ECE 3318
Applied Electricity and Magnetism

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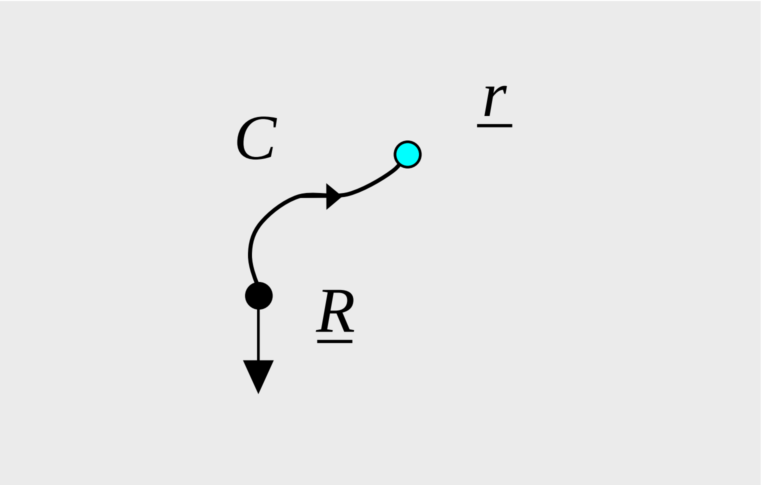
Notes 14
Potential From Field

Potential Calculation from Field

In this set of notes we show how to calculate the potential function $\Phi(x,y,z)$ (assuming that we already know the electric field).

Note:
The electric field must be found first, from either from Coulomb's law or Gauss's law.

Potential Formula



$$V_{AB} \equiv \int_{\underline{A}}^{\underline{B}} \underline{E} \cdot \underline{dr}$$

$$= \Phi(\underline{A}) - \Phi(\underline{B})$$

Choose: $\underline{A} = \underline{r} \quad \underline{B} = \underline{R}$

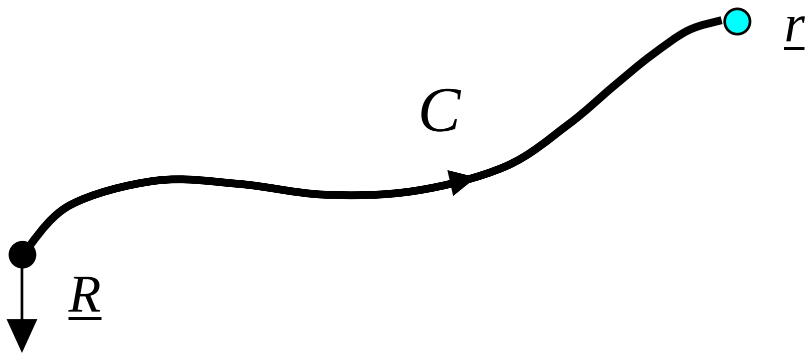
Then we have:

$$\Phi(\underline{r}) - \Phi(\underline{R}) = \int_{\underline{r}}^{\underline{R}} \underline{E} \cdot \underline{dr}$$

or

$$\Phi(\underline{r}) = \Phi(\underline{R}) - \int_{\underline{R}}^{\underline{r}} \underline{E} \cdot \underline{dr}$$

Potential Formula (cont.)



$$\Phi(\underline{r}) = \Phi(\underline{R}) - \int_{\underline{R}}^{\underline{r}} \underline{E} \cdot \underline{dr}$$

Recipe:

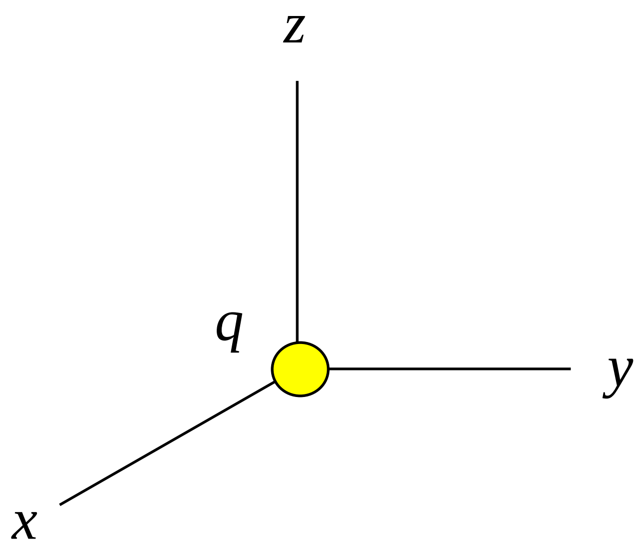
1) Pick a reference point (if it is not already given).
2) Integrate in a coordinate system of your choice (remember that any path can be chosen in statics).

