ECE 3318
Applied Electricity and Magnetism

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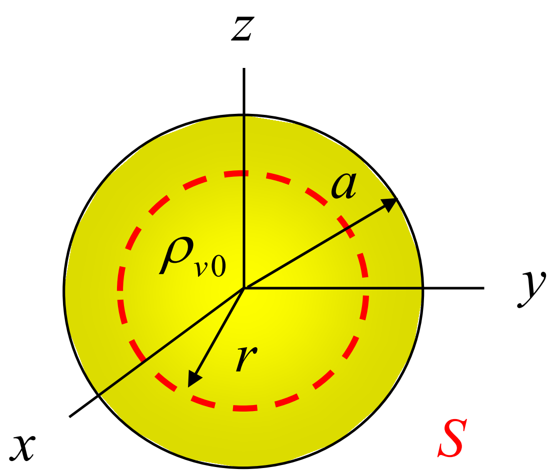
Prof. David R. Jackson
Dept. of ECE

Notes 13
Divergence

Divergence – The Physical Concept

Find the flux going outward through a sphere of radius r .



Spherical region of uniform volume charge density

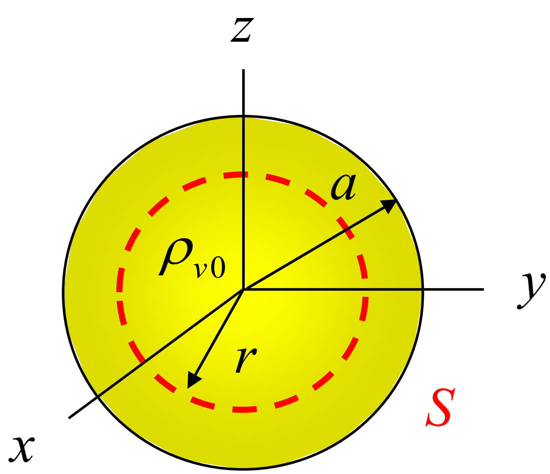
$$\psi = \int_S \underline{D} \cdot \underline{\hat{n}}\, dS$$

$$= \int_S \underline{D} \cdot \underline{\hat{r}}\, dS$$

$$= \int_S D_r \, dS$$

$$= D_r 4\pi r^2$$

Divergence – The Physical Concept



From Gauss's law:

$$D_r = \frac{\rho_{v0}\, r}{3} \quad \left[\mathrm{C/m^2}\right] \qquad (r < a)$$

$$D_r = \frac{\rho_{v0}\, a^3}{3r^2} \quad \left[\mathrm{C/m^2}\right] \qquad (r > a)$$

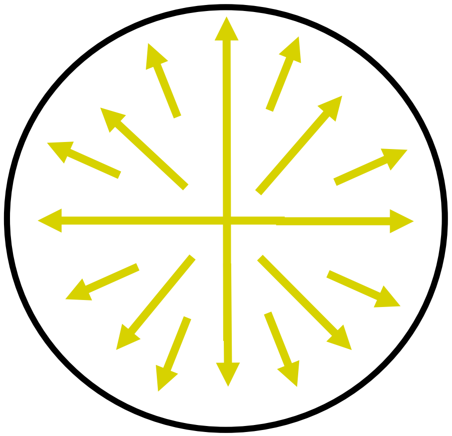
Also, from last slide: $\psi = D_r 4\pi r^2$

Hence

$$\psi = \frac{4}{3}\pi r^3 \rho_{v0} \quad (r < a) \quad \text{(increasing with distance inside sphere)}$$

$$\psi = \frac{4}{3}\pi a^3 \rho_{v0} \quad (r > a) \quad \text{(constant outside sphere)}$$

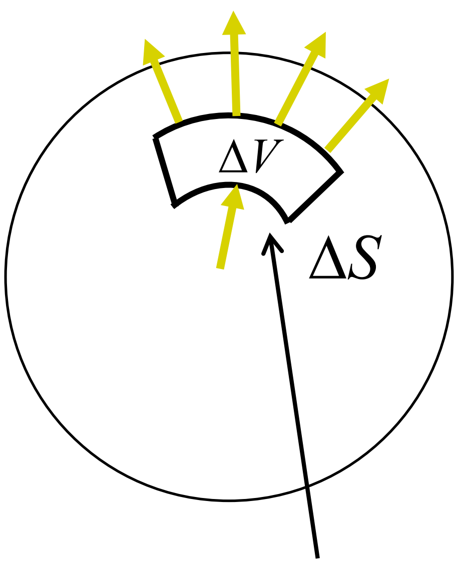
Divergence -- Physical Concept (cont.)



