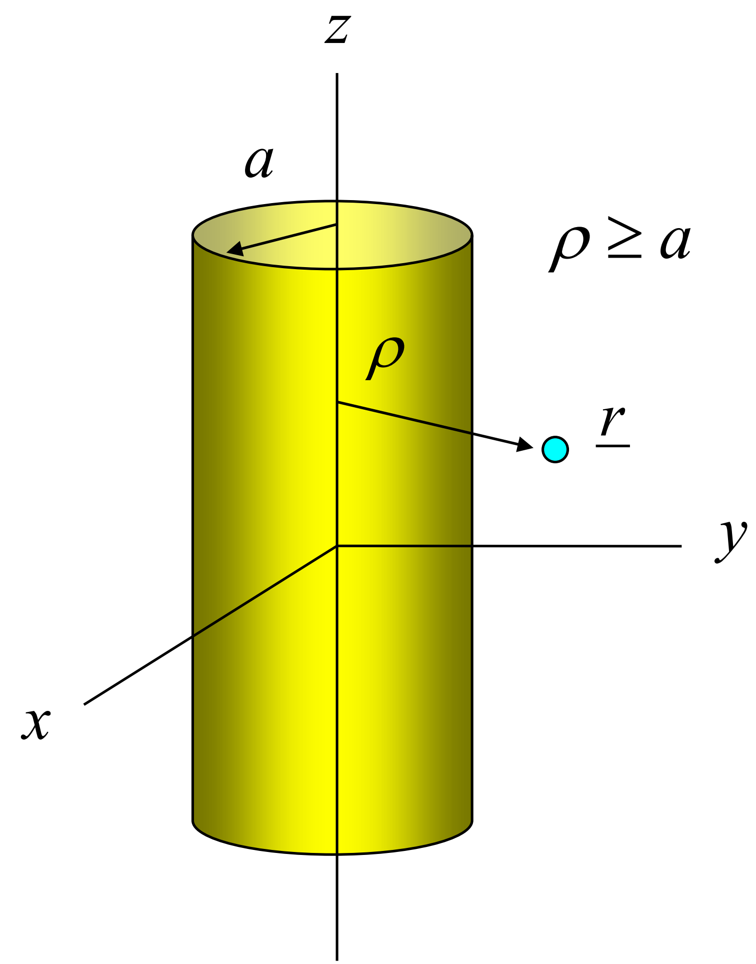
Example (cont.)

$$D_\rho\left(2\pi\rho h\right)=\frac{3}{2}\pi h a^4$$

$$\Rightarrow D_\rho=\frac{\frac{3}{4}a^4}{\rho}$$

Hence, we have

$$\underline{E}=\hat{\underline{\rho}}\left(\frac{3a^4}{4\varepsilon_0\rho}\right)\quad[\mathrm{V/m}],\qquad \rho\geq a$$



Note:
Outside the cylinder, the electric field is the same as that coming from an equivalent line charge located on the z axis at the center.

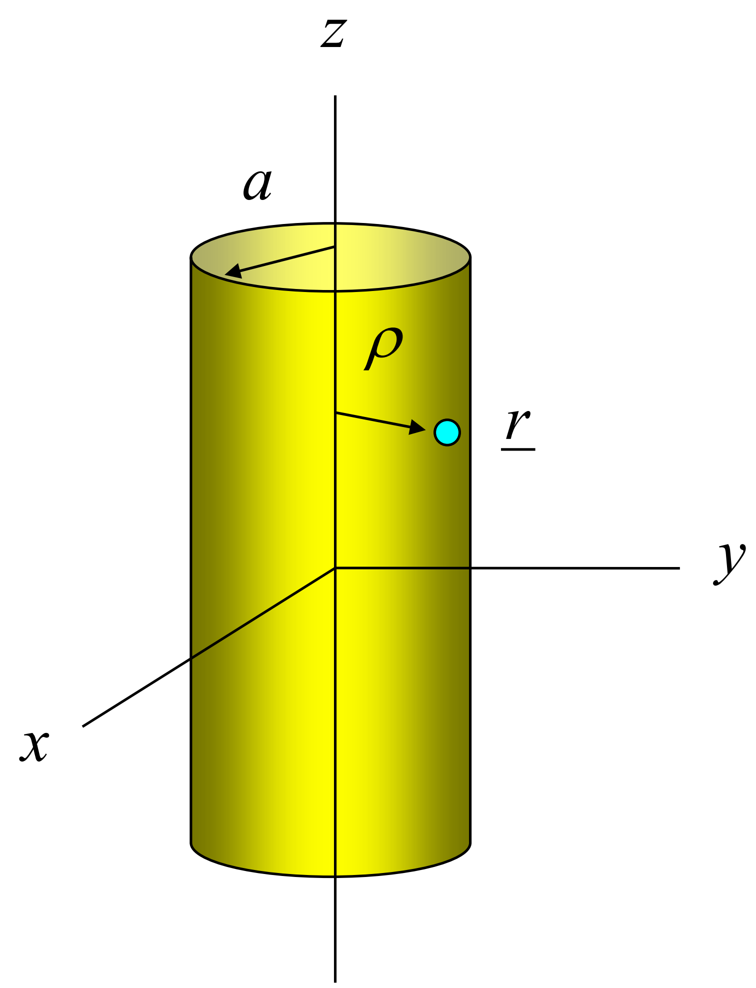
See if you can prove this to yourself!

