ECE 3318
Applied Electricity and Magnetism

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Prof. David R. Jackson
Dept. of ECE

Notes 15
Potential from Charge

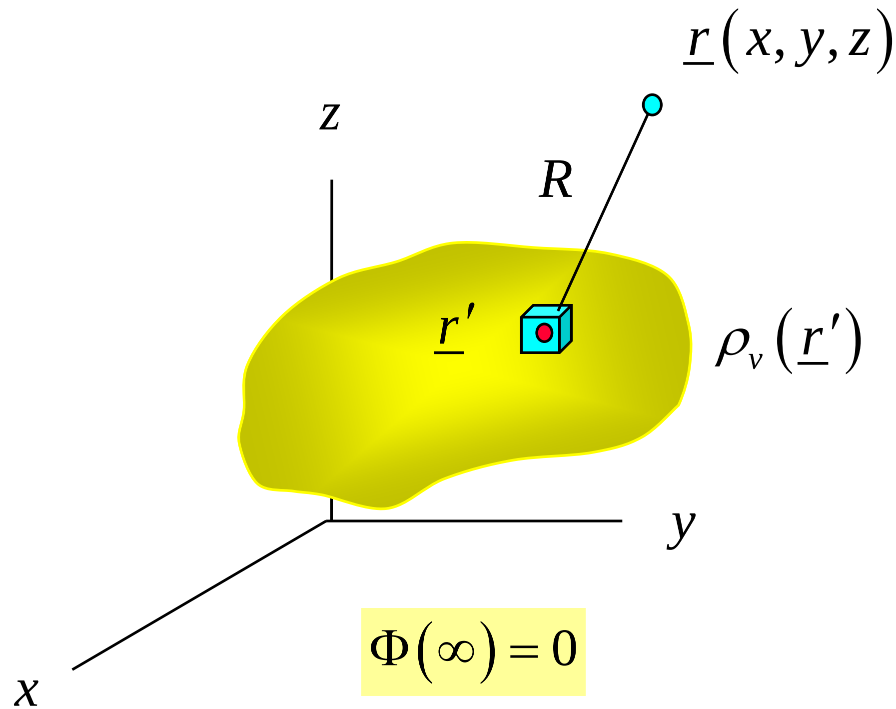
Potential From Charge

In this set of notes we show how to calculate the potential function $\Phi(x,y,z)$ directly from the charge, without having to calculate the electric field first.

- This is often the easiest way to find the potential function (especially when you don't already have the electric field calculated). *There are no vector calculations involved.*
- The method assumes that the potential is zero at infinity. (If this is not so, you must remember to add a constant to the solution.)

Potential From Charge (cont.)

Arbitrary cloud of charge density



$$\Phi(\infty)=0$$

Point charge formula:

$$\Phi = \frac{q}{4\pi\varepsilon_0 r}$$

From the point charge formula:

$$d\Phi = \frac{dQ}{4\pi\varepsilon_0 R} = \frac{\rho_v(\underline{r}')\,dV'}{4\pi\varepsilon_0 R}$$

Integrating, we obtain the following result:

$$\Phi(\underline{r}) = \int_V \frac{\rho_v(\underline{r}')\,dV'}{4\pi\varepsilon_0 R}$$

Summary of Potential From Charge

Summary for all types of charge densities:

$$\Phi(\underline{r}) = \int_V \frac{\rho_v(\underline{r}')\,dV'}{4\pi\varepsilon_0 R}$$

$$\Phi(\underline{r}) = \int_S \frac{\rho_s(\underline{r}')\,dS'}{4\pi\varepsilon_0 R}$$

$$\Phi(\underline{r}) = \int_C \frac{\rho_\ell(\underline{r}')\,dl'}{4\pi\varepsilon_0 R}$$

Note:
The potential is zero at infinity
$(R \to \infty)$
when you use these formulas.

