ECE 3318
Applied Electricity and Magnetism

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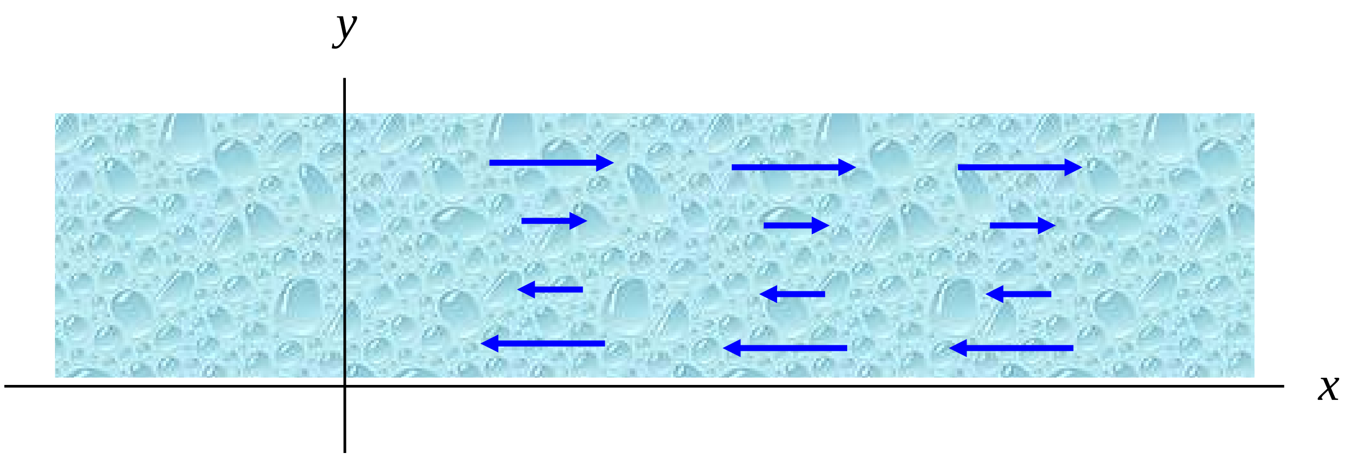
Notes 17
Curl

Curl of a Vector

The curl of a vector function measures the tendency of the vector function to circulate or rotate (or "curl") about an axis.

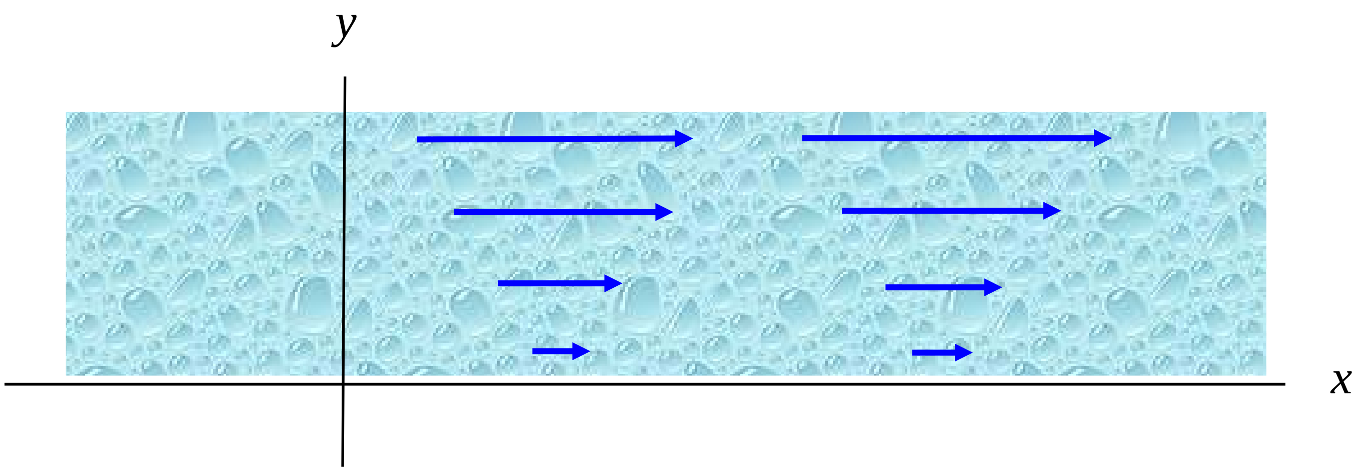
Note the circulation about the z axis in this stream of water.