Example (cont.)

Summary

$$\underline{E} = \underline{\hat{\rho}}\left(\frac{3\rho^3}{4\varepsilon_0}\right) \quad [\text{V/m}], \quad \rho \le a$$

$$\underline{E} = \underline{\hat{\rho}}\left(\frac{3a^4}{4\varepsilon_0 \rho}\right) \quad [\text{V/m}], \quad \rho \ge a$$



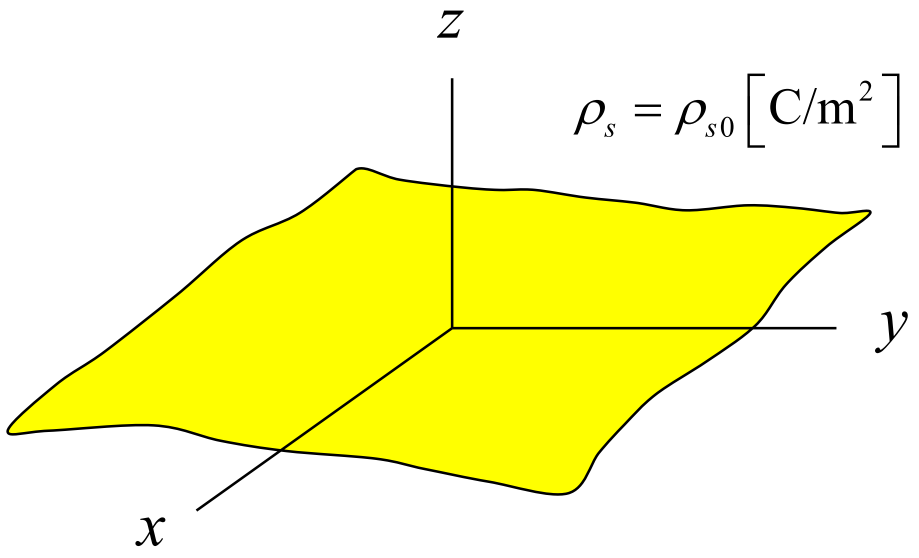
or

$$\underline{E} = \underline{\hat{\rho}}\left(\frac{\rho_{l0}^{eq}}{2\pi\varepsilon_0 \rho}\right) \quad [\text{V/m}], \quad \rho \ge a \qquad \left(\rho_{l0}^{eq} = \frac{3}{2}\pi a^4 \; [\text{C/m}]\right)$$

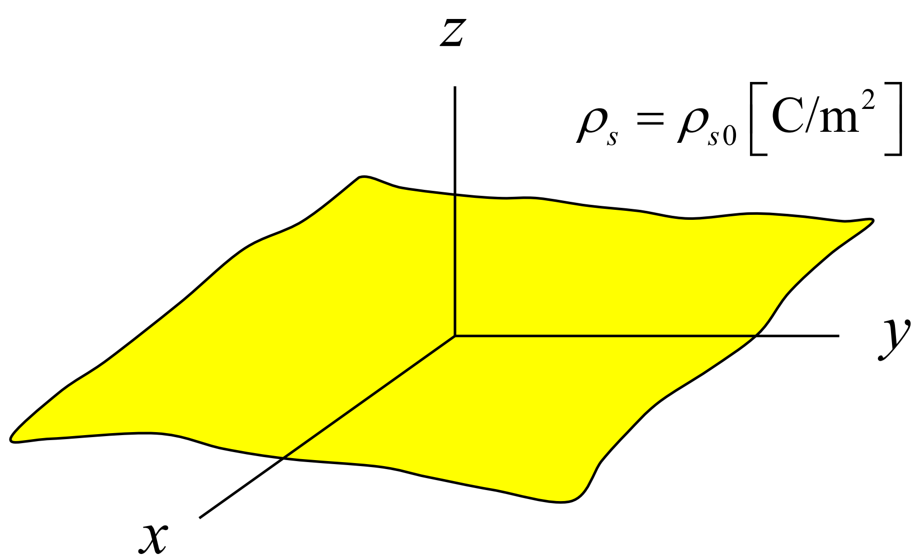
Example

Infinite sheet of uniform surface charge density

Find the electric field vector everywhere



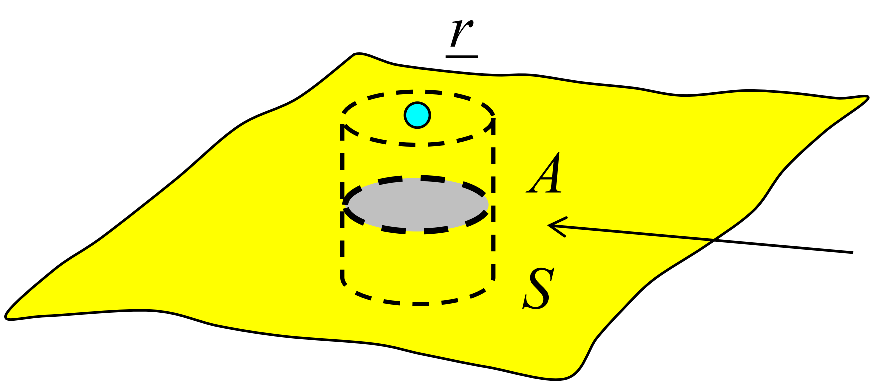
Example (cont.)