Note:

$$R = \left|\underline{R}\right|$$

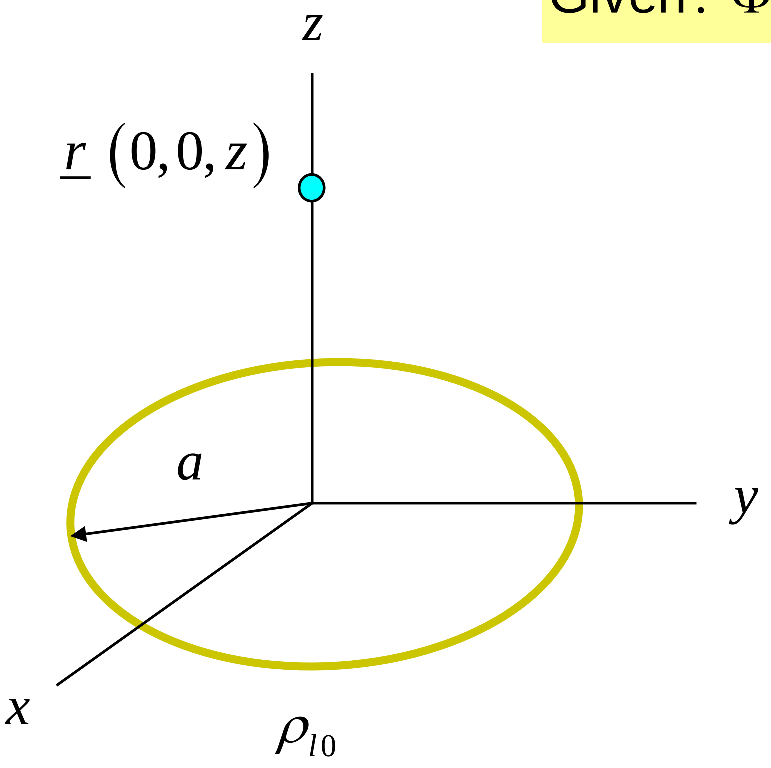
(But we don't need $\underline{R}$.)

Example

Circular ring of line charge

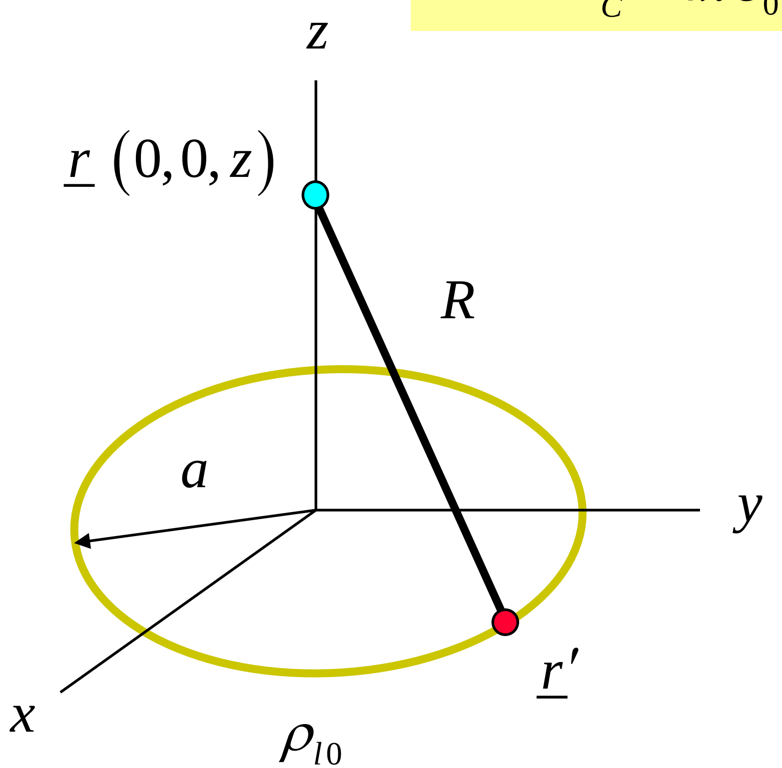
Find $\Phi(0, 0, z)$

Given: $\Phi(\infty) = 0$



Example (cont.)