Example

Point charge

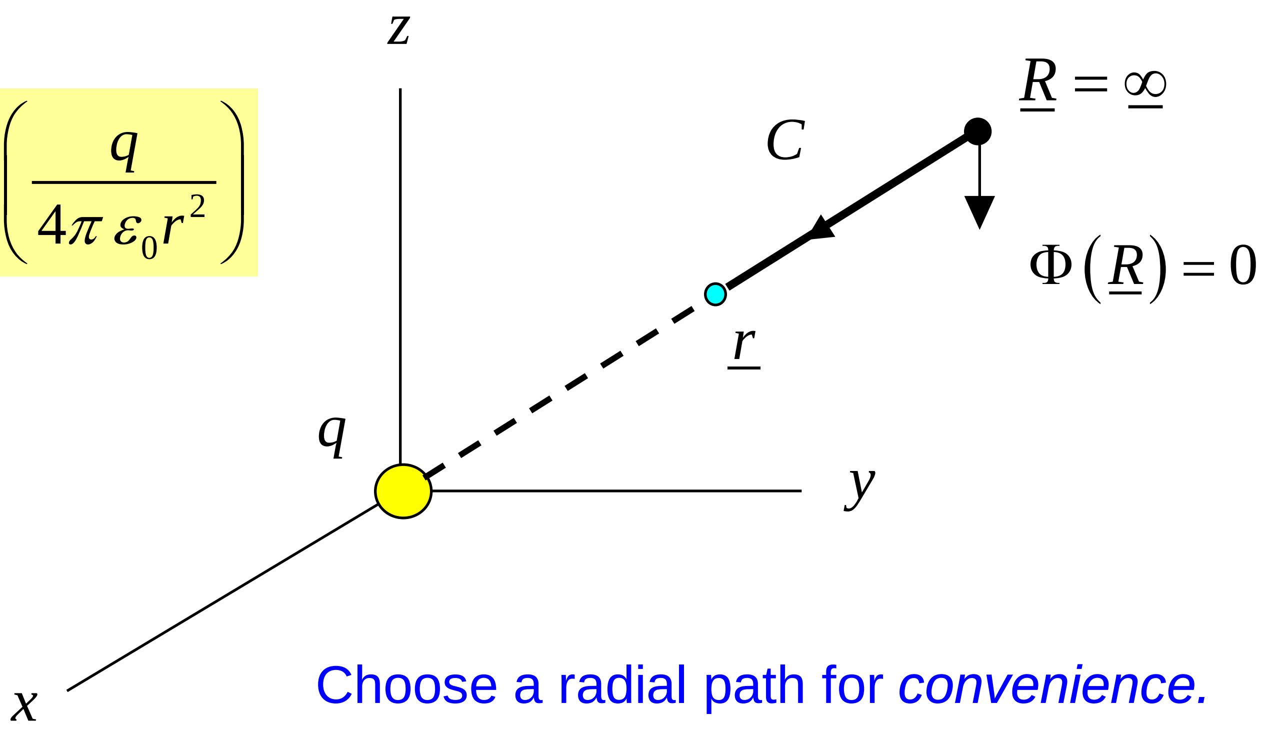
Find the potential function

Given: $\underline{R} = \underline{\infty},\ \Phi\left(\underline{R}\right) = 0$



Example (cont.)

$$\underline{E} = \underline{\hat{r}} \left(\frac{q}{4\pi \varepsilon_0 r^2} \right)$$



Choose a radial path for *convenience.*

$$\Phi(\underline{r}) = \Phi(\underline{R}) - \int_{\underline{R}}^{\underline{r}} \underline{E} \cdot \underline{dr} \quad \Rightarrow \quad \Phi = 0 - \int_{\underline{R}}^{\underline{r}} \underline{E} \cdot \underline{dr}$$

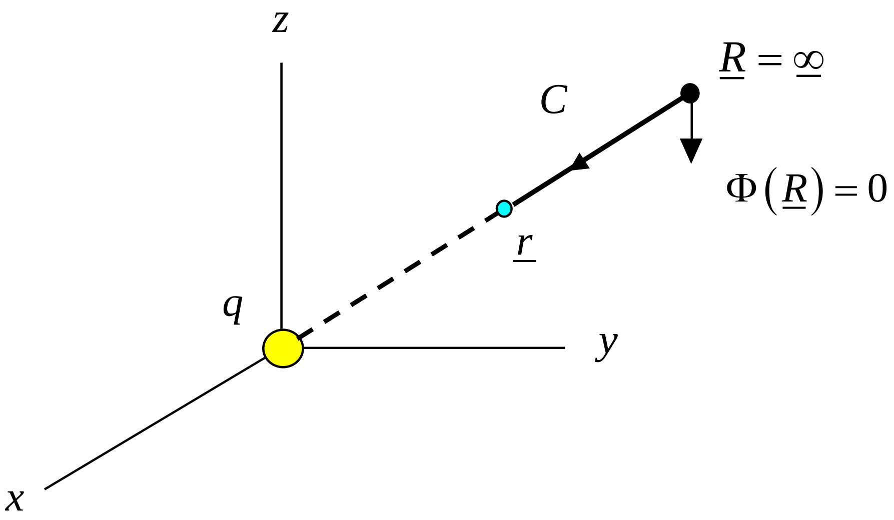
Example (cont.)

$$\Phi = -\int_{\underline{R}}^{\underline{r}} \left(\underline{\hat{r}}\, E_r\right)\cdot\left(\underline{\hat{r}}\, dr + \underline{\hat{\theta}}\, r\, d\theta + \underline{\hat{\phi}}\, r \sin\theta\, d\phi\right)$$

$$= -\int_{R}^{r} E_r\, dr$$

$$= -\int_{\infty}^{r} \frac{q}{4\pi\varepsilon_0 r^2}\, dr$$

$$= \left.\frac{q}{4\pi\varepsilon_0 r}\right|_{\infty}^{r}$$



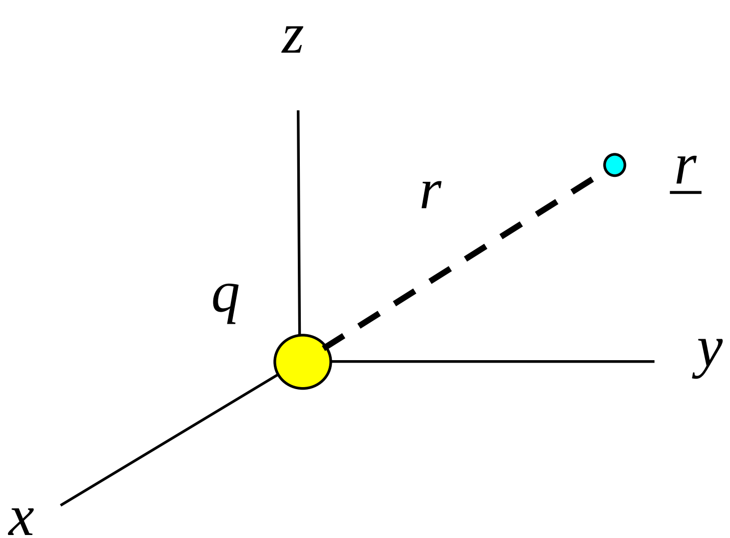
Hence, we have:

$$\Phi(\underline{r}) = \frac{q}{4\pi\varepsilon_0\, r} \quad [\mathrm{V}]$$

Summary for a Point Charge

$$\underline{E}(\underline{r}) = \underline{\hat{r}}\left(\frac{q}{4\pi\varepsilon_0 r^2}\right) \quad [\text{V/m}]$$

$$\Phi(\underline{r}) = \frac{q}{4\pi\varepsilon_0 r} \quad [\text{V}] \quad , \quad \Phi(\underline{\infty}) = 0$$