Observation:

More flux lines are added as the radius increases (as long as we stay <u>inside</u> the charge).

$$\Delta\psi = \int_{\Delta S} \underline{D} \cdot \underline{\hat{n}}\, dS > 0$$



Small "curvilinear cube"

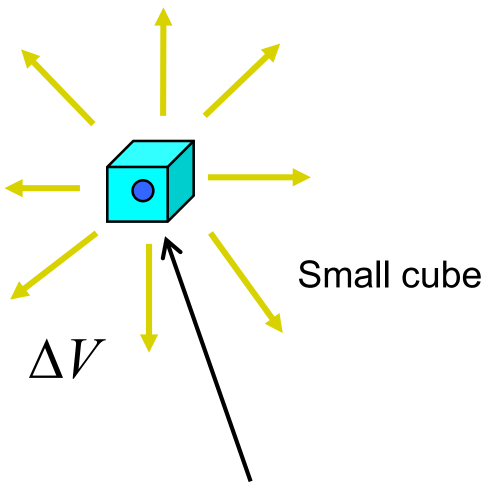
The net flux out of a small volume ΔV inside the charge is <u>not</u> zero.

Divergence is a mathematical way of describing this.

Divergence Definition

Definition of divergence:

$$div\ \underline{D} \equiv \lim_{\Delta V \to 0} \frac{1}{\Delta V} \oint_{\Delta S} \underline{D} \cdot \underline{\hat{n}}\, dS$$



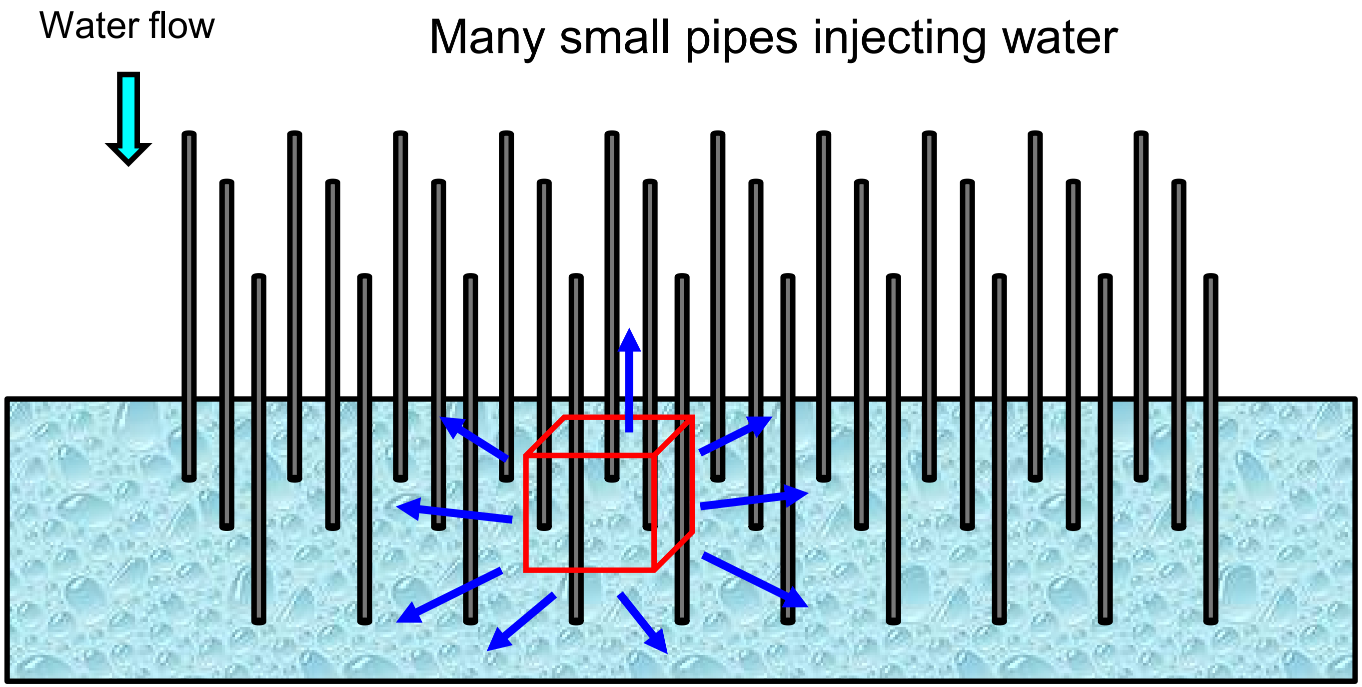
$\underline{\hat{n}}$ = outward normal

Divergence at a given point

Note: The limit exists independent of the shape of the volume (proven later).

A region with a positive divergence acts as a “source” of flux lines.
A region with a negative divergence acts as a “sink” of flux lines.