$$\Phi(\underline{r}) = \int_C \frac{\rho_\ell(\underline{r}')\,dl'}{4\pi\varepsilon_0 R}$$



$$\Phi(\underline{r}) = \int_C \frac{\rho_{\ell 0}\,d\ell'}{4\pi\varepsilon_0 R}$$

$$R = \sqrt{z^2 + a^2}$$

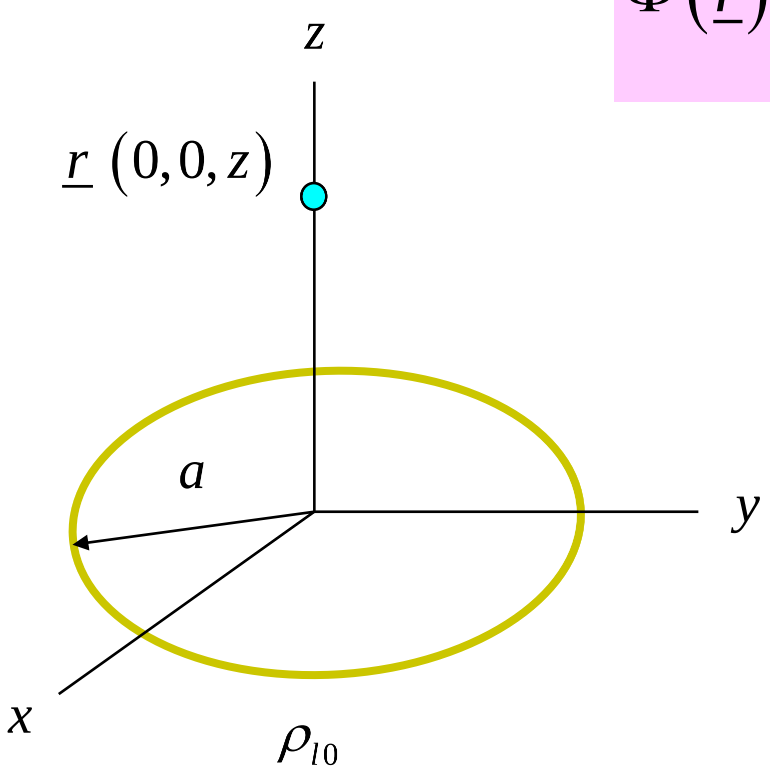
$$\Phi(\underline{r}) = \int_0^{2\pi} \frac{\rho_{\ell 0}\,a\,d\phi'}{4\pi\varepsilon_0\sqrt{z^2+a^2}}$$

$$= \frac{\rho_{\ell 0}\,a}{4\pi\varepsilon_0\sqrt{z^2+a^2}}(2\pi)$$

Note: The upper limit must be larger than the lower limit, to keep dl' positive.

Example (cont.)

Summary



$$\Phi\left(\underline{r}\right)=\left(\frac{\rho_{\ell 0}\,a}{2\varepsilon_0}\right)\frac{1}{\sqrt{z^2+a^2}}\quad[\mathrm{V}]$$