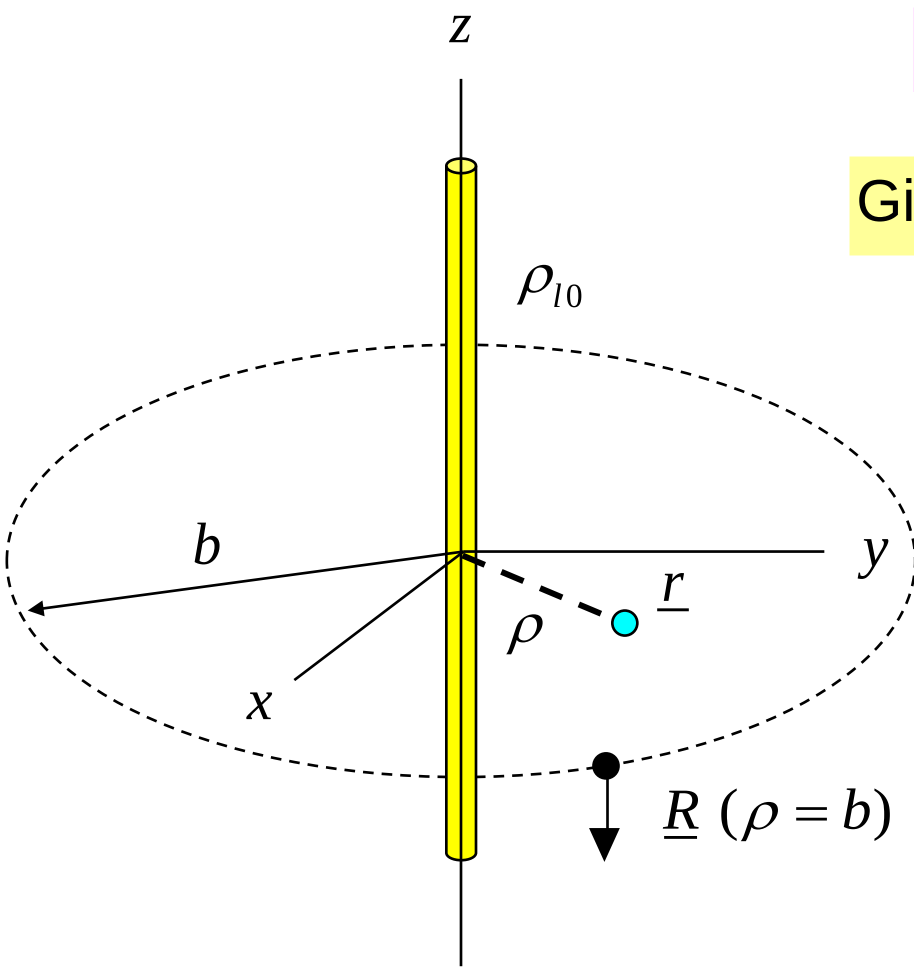


The potential from for a point charge forms the building block for all other charge distributions (using superposition).

This is explained in Notes 15.

Example

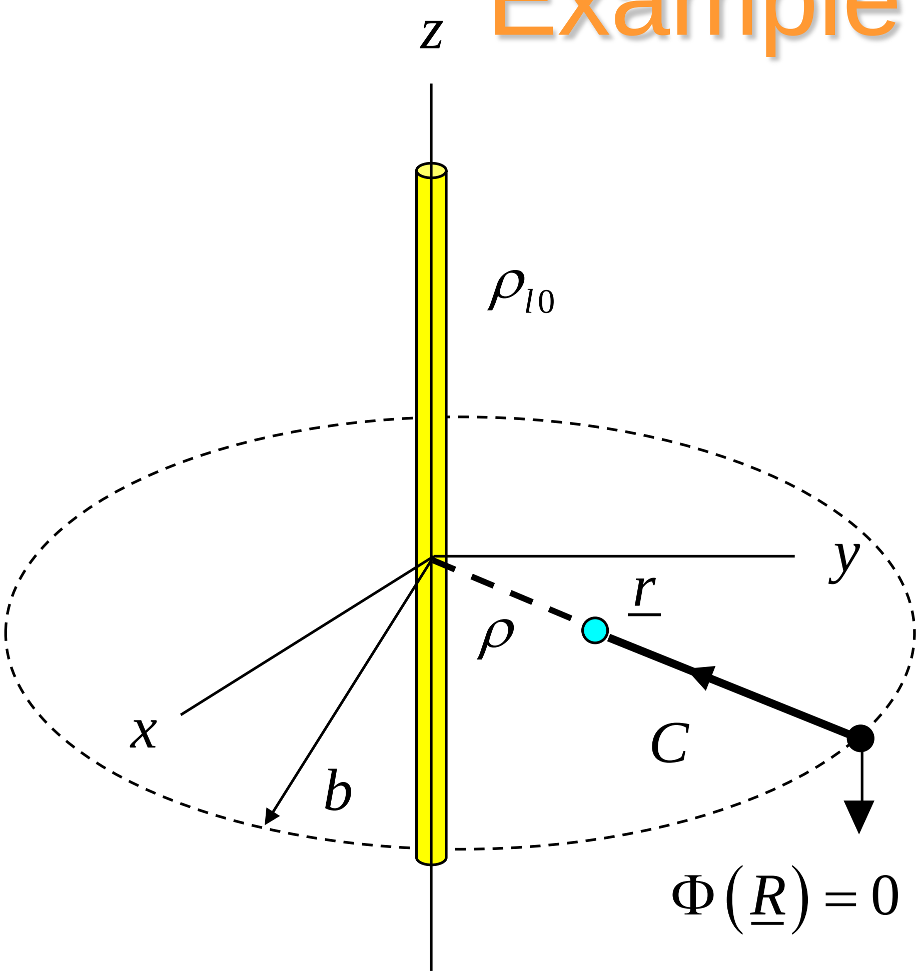
Infinite line charge



Find the potential function

Given: $\underline{R}$ is at $\rho = b,\ \Phi\left(\underline{R}\right) = 0$

Example (cont.)