Assume

$$\underline{D} = \underline{\hat{z}}\, D_z \left(z \right)$$

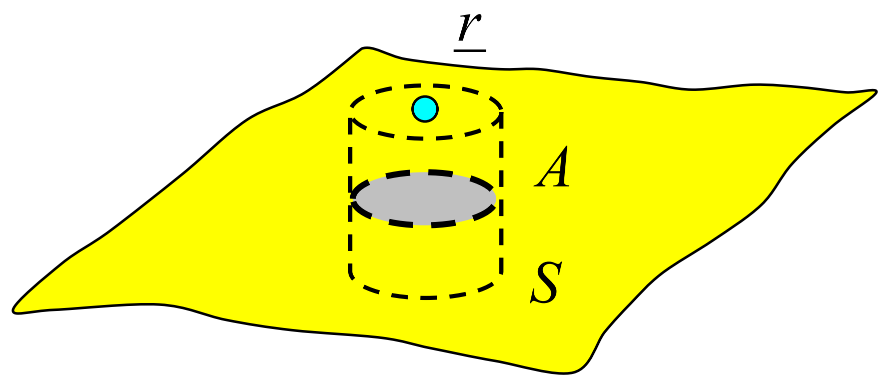
Consider first $z > 0$



$$\oint_S \underline{D} \cdot \underline{\hat{n}} \; dS = Q_{encl}$$

Example (cont.)

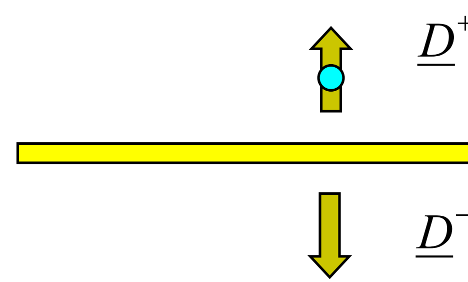
$$\oint_S \left(D_z\, \underline{\hat{z}}\right) \cdot \underline{\hat{n}}\, dS = Q_{encl}$$



$$\text{LHS} = \int_{S_{top}} \left(D_z\, \underline{\hat{z}}\right) \cdot \underline{\hat{z}}\, dS + \int_{S_{bottom}} \left(D_z \underline{\hat{z}}\right) \cdot \left(-\underline{\hat{z}}\right) dS$$

so

$$\text{LHS} = D_z^+ A - D_z^- A$$



Assume $D_z^- = -D_z^+$

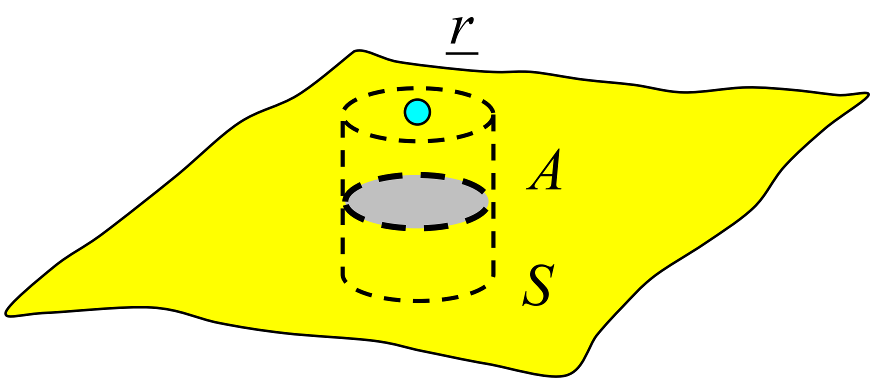
We then have $\text{LHS} = 2D_z^+ A$

Example (cont.)