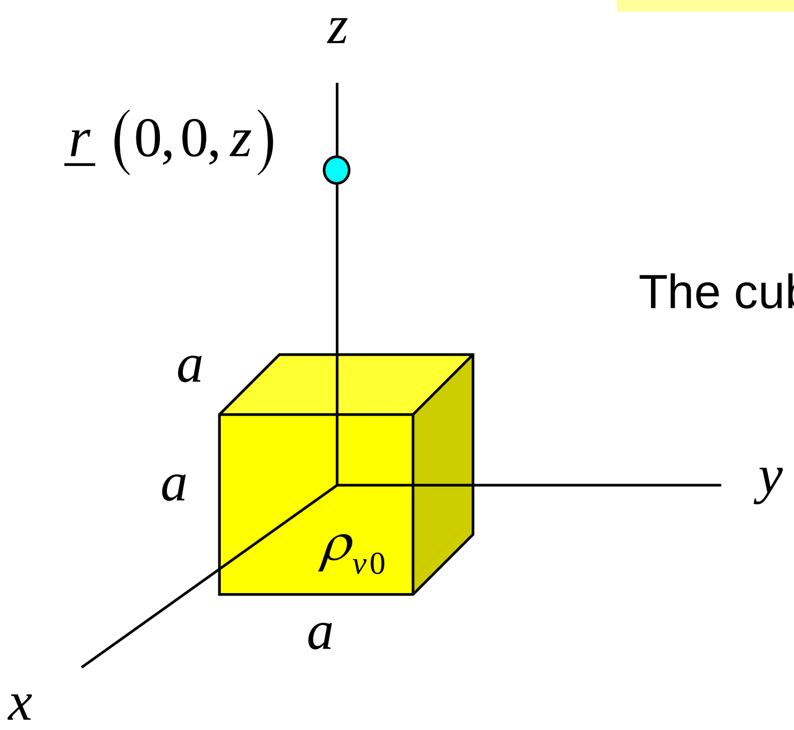
$$\Phi\left(\infty\right)=0$$

Example

Solid cube of uniform charge density

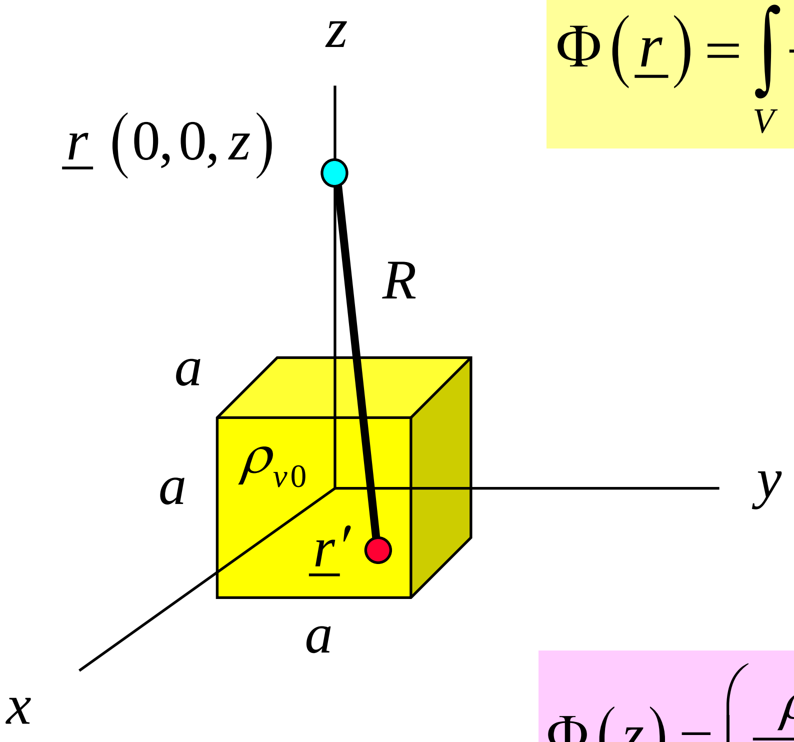
Find $\Phi(0, 0, z)$

Given: $\Phi(\infty) = 0$



The cube is centered at the origin.

Example (cont.)



$$\Phi(\underline{r}) = \int_V \frac{\rho_v(\underline{r}')\,dV'}{4\pi\varepsilon_0 R}$$