Choose a radial path

$$\Phi\left(\underline{r}\right)=\Phi\left(\underline{R}\right)-\int_{\underline{R}}^{\underline{r}}\underline{E}\cdot\underline{dr}$$

$$=0-\int_{\underline{R}}^{\underline{r}}\underline{E}\cdot\underline{dr}$$

$$\underline{E}=\underline{\hat{\rho}}\left(\frac{\rho_{\ell 0}}{2\pi\,\varepsilon_0\rho}\right)$$

$$\Phi\left(\rho\right)=-\int_{\underline{R}}^{\underline{r}}\left(\underline{\hat{\rho}}\,E_\rho\right)\cdot\left(\underline{\hat{\rho}}\,d\rho+\underline{\hat{z}}\,dz+\underline{\hat{\phi}}\,\rho\,d\phi\right)$$

$$=-\int_{b}^{\rho}\left(E_\rho\,d\rho\right)$$

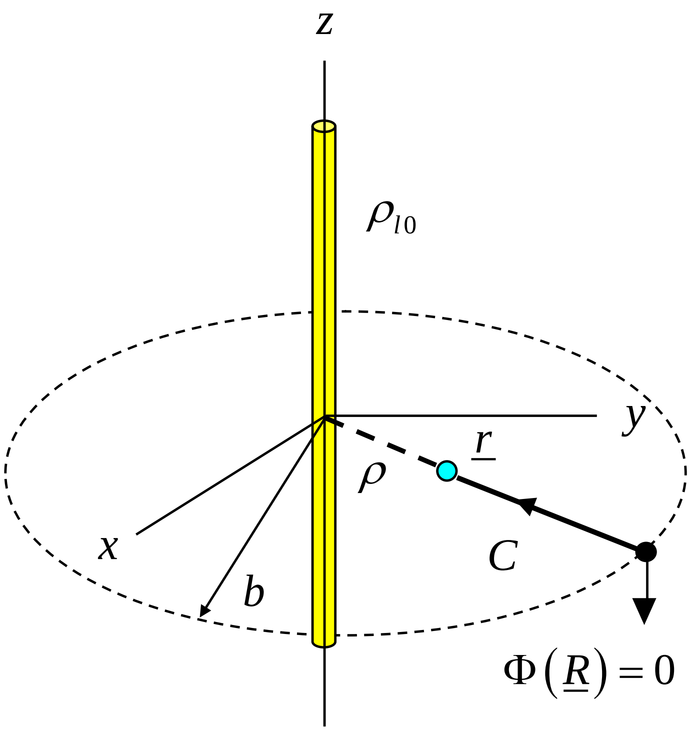
Example (cont.)

Performing the integration:

$$\Phi(\rho) = -\int_b^{\rho} \frac{\rho_{\ell 0}}{2\pi\,\varepsilon_0 \rho}\, d\rho$$

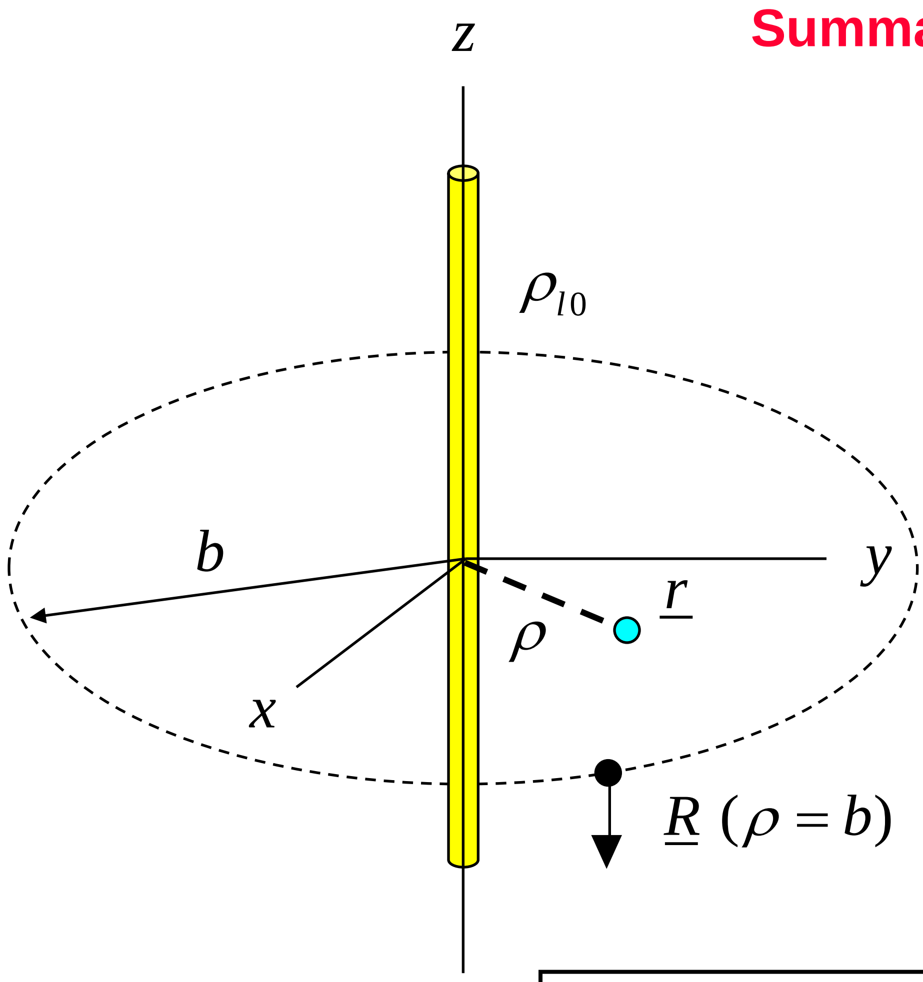
$$= -\frac{\rho_{\ell 0}}{2\pi\,\varepsilon_0} \ln \rho \Big|_b^{\rho}$$

$$= -\frac{\rho_{\ell 0}}{2\pi\,\varepsilon_0} \ln\left(\frac{\rho}{b}\right)$$



Example (cont.)