For the charge enclosed we have

$$\text{RHS} = Q_{encl} = \rho_{s0} A$$

Hence, from Gauss's law we have



$$\text{LHS} = \text{RHS}$$

$$\Rightarrow \quad 2AD_z^{\,+} = \rho_{s0} A$$

so

$$D_z^{+} = \frac{\rho_{s0} A}{2A} = \frac{\rho_{s0}}{2}$$

Therefore

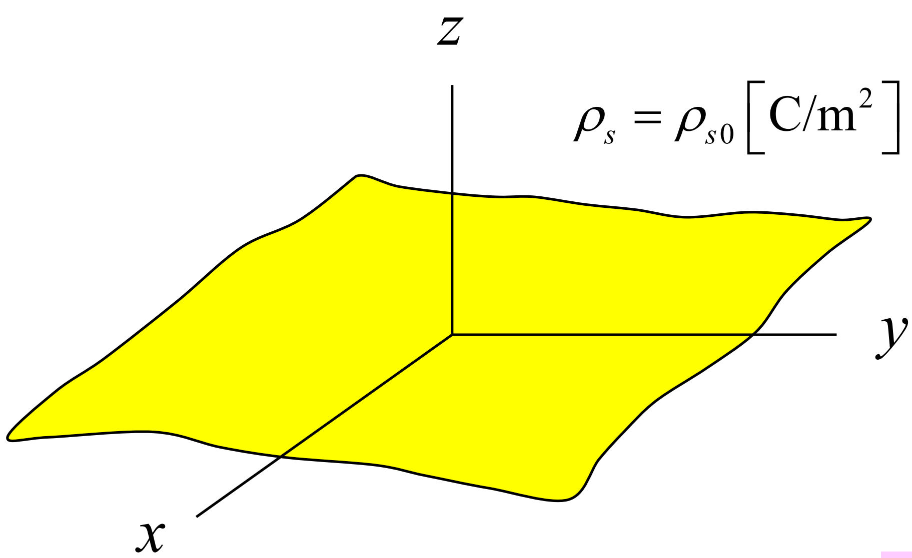
$$\underline{E}^{+} = \hat{\underline{z}}\left(\frac{\rho_{s0}}{2\varepsilon_0}\right)$$

We then also have:

$$\underline{E}^{-} = -\hat{\underline{z}}\left(\frac{\rho_{s0}}{2\varepsilon_0}\right)$$

Example (cont.)

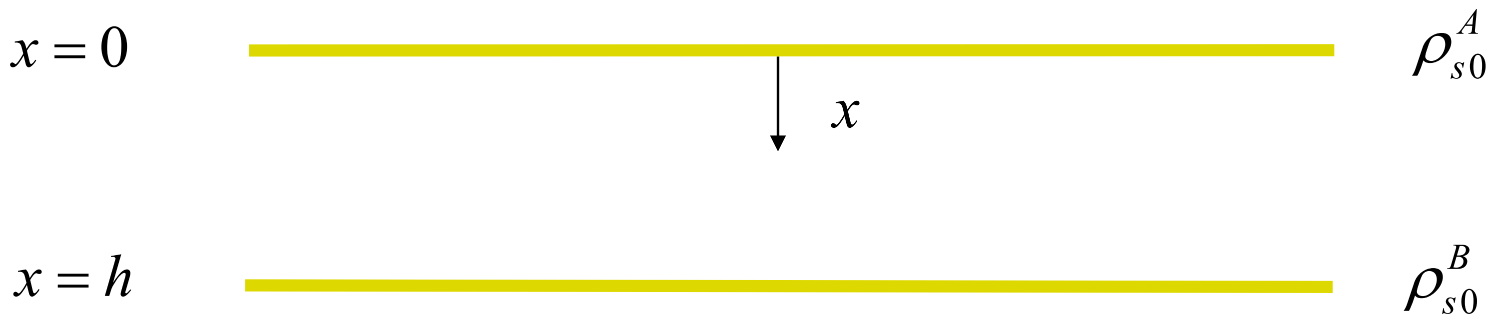
Summary



$$\underline{E} = \pm \underline{\hat{z}} \left(\frac{\rho_{s0}}{2\varepsilon_0} \right) \quad [\mathrm{V/m}];$$

$+$ for $z > 0$, $-$ for $z < 0$

Example



From superposition:

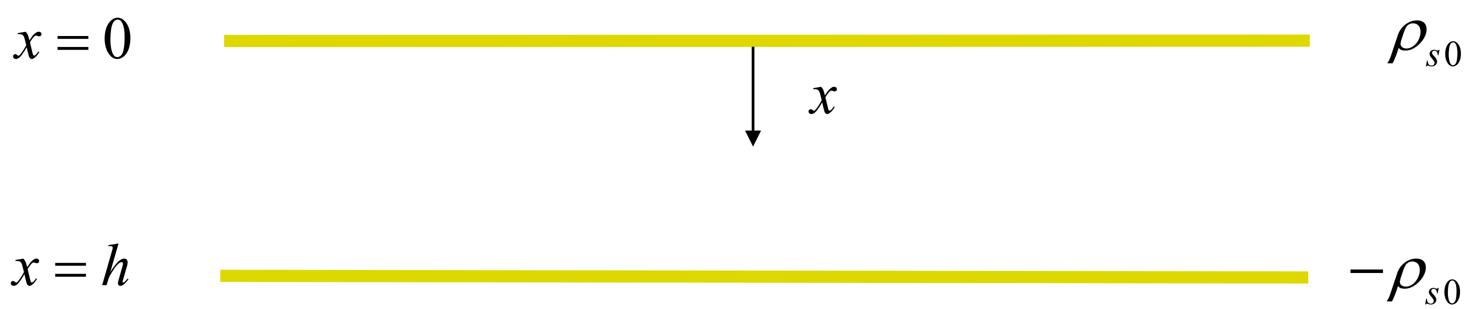
(a) $x > h$ $\qquad \underline{E} = \underline{\hat{x}}\left(\dfrac{\rho_{s0}^{A}}{2\varepsilon_0} + \dfrac{\rho_{s0}^{B}}{2\varepsilon_0}\right)$

(b) $0 < x < h$ $\qquad \underline{E} = \underline{\hat{x}}\left(\dfrac{\rho_{s0}^{A}}{2\varepsilon_0} - \dfrac{\rho_{s0}^{B}}{2\varepsilon_0}\right)$

(c) $x < 0$ $\qquad \underline{E} = -\underline{\hat{x}}\left(\dfrac{\rho_{s0}^{A}}{2\varepsilon_0} + \dfrac{\rho_{s0}^{B}}{2\varepsilon_0}\right)$

Example (cont.)