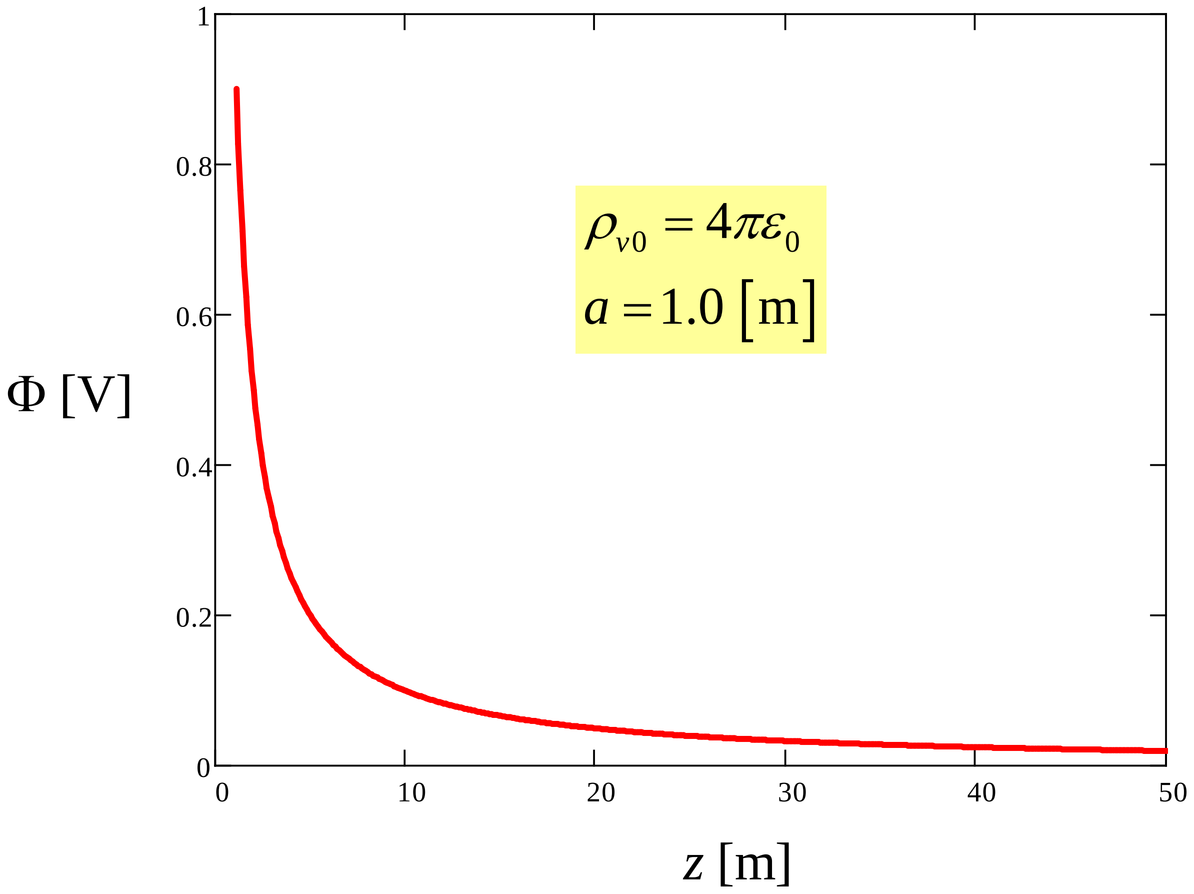
$$\Phi(\underline{r}) = \int_V \frac{\rho_{v0}\,dV'}{4\pi\varepsilon_0 R}$$

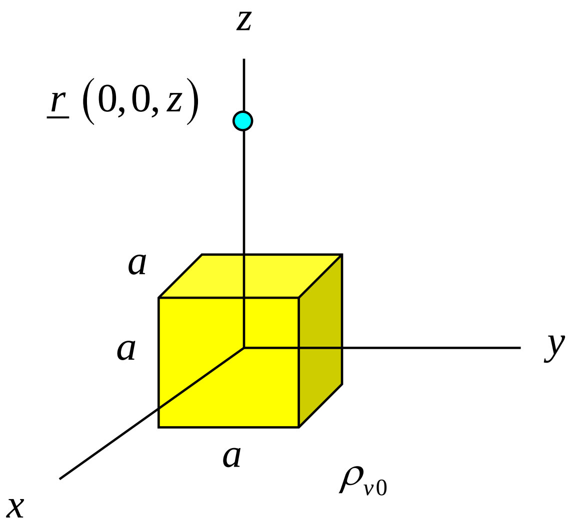
$$R = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$$

$$\Phi(z) = \left(\frac{\rho_{v0}}{4\pi\varepsilon_0}\right) \int_{-a/2}^{a/2}\int_{-a/2}^{a/2}\int_{-a/2}^{a/2} \frac{dx'dy'dz'}{\sqrt{x'^2 + y'^2 + (z-z')^2}}$$

The integral can be evaluated <u>numerically</u>.