Divergence Definition



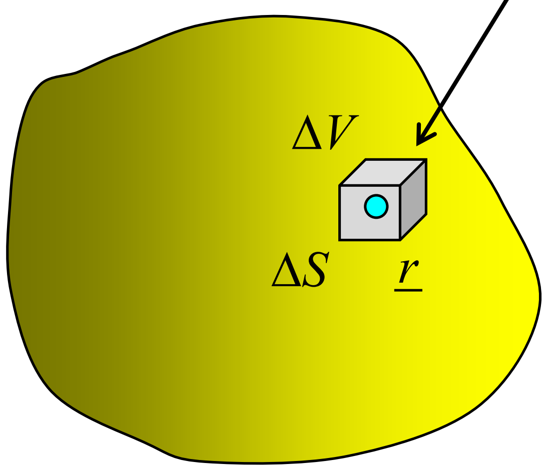
Tub of water

$\underline{V}(x, y, z) =$ velocity vector of water inside tub

$$\nabla \cdot \underline{V} > 0 \ \ (\text{inside tub})$$

Gauss's Law -- Differential Form

Apply divergence definition to a small volume ΔV inside a region of charge:



$$div\ \underline{D} \equiv \lim_{\Delta V \to 0} \frac{1}{\Delta V} \oint_{\Delta S} \underline{D} \cdot \underline{\hat{n}}\, dS$$

Gauss's law:

$$\oint_{\Delta S} \underline{D} \cdot \underline{\hat{n}}\, dS = Q_{encl} \approx \rho_v(\underline{r})\, \Delta V$$

Hence

$$div\ \underline{D}(\underline{r}) = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \left(\rho_v(\underline{r}) \Delta V \right)$$

$$= \rho_v(\underline{r})$$

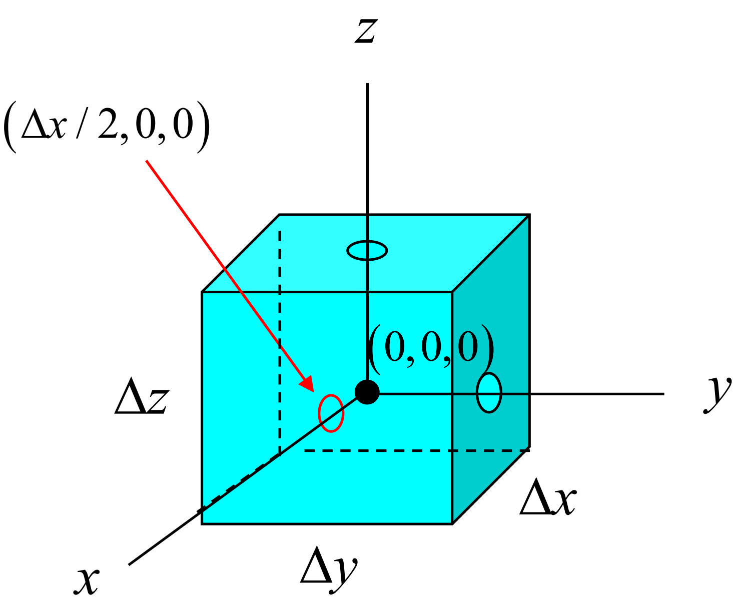
Gauss's Law -- Differential Form (cont.)

The electric Gauss law in point (differential) form:

$$div\,\underline{D}\left(\underline{r}\right) = \rho_v\left(\underline{r}\right)$$

This is one of Maxwell's equations.

Calculation of Divergence



$$div\ \underline{D} \equiv \lim_{\Delta V \to 0} \frac{1}{\Delta x\, \Delta y\, \Delta z} \oint_S \underline{D} \cdot \underline{\hat{n}}\, dS$$

$$\oint_S \underline{D} \cdot \underline{\hat{n}}\, dS \approx D_x\left(\frac{\Delta x}{2}, 0, 0\right)\Delta y\, \Delta z$$
$$- D_x\left(-\frac{\Delta x}{2}, 0, 0\right)\Delta y\, \Delta z$$
$$+ D_y\left(0, \frac{\Delta y}{2}, 0\right)\Delta x\, \Delta z$$
$$- D_y\left(0, -\frac{\Delta y}{2}, 0\right)\Delta x\, \Delta z$$
$$+ D_z\left(0, 0, \frac{\Delta z}{2}\right)\Delta x\, \Delta y$$
$$- D_z\left(0, 0, -\frac{\Delta z}{2}\right)\Delta x\, \Delta y$$

Assume that the point of interest is at the origin for simplicity (the center of the cube).

The integrals over the 6 faces are approximated by "sampling" at the centers of the faces.

Calculation of Divergence (cont.)

$$\operatorname{div} \underline{D} \equiv \lim_{\Delta V \to 0} \frac{1}{\Delta x \, \Delta y \, \Delta z} \oint_S \underline{D} \cdot \underline{\hat{n}} \, dS$$

$$\begin{aligned}
\oint_S \underline{D} \cdot \underline{\hat{n}} \, dS \approx\ & D_x\left(\frac{\Delta x}{2}, 0, 0\right) \Delta y \, \Delta z \\
& - D_x\left(-\frac{\Delta x}{2}, 0, 0\right) \Delta y \, \Delta z \\
& + D_y\left(0, \frac{\Delta y}{2}, 0\right) \Delta x \, \Delta z \\
& - D_y\left(0, -\frac{\Delta y}{2}, 0\right) \Delta x \, \Delta z \\
& + D_z\left(0, 0, \frac{\Delta z}{2}\right) \Delta x \, \Delta y \\
& - D_z\left(0, 0, -\frac{\Delta z}{2}\right) \Delta x \, \Delta y
\end{aligned}$$