Choose: $\rho_{s0}^{A} = \rho_{s0}\,,\;\; \rho_{s0}^{B} = -\rho_{s0}$



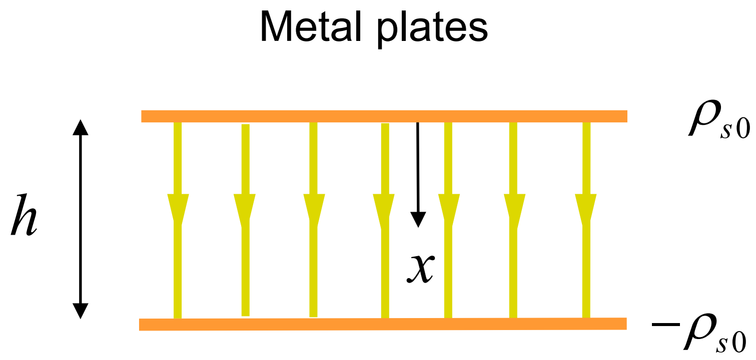
(a) $x > h$ $\qquad \underline{E} = \underline{\hat{x}}\left(\dfrac{\rho_{s0}^{A}}{2\varepsilon_0} + \dfrac{\rho_{s0}^{B}}{2\varepsilon_0}\right) \;\Rightarrow\; \underline{E} = \underline{0}$

(b) $0 < x < h$ $\qquad \underline{E} = \underline{\hat{x}}\left(\dfrac{\rho_{s0}^{A}}{2\varepsilon_0} - \dfrac{\rho_{s0}^{B}}{2\varepsilon_0}\right) \;\Rightarrow\; \underline{E} = \underline{\hat{x}}\left(\dfrac{\rho_{s0}}{\varepsilon_0}\right)$

(c) $x < 0$ $\qquad \underline{E} = -\underline{\hat{x}}\left(\dfrac{\rho_{s0}^{A}}{2\varepsilon_0} + \dfrac{\rho_{s0}^{B}}{2\varepsilon_0}\right) \;\Rightarrow\; \underline{E} = \underline{0}$

Example (cont.)

Ideal parallel-plate capacitor



Note:
The metal plates support the charge, and the charge produces the electric field.

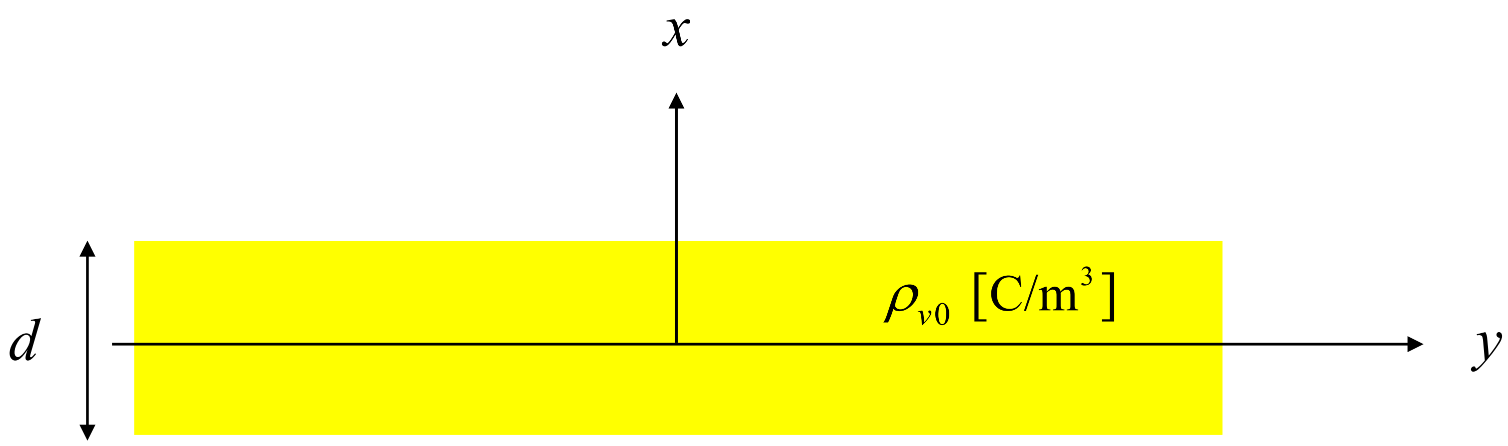
$$\underline{E} = \underline{\hat{x}}\left(\frac{\rho_{s0}}{\varepsilon_0}\right) \qquad 0 < x < h$$