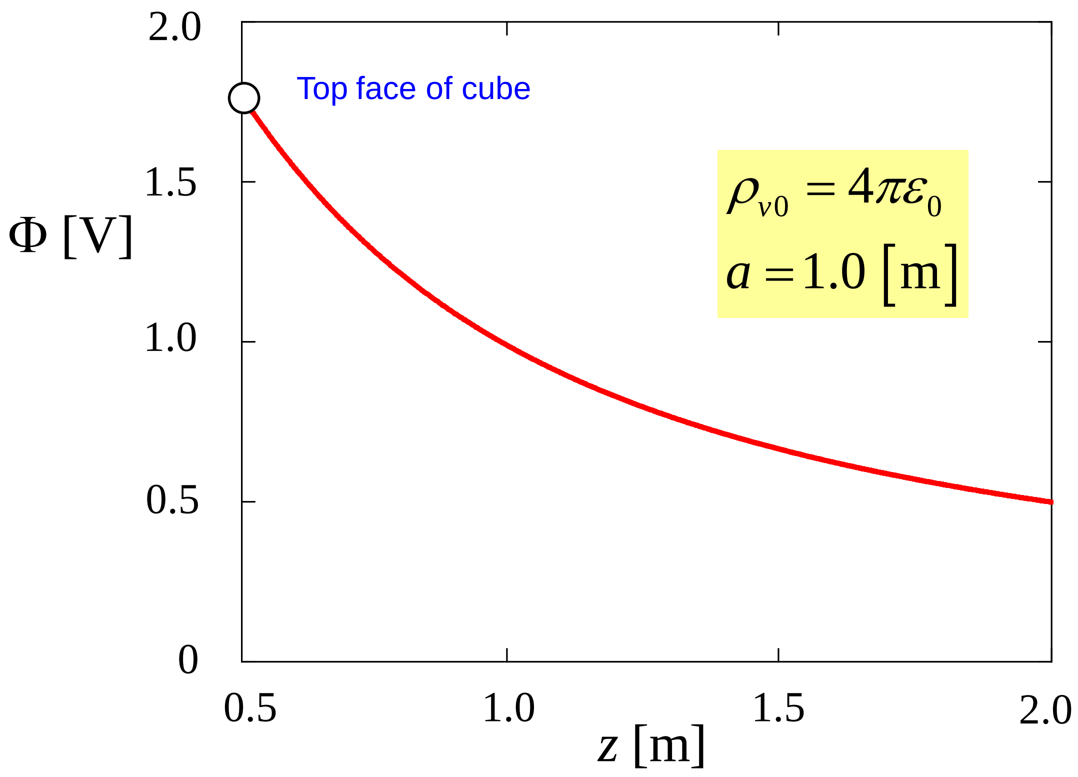
Example (cont.)

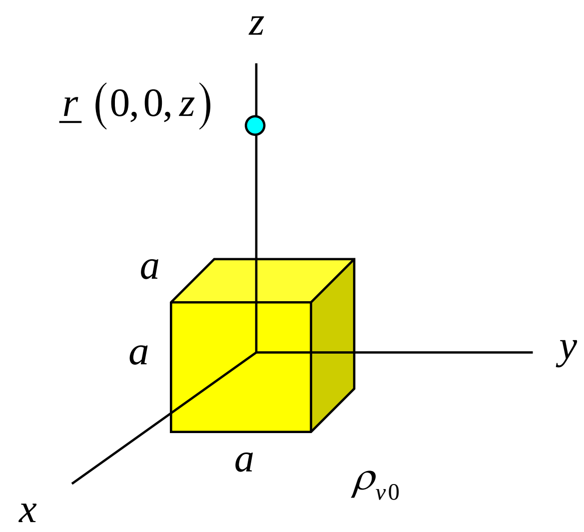




Result from Mathcad

Example (cont.)