Summary



$$\Phi(\rho) = \frac{\rho_{\ell 0}}{2\pi\,\varepsilon_0} \ln\left(\frac{b}{\rho}\right) \quad [\mathrm{V}]$$

$$\Phi(b) = 0$$

Note:

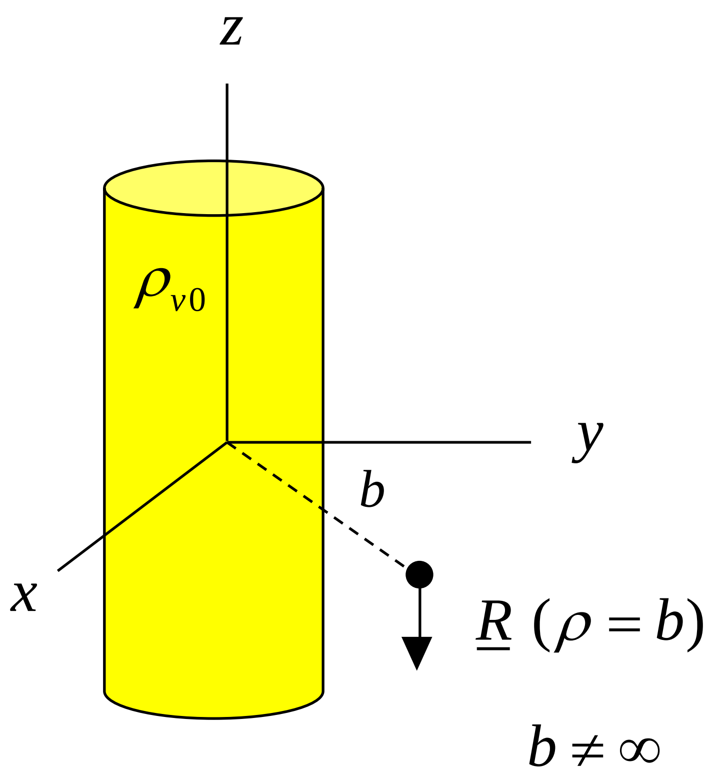
The radius b cannot be chosen as ∞

(There is an infinite voltage drop between $\rho = \rho$ and $\rho = \infty$.)

Example (cont.)

Note on 2D Problems

In any 2D problem, the reference point <u>cannot</u> be chosen at infinity.

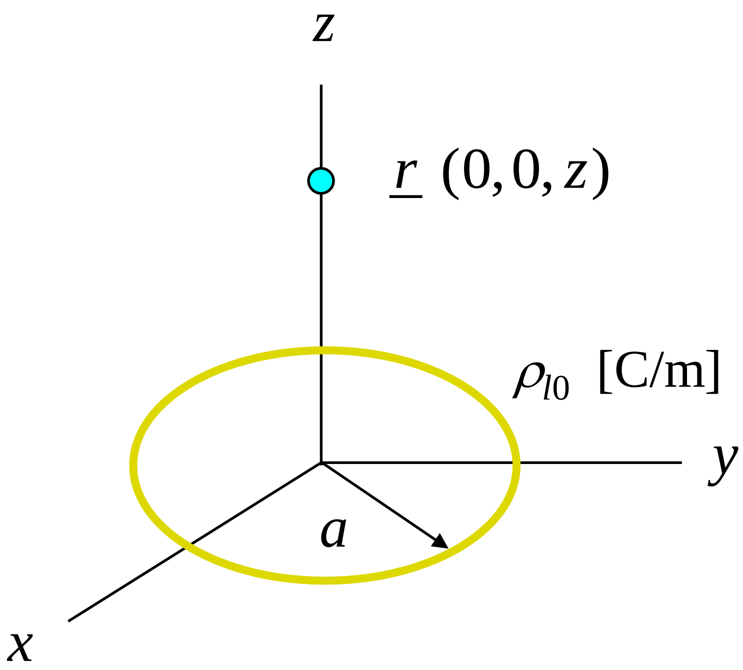


Example

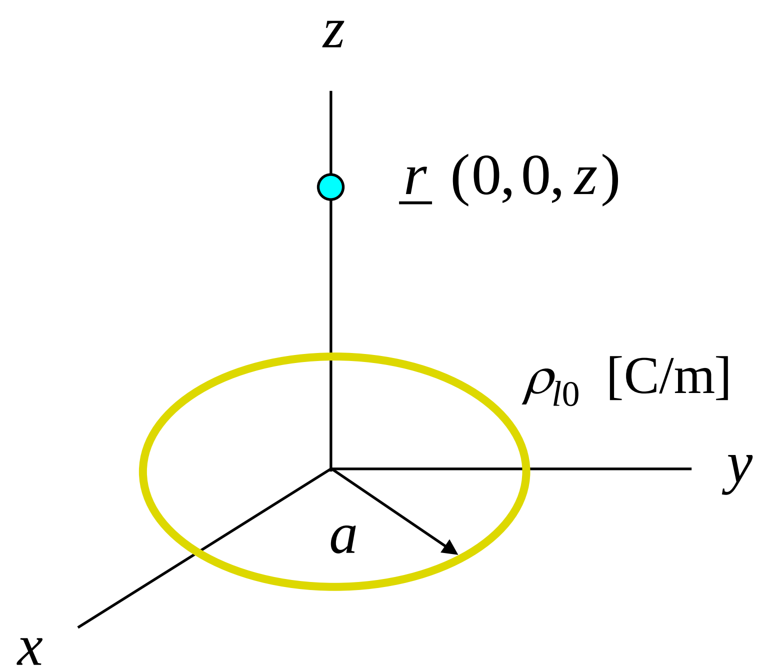
Circular ring of line charge

Find the potential function on the z axis

Given: $\underline{R} = \underline{\infty}, \quad \Phi\left(\underline{R}\right) = 0$



Example (cont.)