The curl of the water velocity vector has a z component.

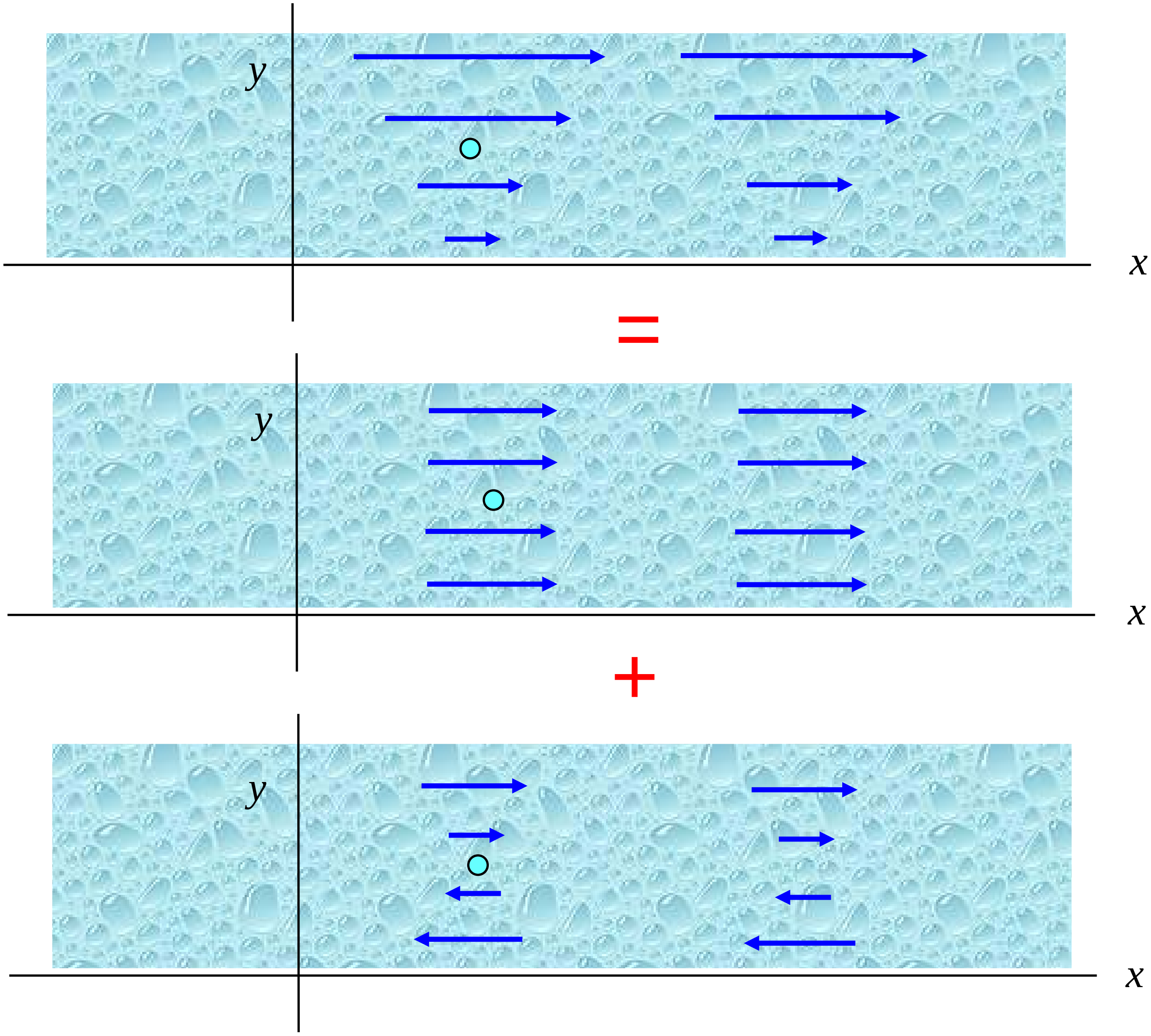
Curl of a Vector (cont.)