$$\Phi(z) \to 1.793 \text{ [V]}$$

as $z \to 0.5$ [m]

Result from Mathcad

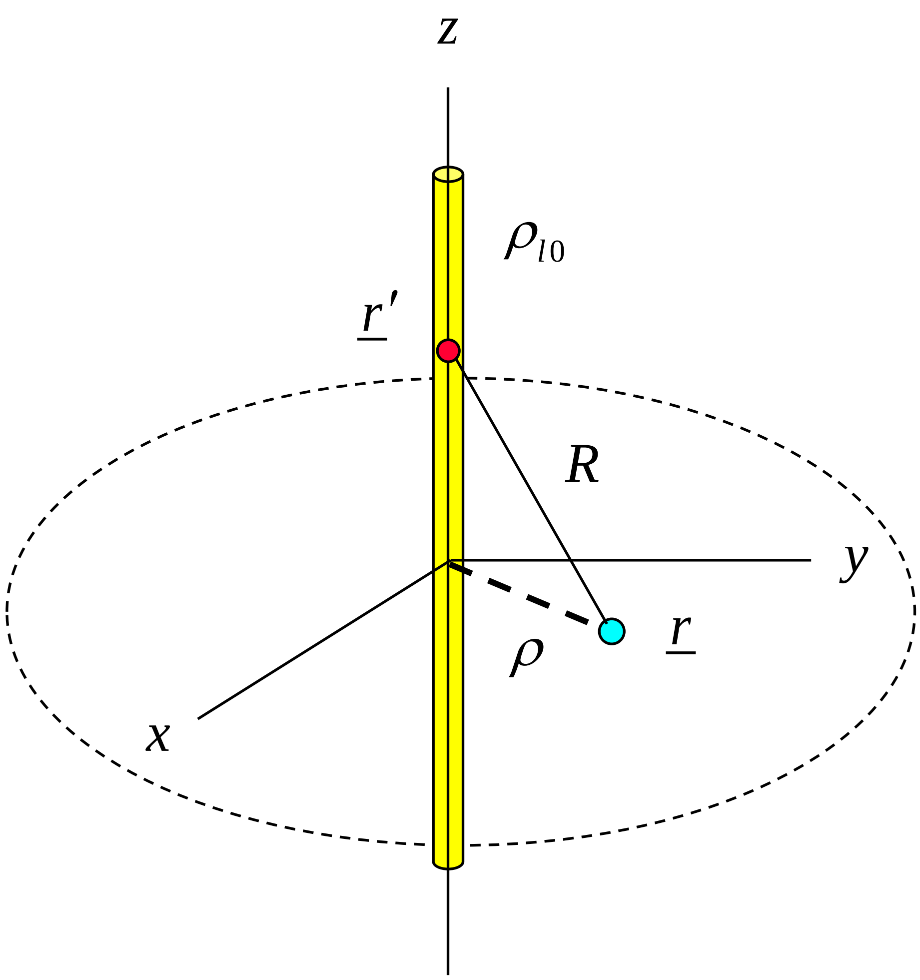
Limitation of Potential-Charge Formula

This method always works for a bounded charge density; that is, one that may be completely enclosed by a closed surface.

- ❖ The method will fail for 2D problems (it is never possible to have zero volts at infinity).

Example of Limitation (cont.)