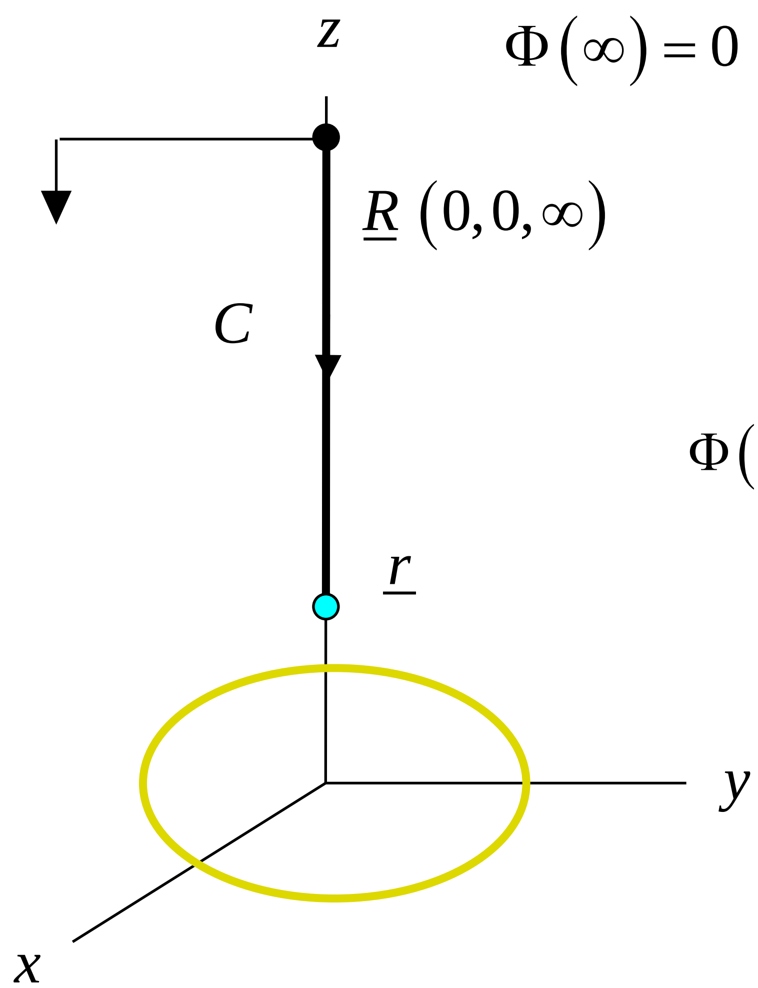


$$\Phi\left(\underline{r}\right) = \Phi\left(\underline{R}\right) - \int_{\underline{R}}^{\underline{r}} \underline{E} \cdot \underline{dr}$$

Note:
We only know $\underline{E}$ on the z axis (from a previous Coulomb's law example), so we must choose a path on the z axis.

Example (cont.)

Choice of path:



$$\Phi(\underline{r})=\Phi(\underline{R})-\int_{\underline{R}}^{\underline{r}}\underline{E}\cdot\underline{dr}$$

Example (cont.)

$$\Phi\left(\underline{r}\right) = \Phi\left(\underline{R}\right) - \int_{\underline{R}}^{\underline{r}} \underline{E} \cdot \underline{dr}$$

$$= 0 - \int_{\underline{R}}^{\underline{r}} \underline{E} \cdot \underline{dr}$$

Hence, we have:

$$\Phi\left(\underline{r}\right) = -\int_{\underline{R}}^{\underline{r}} \left(\underline{\hat{z}}\, E_z\right) \cdot \left(\cancel{\underline{\hat{x}}dx} + \cancel{\underline{\hat{y}}\,dy} + \underline{\hat{z}}\, dz\right)$$

$$= -\int_{\infty}^{z} E_z\, dz$$

From Coulomb's law, we know that on the *z* axis the field is

$$E_z\left(0,0,z\right) = \left(\frac{\rho_{\ell 0}\, a}{2\varepsilon_0}\right)\left(\frac{z}{\left(z^2 + a^2\right)^{3/2}}\right)$$

Example (cont.)

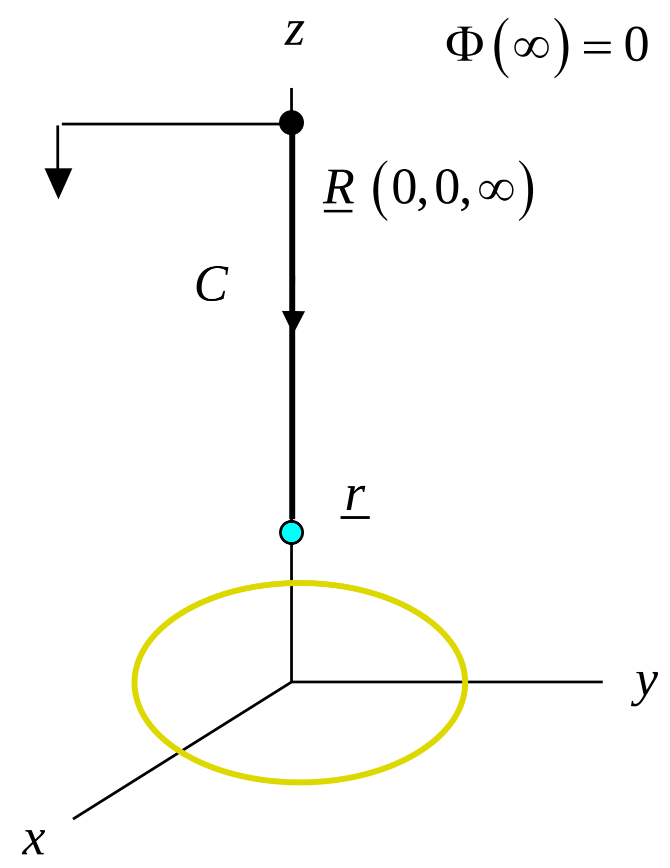
Hence, we have

$$\Phi(0,0,z) = -\int_{\infty}^{z}\left(\frac{\rho_{\ell 0}\,a}{2\varepsilon_0}\right)\frac{z}{\left(z^2+a^2\right)^{3/2}}\,dz$$

$$-\left(\frac{\rho_{\ell 0}\,a}{2\varepsilon_0}\right)\int_{\infty}^{z}\frac{z}{\left(z^2+a^2\right)^{3/2}}\,dz$$