Here the water also has a circulation about the z axis.



This is more obvious if we subtract a constant velocity vector from the water, as seen on the next slide.

Curl of a Vector (cont.)



Curl of a Vector (cont.)

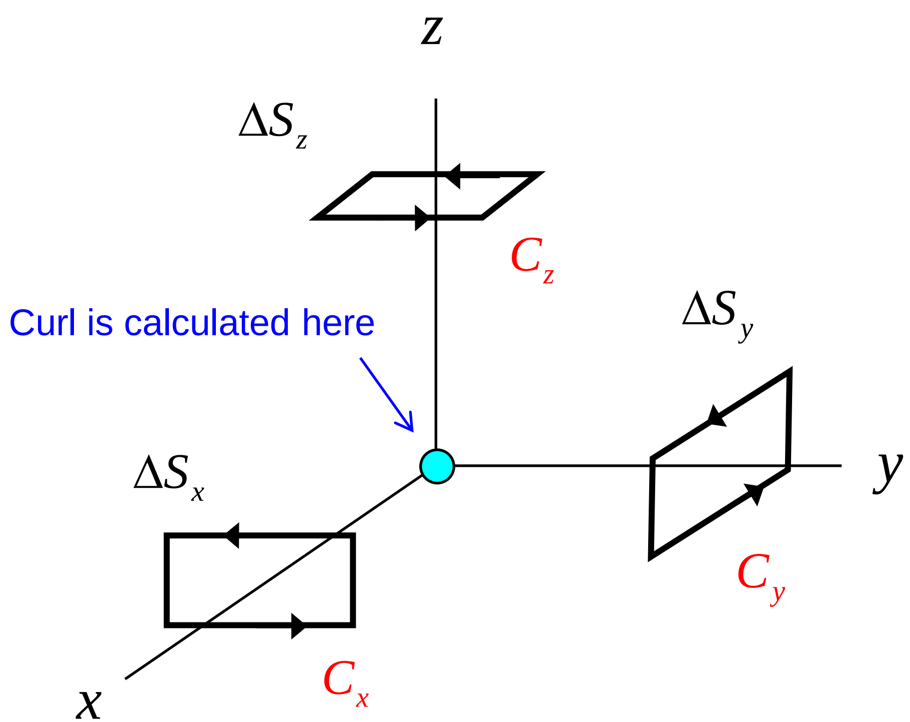
$$\underline{V}(x, y, z) = \text{vector function}$$

$$curl\ \underline{V} = \text{vector function}$$

$$\underline{\hat{x}} \cdot curl\ \underline{V} \equiv \lim_{\Delta s_x \to 0} \frac{1}{\Delta S_x} \oint_{C_x} \underline{V} \cdot \underline{dr}$$

$$\underline{\hat{y}} \cdot curl\ \underline{V} \equiv \lim_{\Delta s_y \to 0} \frac{1}{\Delta S_y} \oint_{C_y} \underline{V} \cdot \underline{dr}$$

$$\underline{\hat{z}} \cdot curl\ \underline{V} \equiv \lim_{\Delta s_z \to 0} \frac{1}{\Delta S_z} \oint_{C_z} \underline{V} \cdot \underline{dr}$$



Note: It turns out that the results are independent of the shape of the paths, but rectangular paths are chosen for simplicity.

The paths are defined according to the “right-hand rule”.