$$\frac{1}{\Delta x \, \Delta y \, \Delta z} \oint_S \underline{D} \cdot \underline{\hat{n}} \, dS \approx \frac{D_x\left(\frac{\Delta x}{2}, 0, 0\right) - D_x\left(-\frac{\Delta x}{2}, 0, 0\right)}{\Delta x} + \frac{D_y\left(0, \frac{\Delta y}{2}, 0\right) - D_y\left(0, -\frac{\Delta y}{2}, 0\right)}{\Delta y} + \frac{D_z\left(0, 0, \frac{\Delta z}{2}\right) - D_z\left(0, 0, -\frac{\Delta z}{2}\right)}{\Delta z}$$

Calculation of Divergence (cont.)

$$div\ \underline{D} \equiv \lim_{\Delta V \to 0} \left\{ \frac{D_x\left(\frac{\Delta x}{2},0,0\right) - D_x\left(-\frac{\Delta x}{2},0,0\right)}{\Delta x} + \frac{D_y\left(0,\frac{\Delta y}{2},0\right) - D_y\left(0,-\frac{\Delta y}{2},0\right)}{\Delta y} + \frac{D_z\left(0,0,\frac{\Delta z}{2}\right) - D_z\left(0,0,-\frac{\Delta z}{2}\right)}{\Delta z} \right.$$

Hence

$$div\ \underline{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

Calculation of Divergence (cont.)

Final result in rectangular coordinates:

$$div\ \underline{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

The "del" operator

$$\nabla \equiv \underline{\hat{x}}\frac{\partial}{\partial x} + \underline{\hat{y}}\frac{\partial}{\partial y} + \underline{\hat{z}}\frac{\partial}{\partial z}$$

This is *a vector operator*.

Examples of *derivative operators*:

Note:
The del operator is only defined in rectangular coordinates.

$$\nabla \neq \underline{\hat{\rho}}\frac{\partial}{\partial \rho} + \underline{\hat{\phi}}\frac{\partial}{\partial \phi} + \underline{\hat{z}}\frac{\partial}{\partial z}$$

$$\nabla \neq \underline{\hat{r}}\frac{\partial}{\partial r} + \underline{\hat{\theta}}\frac{\partial}{\partial \theta} + \underline{\hat{\phi}}\frac{\partial}{\partial \phi}$$

scalar $\frac{d}{dx}$:

$$\frac{d}{dx}(\sin x) = \cos x$$

vector $\underline{\hat{x}}\frac{d}{dx}$:

$$\underline{\hat{x}}\frac{d}{dx}(\sin x) = \underline{\hat{x}}\cos x$$

$$\left(\underline{\hat{x}}\frac{d}{dx}\right)\cdot\left(\underline{\hat{x}}\sin x\right) = \underline{\hat{x}}\cdot\underline{\hat{x}}\frac{d}{dx}(\sin x) = \cos x$$

$$\left(\underline{\hat{x}}\frac{d}{dx}\right)\times\left(\underline{\hat{y}}\sin x\right) = \underline{\hat{x}}\times\underline{\hat{y}}\frac{d}{dx}(\sin x) = \underline{\hat{z}}\cos x$$

Divergence Expressed with del Operator

Now consider:

$$\nabla \cdot \underline{D} = \left(\underline{\hat{x}} \frac{\partial}{\partial x} + \underline{\hat{y}} \frac{\partial}{\partial y} + \underline{\hat{z}} \frac{\partial}{\partial z} \right) \cdot \left(\underline{\hat{x}} D_x + \underline{\hat{y}} D_y + \underline{\hat{z}} D_z \right)$$

$$= \underline{\hat{x}} \cdot \underline{\hat{x}} \frac{\partial D_x}{\partial x} + \underline{\hat{y}} \cdot \underline{\hat{y}} \frac{\partial D_y}{\partial y} + \underline{\hat{z}} \cdot \underline{\hat{z}} \frac{\partial D_z}{\partial z}$$

Hence

$$\nabla \cdot \underline{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

This is the same as the divergence.

Divergence with del Operator (cont.)

$$\nabla \cdot \underline{D} = div\ \underline{D}$$

Note that the dot *after* the del operator is important; any symbol following it tells us how it is to be used and how it is read:

$\nabla \Phi$ = "gradient"

$\nabla \cdot \underline{V}$ = "divergence"

$\nabla \times \underline{V}$ = "curl"

Summary of Divergence Formulas

Rectangular:

$$\nabla \cdot \underline{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

See Appendix A.2 in the Hayt & Buck book for a general derivation that holds in <u>any</u> coordinate system.