$$\underline{D} = \underline{\hat{x}}\,\rho_{s0}$$

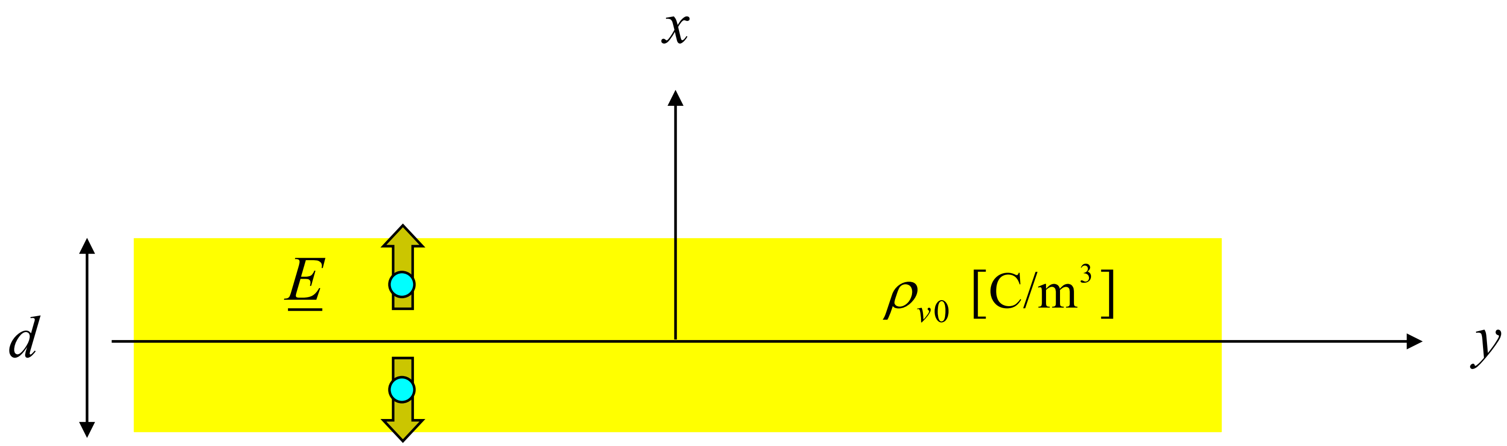
Example

Infinite slab of uniform volume charge density

Find the electric field vector everywhere



Example (cont.)

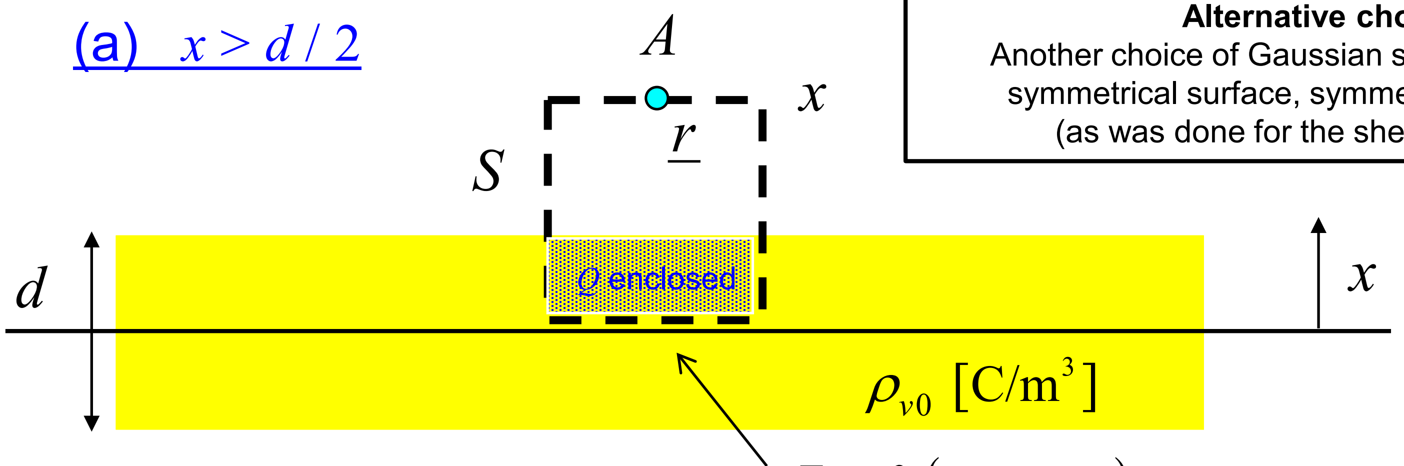


Assume $\begin{cases} \underline{E} = \underline{\hat{x}}\, E_x(x) \\ E_x(-x) = -E_x(x) \end{cases}$ (E_x is an odd function.)

Also $E_x(0) = 0$ (symmetry)

Example (cont.)