$$= -\left(\frac{\rho_{\ell 0}\,a}{2\varepsilon_0}\right)\left[-\left(z^2+a^2\right)^{-1/2}\right]_{\infty}^{z}$$

$$= \frac{\rho_{\ell 0}\,a}{2\varepsilon_0}\frac{1}{\sqrt{z^2+a^2}}$$

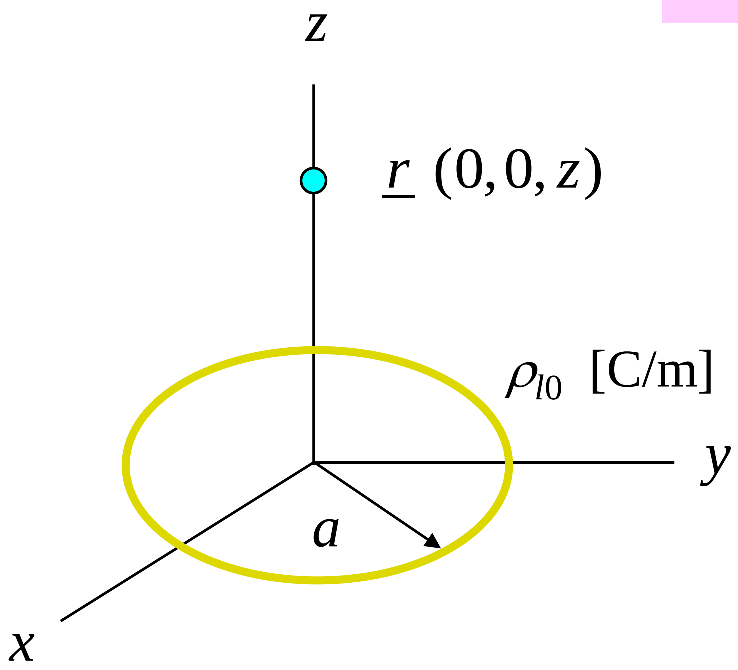


Example (cont.)

Summary

$$\Phi(0,0,z) = \frac{\rho_{\ell 0}\, a}{2\varepsilon_0} \frac{1}{\sqrt{z^2 + a^2}} \quad [\mathrm{V}]$$

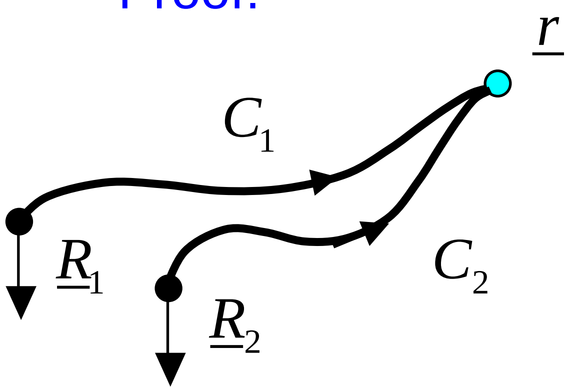
$$\Phi(\infty) = 0$$



Adding a Constant to Φ

Two different solutions for the potential function can only differ by a constant.

Proof:



Assume two solutions: Φ_1 and Φ_2

$$\Phi_1\left(\underline{R}_1\right) = \Phi_{01} \quad , \quad \Phi_2\left(\underline{R}_2\right) = \Phi_{02}$$

$$\Phi_1\left(\underline{r}\right) = \Phi_1\left(\underline{R}_1\right) - \int_{\underline{R}_1}^{\underline{r}} \underline{E} \cdot \underline{dr}$$

$$\Phi_2\left(\underline{r}\right) = \Phi_2\left(\underline{R}_2\right) - \int_{\underline{R}_2}^{\underline{r}} \underline{E} \cdot \underline{dr}$$

$$\Phi_1\left(\underline{r}\right) - \Phi_2\left(\underline{r}\right) = \left[\Phi_1\left(\underline{R}_1\right) - \Phi_2\left(\underline{R}_2\right)\right] + \int_{\underline{R}_2}^{\underline{r}} \underline{E} \cdot \underline{dr} + \int_{\underline{r}}^{\underline{R}_1} \underline{E} \cdot \underline{dr}$$

$$= \left[\Phi_1\left(\underline{R}_1\right) - \Phi_2\left(\underline{R}_2\right)\right] + \int_{\underline{R}_2}^{\underline{R}_1} \underline{E} \cdot \underline{dr} = \text{constant}$$

Adding a Constant to Φ (cont.)

Conclusion:

Valid potential functions can only differ by the addition of a constant.

Adding a constant to a valid potential function gives a another valid potential function (this does not change the electric field).

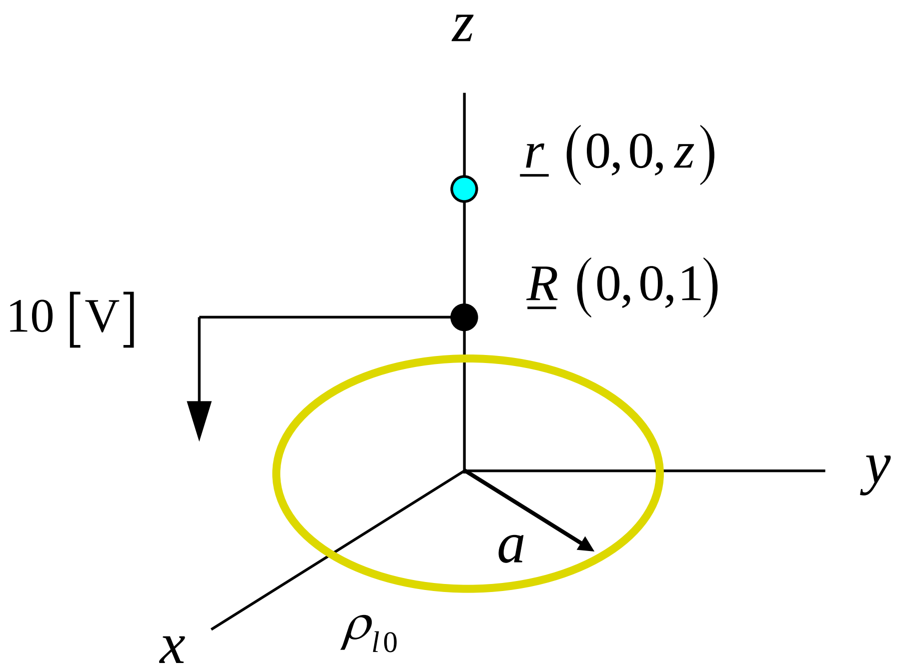
$$\Phi_2(x, y, z) = \Phi_1(x, y, z) + C$$