Cylindrical:

$$\nabla \cdot \underline{D} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho D_\rho\right) + \frac{1}{\rho}\frac{\partial D_\phi}{\partial \phi} + \frac{\partial D_z}{\partial z}$$

Spherical:

$$\nabla \cdot \underline{D} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 D_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(D_\theta \sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial \phi}$$

Note on ∇ Operator

Divergence is defined in <u>any</u> coordinated system, but the ∇ operator is only defined in <u>rectangular</u> coordinates:

$$\nabla \equiv \underline{\hat{x}}\frac{\partial}{\partial x}+\underline{\hat{y}}\frac{\partial}{\partial y}+\underline{\hat{z}}\frac{\partial}{\partial z}$$

$$\nabla \neq \underline{\hat{\rho}}\frac{\partial}{\partial \rho}+\underline{\hat{\phi}}\frac{\partial}{\partial \phi}+\underline{\hat{z}}\frac{\partial}{\partial z}$$

$$\nabla \neq \underline{\hat{r}}\frac{\partial}{\partial r}+\underline{\hat{\theta}}\frac{\partial}{\partial \theta}+\underline{\hat{\phi}}\frac{\partial}{\partial \phi}$$

For example, in spherical coordinates:

$$div\,\underline{D} \neq \left(\underline{\hat{r}}\frac{\partial}{\partial r}+\underline{\hat{\theta}}\frac{\partial}{\partial \theta}+\underline{\hat{\phi}}\frac{\partial}{\partial \phi}\right)\cdot\left(\underline{\hat{r}}D_r+\underline{\hat{\theta}}D_\theta+\underline{\hat{\phi}}D_\phi\right)$$

Electric Gauss Law (Point or Differential Form)

We now have, in the notation of the “del” operator:

$$\nabla \cdot \underline{D} = \rho_v$$

Electric Gauss law (point form)

Putting back the coordinate variables in the notation, it looks like:

$$\nabla \cdot \underline{D}(x, y, z) = \rho_v(x, y, z)$$

Note:
There is only one form of this equation, which has volume charge density. There is no form that has surface charge density or line charge density.

Maxwell's Equations

(Maxwell's equations in point or differential form)

$$\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}$$ Faraday's law

$$\nabla \times \underline{H} = \underline{J} + \frac{\partial \underline{D}}{\partial t}$$ Ampere's law

$$\nabla \cdot \underline{D} = \rho_v$$ Electric Gauss law

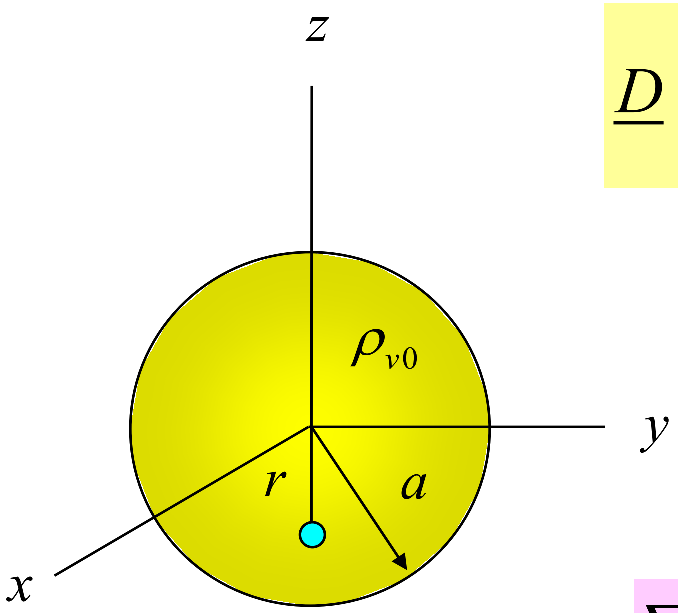
$$\nabla \cdot \underline{B} = 0$$ Magnetic Gauss law

Divergence appears in two of Maxwell's equations.

Note:
There is no magnetic charge density!
(Magnetic lines of flux must therefore form closed loops.)

Example

Evaluate the divergence of the electric flux vector inside and outside a sphere of uniform volume charge density, and verify that the answer is what is expected from the electric Gauss law.



$\underline{r < a}$

$$\underline{D} = \underline{\hat{r}}\left(\frac{\rho_{v0} r}{3}\right)$$

$$\nabla \cdot \underline{D} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 D_r\right)$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\left[\frac{\rho_{v0} r}{3}\right]\right)$$

$$= \frac{\rho_{v0}}{3}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^3\right)$$