(a) $x > d/2$



$E_x = 0$ (on bottom)

Alternative choice:
Another choice of Gaussian surface would be a symmetrical surface, symmetrical about $x = 0$
(as was done for the sheet of charge).

$$\oint_S \left(D_x\,\underline{\hat{x}}\right)\cdot\underline{\hat{n}}\;dS = Q_{encl}$$

$$\Rightarrow \int_{S_{top}} \left(D_x\,\underline{\hat{x}}\right)\cdot\underline{\hat{x}}\;dS \;+\; \int_{S_{bottom}} \left(D_x\,\underline{\hat{x}}\right)\cdot\left(-\underline{\hat{x}}\right)\,dS = Q_{encl}$$

$$\Rightarrow D_x\left(x\right)A - D_x\left(0\right)A = Q_{encl} = \rho_{v0}\,A(d/2)$$

$$\Rightarrow D_x\left(x\right) = \rho_{v0}\,d/2$$

$$\underline{E} = \underline{\hat{x}}\left(\frac{\rho_{v0}d}{2\varepsilon_0}\right)\quad [\mathrm{V/m}],\qquad x \geq (d/2)$$

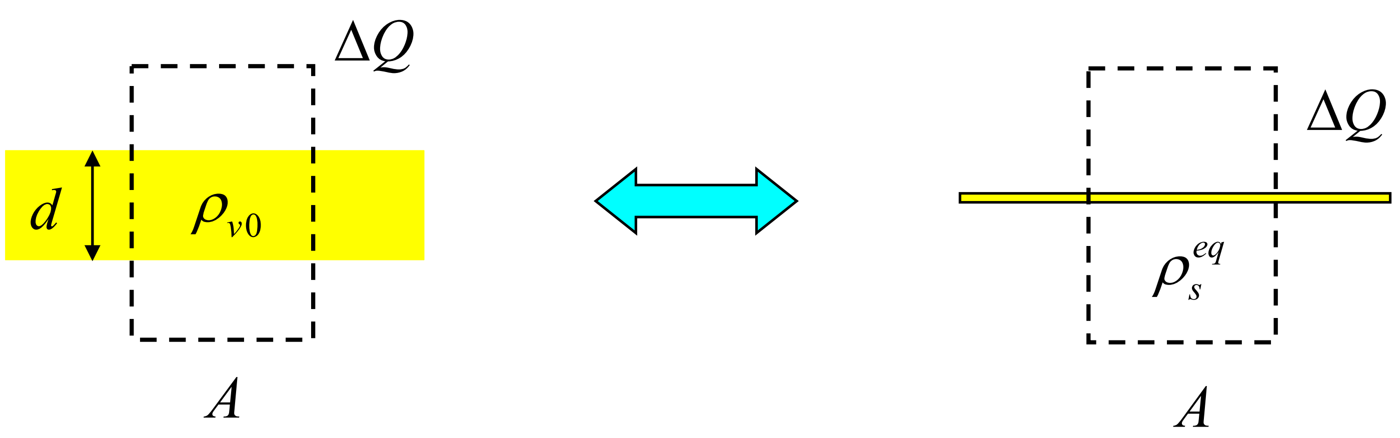
Example (cont.)

Note: If we define $\rho_s^{eq} = \rho_{v0}\, d$

then $\underline{E} = \underline{\hat{x}} \left(\dfrac{\rho_s^{eq}}{2\varepsilon_0} \right) \; [\text{V/m}]$ (sheet formula)

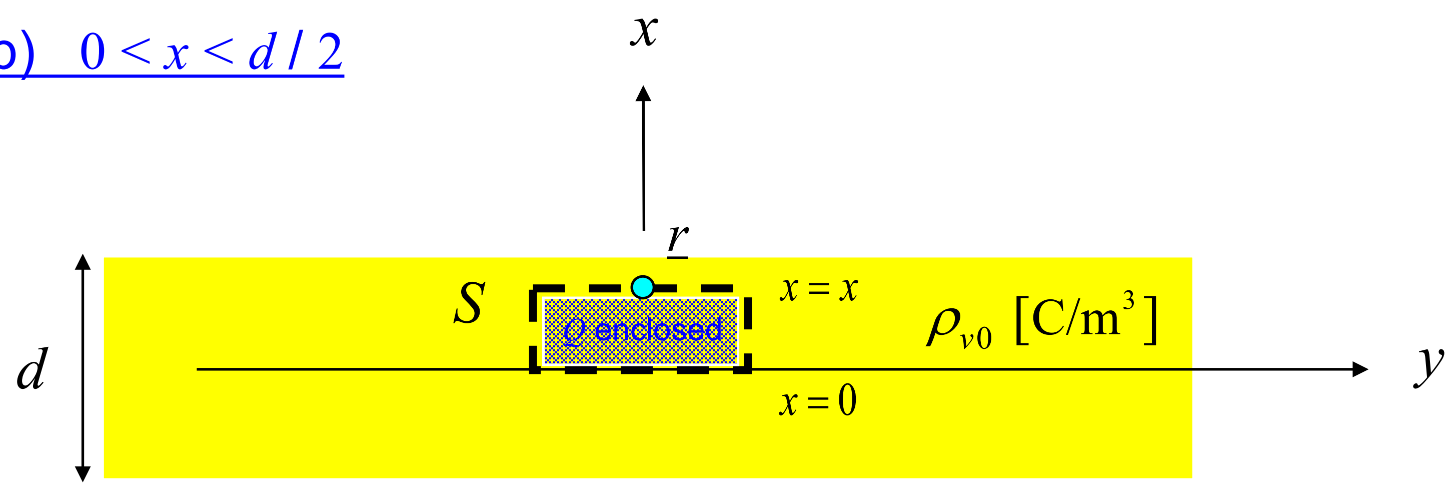
Note: $\Delta Q = \rho_{v0}\, A\, d = \rho_s^{eq} A$

so $\rho_s^{eq} = \rho_{v0}\, d$



Example (cont.)