Example

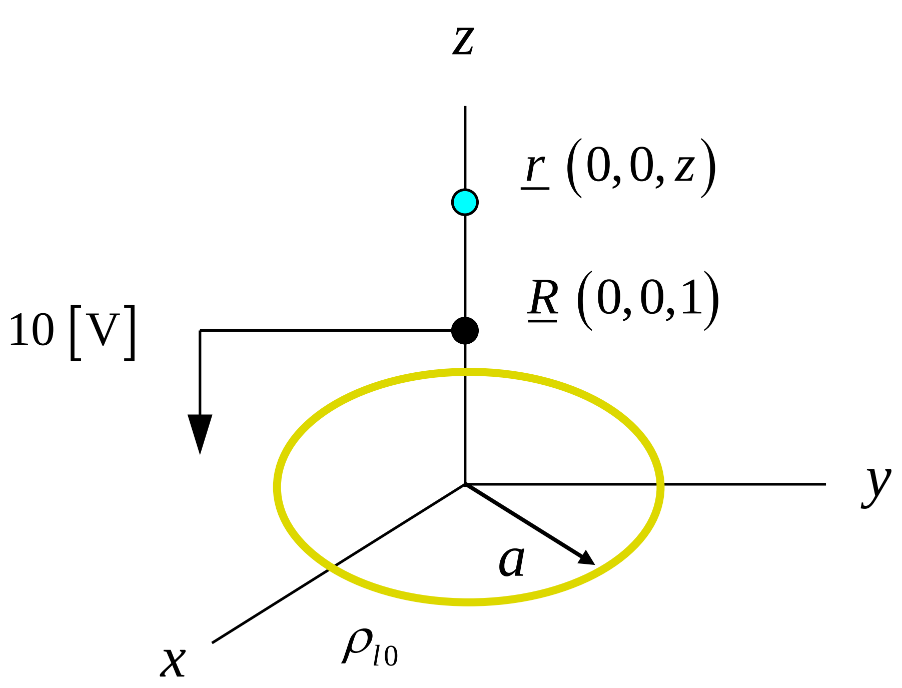
Circular ring of line charge

Find the potential function on the z axis

Given: $\Phi(0,0,1)\,[\text{m}] = 10\,[\text{V}]$



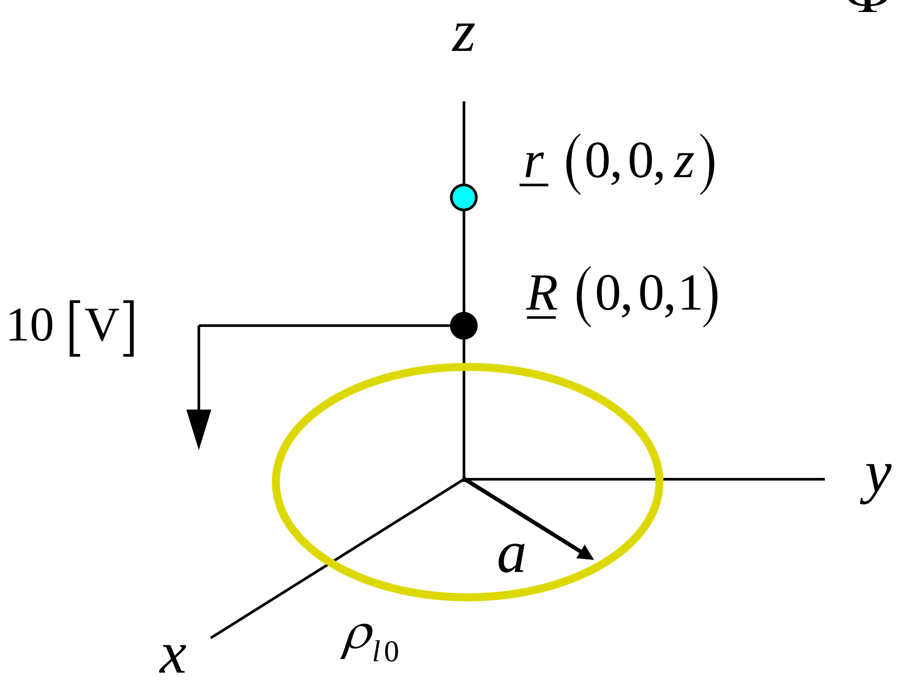
Example (cont.)



Start with our previous example, which has zero volts at infinity, and add a constant to it:

$$\Phi(0,\,0,\,z) = \frac{\rho_{\ell 0} a}{2\varepsilon_0} \frac{1}{\sqrt{z^2 + a^2}} + C$$

Example (cont.)



$$\Phi(0,0,z) = \frac{\rho_{\ell 0} a}{2\varepsilon_0} \frac{1}{\sqrt{z^2 + a^2}} + C$$

Set: $\Phi(0,0,1) = 10$ [V]

$$\Rightarrow \quad \frac{\rho_{\ell 0} a}{2\varepsilon_0} \frac{1}{\sqrt{1^2 + a^2}} + C = 10$$

Hence we have:

$$C = 10 - \frac{\rho_{\ell 0} a}{2\varepsilon_0} \frac{1}{\sqrt{1^2 + a^2}} \text{ [V]}$$

Example (cont.)

Summary

$$\Phi(0,0,z) = 10 + \frac{\rho_{\ell 0} a}{2\varepsilon_0}\left(\frac{1}{\sqrt{z^2+a^2}} - \frac{1}{\sqrt{1^2+a^2}}\right) \quad [\mathrm{V}]$$

$$\Phi(0,0,1)\,[\mathrm{m}] = 10\,[\mathrm{V}]$$