$$= \frac{1}{r^2}\rho_{v0} r^2$$

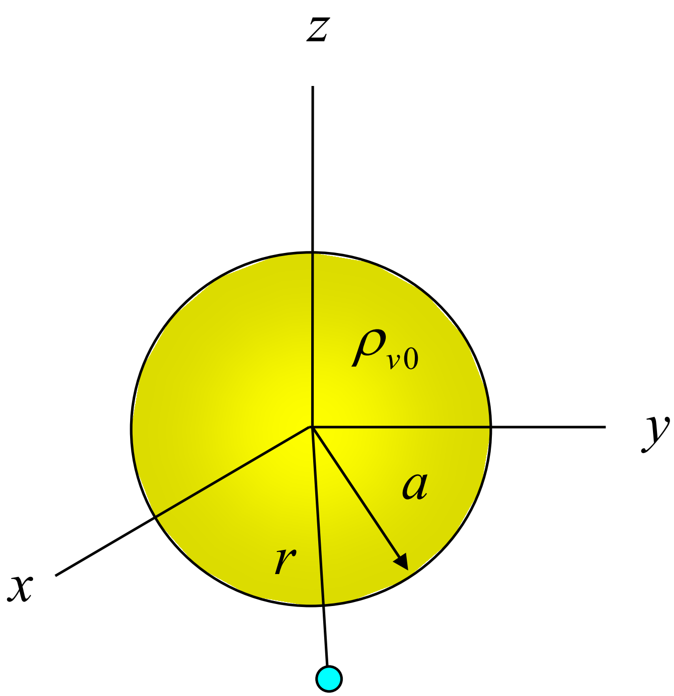
$$\nabla \cdot \underline{D} = \rho_{v0}$$

This agrees with the electric Gauss law.

Example (cont.)

$\underline{r > a}$

$$\underline{D} = \underline{\hat{r}}\left(\frac{\rho_{v0}a^3}{3r^2}\right)$$



$$\nabla \cdot \underline{D} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 D_r\right)$$

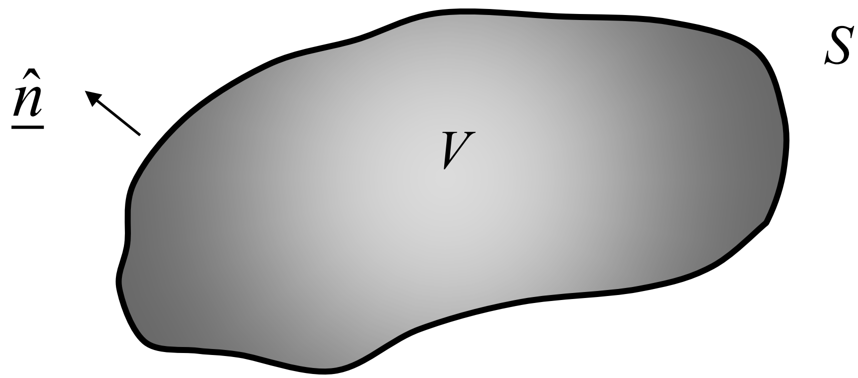
$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\left[\frac{\rho_{v0}a^3}{3r^2}\right]\right)$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{\rho_{v0}a^3}{3}\right) = 0$$

$$\nabla \cdot \underline{D} = 0$$

This agrees with the electric Gauss law.

Divergence Theorem



$\underline{\hat{n}}$ = outward normal

$$\int_V \nabla \cdot \underline{A}\; dV = \oint_S \underline{A} \cdot \underline{\hat{n}}\; dS$$

Please see the Appendix for a proof.

$\underline{A}$ = arbitrary vector function

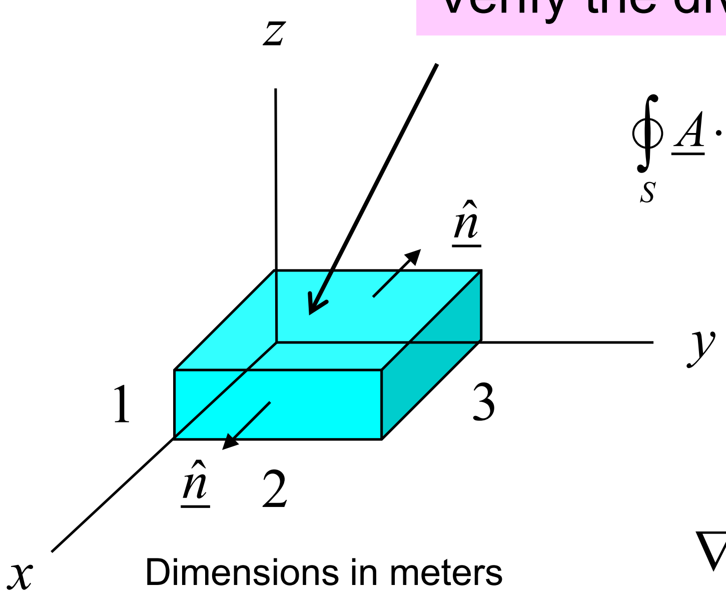
In words:

The volume integral of "flux per volume" equals the total flux!