Here the potential integral formula fails.



$$\Phi(\rho) = \int_{-\infty}^{\infty} \frac{\rho_{l0}}{4\pi\varepsilon_0 R} dz'$$

$$= \frac{\rho_{l0}}{4\pi\varepsilon_0} \int_{-\infty}^{\infty} \frac{1}{\sqrt{\rho^2 + z'^2}} dz'$$

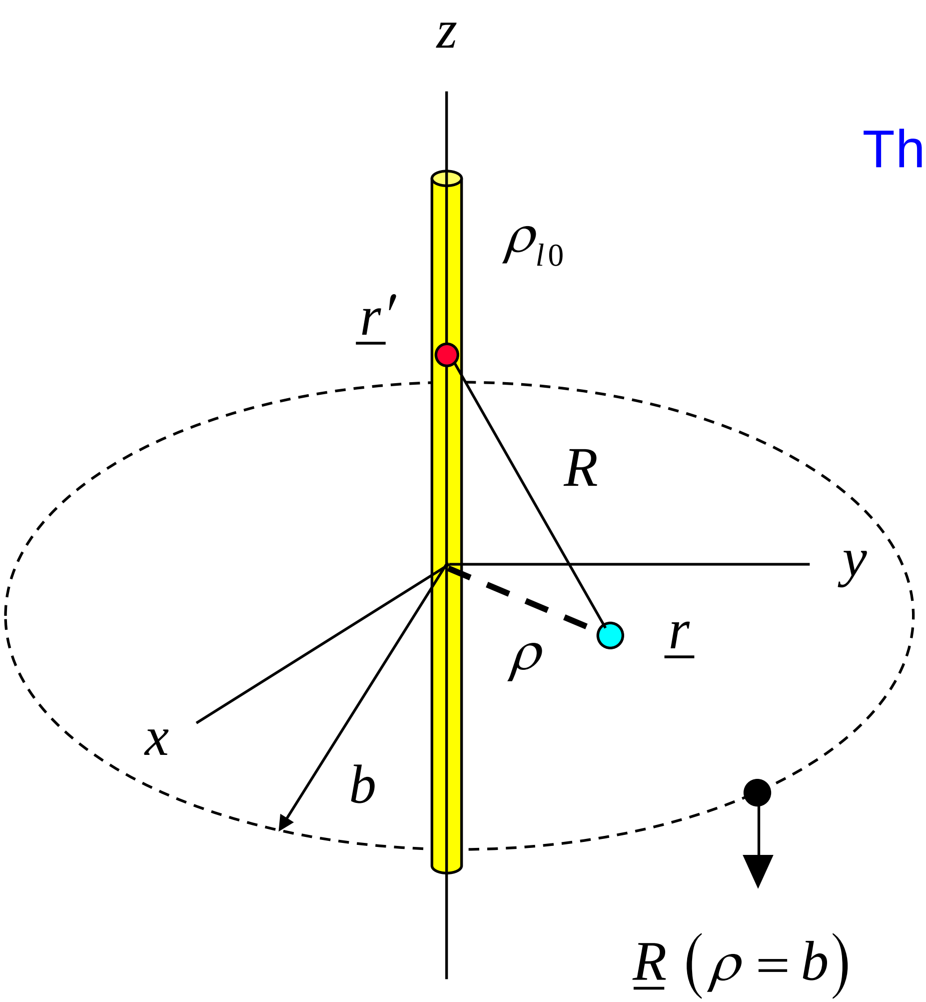
$$= (2)\frac{\rho_{l0}}{4\pi\varepsilon_0} \int_{0}^{\infty} \frac{1}{\sqrt{\rho^2 + z'^2}} dz'$$

$$= \frac{\rho_{l0}}{2\pi\varepsilon_0} \ln\left(z' + \sqrt{z'^2 + \rho^2}\right)_0^{\infty}$$

The integral does not converge!

Infinite line charge (2D problem)

Example of Limitation (cont.)



Infinite line charge

The field-integration method still works:

$$\Phi(\rho) = \frac{\rho_{\ell 0}}{2\pi\,\varepsilon_0} \ln\left(\frac{b}{\rho}\right) \quad [\text{V}]$$

(From Notes 14)

Note:
We can still use the "potential from charge" method if we assume a finite length of line charge first, add a constant to make Φ zero at $\rho = b$, and then let the length tend to infinity after solving the problem.

(This will be a homework problem.)