(b) $0 < x < d/2$



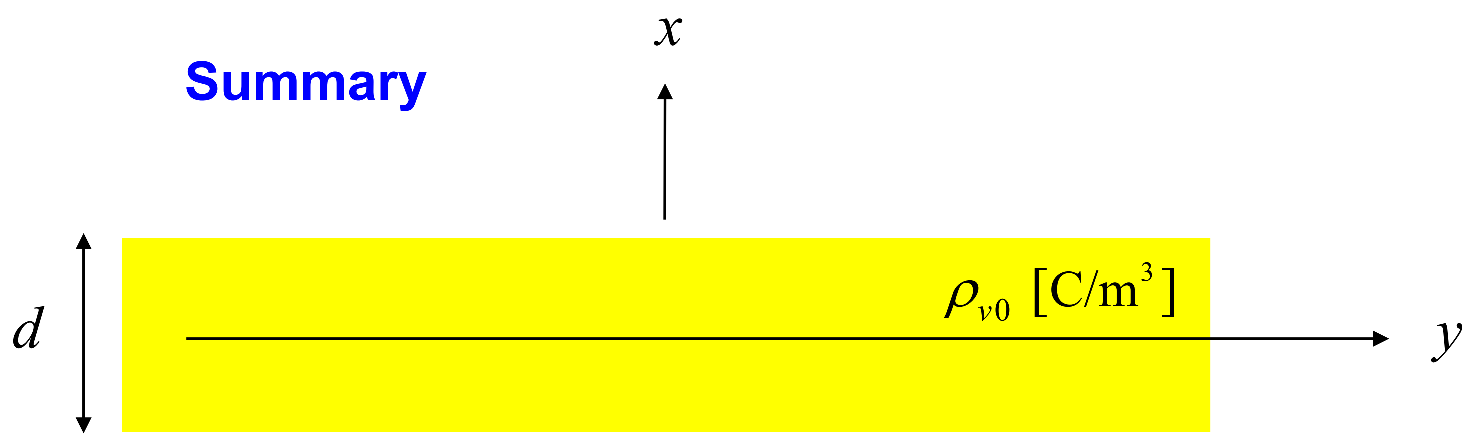
$$D_x(x)A - D_x(0)A = Q_{encl} = \rho_{v0}\, A\, x$$

$$\Rightarrow D_x = \rho_{v0}\, x$$

$$\underline{E} = \underline{\hat{x}}\left(\frac{\rho_{v0} x}{\varepsilon_0}\right) \quad [\text{V/m}], \qquad 0 \le x \le d/2$$

Example (cont.)

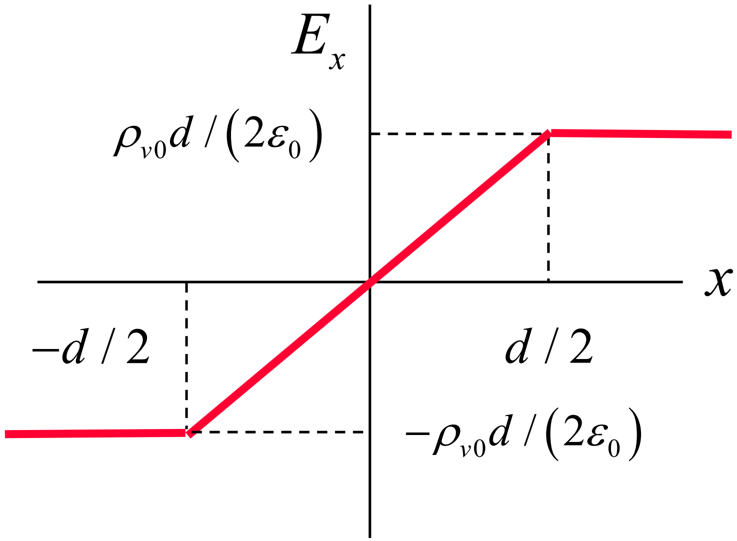
Summary



$$\underline{E} = \underline{\hat{x}}\left(\frac{\rho_{v0} d}{2\varepsilon_0}\right) \quad [\mathrm{V/m}], \quad x \ge (d/2)$$

$$\underline{E} = -\underline{\hat{x}}\left(\frac{\rho_{v0} d}{2\varepsilon_0}\right) \quad [\mathrm{V/m}], \quad x \le -(d/2)$$

$$\underline{E} = \underline{\hat{x}}\left(\frac{\rho_{v0} x}{\varepsilon_0}\right) \quad [\mathrm{V/m}], \quad -d/2 \le x \le d/2$$



Note:
In the second formula we had to introduce a minus sign, while in the third one we did not.