Example

Given: $\underline{A} = \underline{\hat{x}}\left(3x\right)$

Verify the divergence theorem using this region.



$$\oint_S \underline{A}\cdot\underline{\hat{n}}\,dS = \left(\underline{\hat{x}}\,3(3)\cdot\underline{\hat{x}}\right)(2) + \left(\underline{\hat{x}}\,3(0)\cdot\left(-\underline{\hat{x}}\right)\right)(2) = 18$$

Note:
Only the front and back faces contribute.
$\underline{A}$ is constant on the front and back faces, and the area of these faces is 2 [m²].

$$\nabla\cdot\underline{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$= \frac{\partial}{\partial x}\left(3x\right) = 3$$

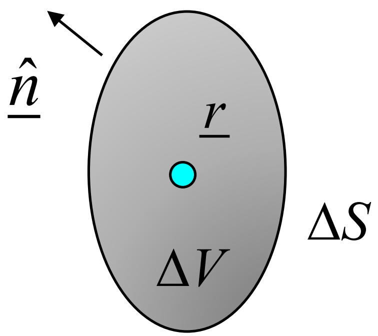
so

$$\int_V \nabla\cdot\underline{A}\;dV = \int_V 3\;dV = 3V = 3\left(1\cdot 2\cdot 3\right) = 18$$

Note on Divergence Definition

$$div\,\underline{D} \equiv \lim_{\Delta V \to 0} \frac{1}{\Delta V} \oint_{\Delta S} \underline{D} \cdot \underline{\hat{n}}\, dS$$

Is this limit independent of the shape of the volume?



Small arbitrary-shaped volume

Use the divergence theorem for RHS:

$$\oint_{\Delta S} \underline{D} \cdot \underline{\hat{n}}\; dS = \int_{\Delta V} \nabla \cdot \underline{D}\, dV$$

$$div\,\underline{D} = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \int_{\Delta V} \nabla \cdot \underline{D}\; dV$$

$$= \lim_{\Delta V \to 0} \frac{1}{\cancel{\Delta V}} \left(\left(\nabla \cdot \underline{D} \right)\Big|_{\underline{r}} \cancel{\Delta V} \right) = \nabla \cdot \underline{D}\Big|_{\underline{r}}$$

Hence, the limit is the same regardless of the shape of the limiting volume.

Gauss's Law (Differential to integral form)

We can convert the differential form into the integral form by using the divergence theorem.

$$\nabla \cdot \underline{D} = \rho_v$$

Integrate both sides over a volume:

$$\int_V \nabla \cdot \underline{D}\, dV = \int_V \rho_v\, dV$$

Apply the divergence theorem to the LHS:

$$\oint_S \underline{D} \cdot \underline{\hat{n}}\, dS = \int_V \rho_v\, dV$$

Use the definition of Q_{encl}:

$$\oint_S \underline{D} \cdot \underline{\hat{n}}\; dS = Q_{encl}$$

Gauss's Law (Summary of two forms)

$$\oint_S \underline{D} \cdot \underline{\hat{n}} \; dS = Q_{encl}$$

Integral (volume) form of Gauss's law

Definition of divergence (↓)

Divergence theorem (↑)

$$\nabla \cdot \underline{D} = \rho_v$$

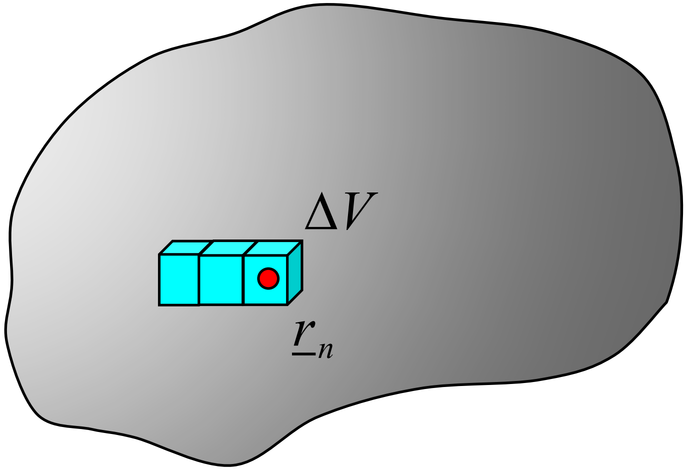
Differential (point) form of Gauss's law

Note: All of Maxwell's equations have both a point (differential) and an integral form.

Appendix: Proof of Divergence Theorem

Proof

The volume is divided up into many small cubes.



$$\int_V \nabla \cdot \underline{A} \, dV = \lim_{\Delta V \to 0} \sum_{n=1}^{N} \left(\nabla \cdot \underline{A} \right)_{\underline{r}_n} \Delta V$$

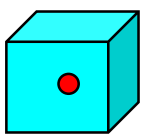
Note: The point $\underline{r}_n$ is the center of cube n.

Proof of Divergence Theorem (cont.)

From the definition of divergence:

$$\left(\nabla \cdot \underline{A}\right)_{\underline{r}_n} = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \oint_{\Delta S_n} \underline{A} \cdot \underline{\hat{n}}\, dS$$

$$\approx \frac{1}{\Delta V} \oint_{\Delta S_n} \underline{A} \cdot \underline{\hat{n}}\, dS$$



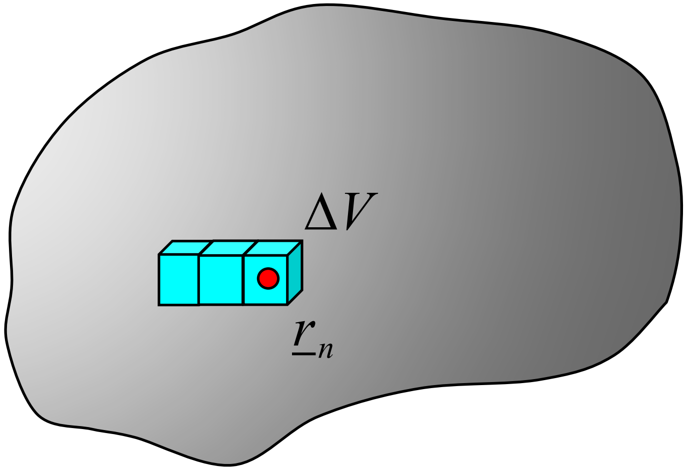
$\underline{r}_n$

ΔS_n = surface of cube n

Hence:

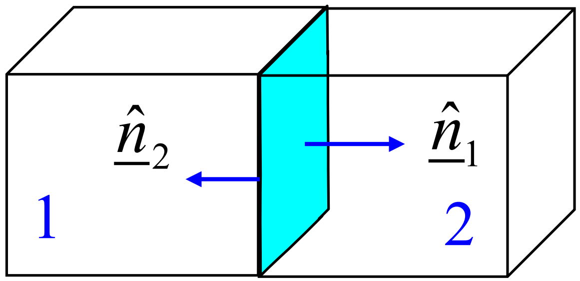
$$\int_V \nabla \cdot \underline{A}\, dV = \lim_{\Delta V \to 0} \sum_{n=1}^{N} \left(\nabla \cdot \underline{A}\right)_{\underline{r}_n} \Delta V = \lim_{\Delta V \to 0} \sum_{n=1}^{N} \left(\frac{1}{\Delta V} \oint_{\Delta S_n} \underline{A} \cdot \underline{\hat{n}}\, dS \right) \Delta V = \lim_{\Delta V \to 0} \sum_{n=1}^{N} \oint_{\Delta S_n} \underline{A} \cdot \underline{\hat{n}}\, dS$$

Proof of Divergence Theorem (cont.)



$$\int_V \nabla \cdot \underline{A}\, dV = \lim_{\Delta V \to 0} \sum_{n=1}^{N} \oint_{\Delta S_n} \underline{A} \cdot \underline{\hat{n}}\, dS$$

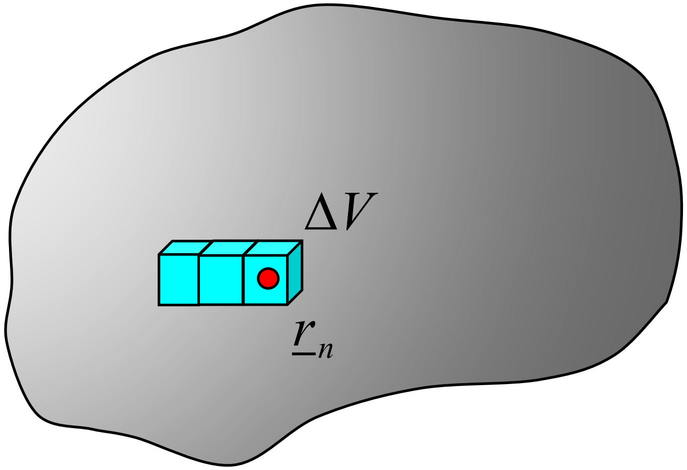
Consider two adjacent cubes:



$\underline{A} \cdot \underline{\hat{n}}$ is opposite on the two faces.

Hence, the surface integral cancels on all INTERIOR faces.

Proof of Divergence Theorem (cont.)



$$\int_V \nabla \cdot \underline{A}\; dV = \lim_{\Delta V \to 0} \sum_{n=1}^{N} \oint_{\Delta S_n} \underline{A} \cdot \underline{\hat{n}}\; dS$$

$$= \lim_{\Delta V \to 0} \sum_{\substack{\text{outside} \\ \text{faces}}} \int_{\Delta S_n} \underline{A} \cdot \underline{\hat{n}}\; dS$$

But

$$\lim_{\Delta V \to 0} \sum_{\substack{\text{outside} \\ \text{faces}}} \int_{\Delta S_n} \underline{A} \cdot \underline{\hat{n}}\; dS = \oint_S \underline{A} \cdot \underline{\hat{n}}\; dS$$

Hence

$$\int_V \nabla \cdot \underline{A}\, dV = \oint_S \underline{A} \cdot \underline{\hat{n}}\; dS \qquad \text{(proof complete)}$$