The paths are all centered at the point of interest, taken here as the origin for simplicity.
(This is an “exploded view”; a separation between the paths is shown for clarity).

Curl of a Vector (cont.)

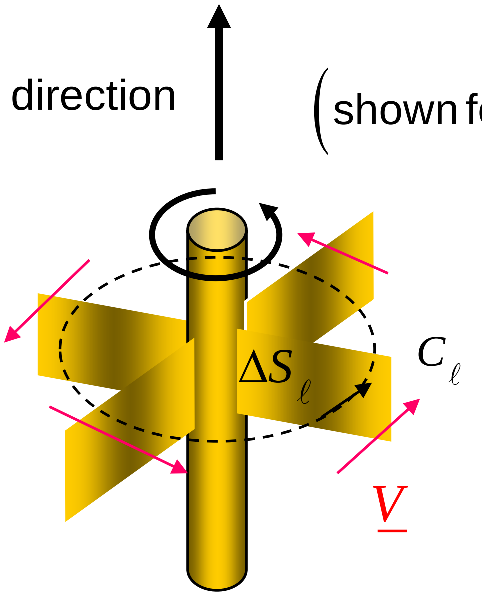
"Curl meter"

$$\hat{\underline{\ell}} = \hat{\underline{x}},\ \hat{\underline{y}},\ \text{or}\ \hat{\underline{z}}$$

Assume that $\underline{V}$ represents the velocity of a fluid.

$T_l =$ torque in l direction

$\left(\text{shown for } \hat{\underline{\ell}} = \hat{\underline{z}}\right)$



$$\hat{\underline{\ell}} \cdot curl\ \underline{V} \equiv \lim_{\Delta s_\ell \to 0} \frac{1}{\Delta S_\ell} \oint_{C_\ell} \underline{V} \cdot \underline{dr}$$

The term $\underline{V} \cdot \underline{dr}$ measures the force on the paddles.

Hence,

$\hat{\underline{\ell}} \cdot curl\ \underline{V}$ is a measure of T_l, the component of torque in the $\hat{\underline{\ell}}$ direction.

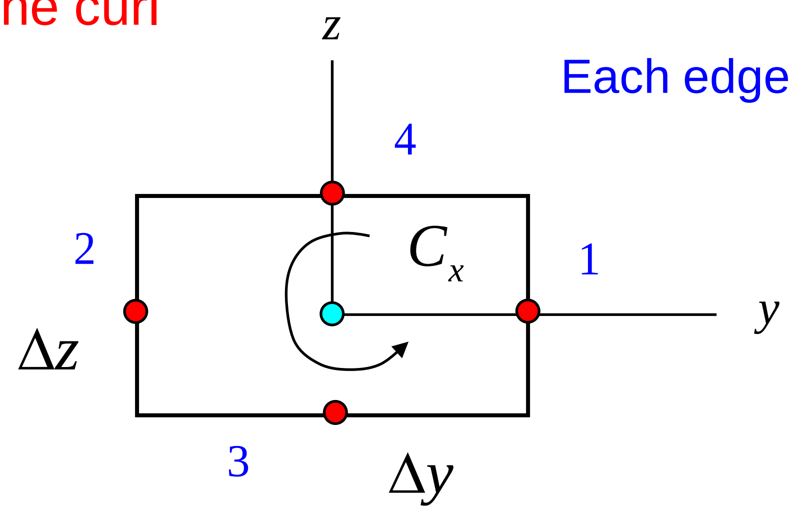
(If this component is positive, the paddle wheel will spin counterclockwise.)

Curl Calculation

The x component of the curl

Each edge is numbered.

Path C_x :



$$\oint_{C_x} \underline{V}\cdot\underline{dr} = \oint_{C_x} V_x\,dx + V_y\,dy + V_z\,dz \approx V_z\left(0, \frac{\Delta y}{2}, 0\right)\Delta z \quad (1)$$

$$-V_z\left(0, -\frac{\Delta y}{2}, 0\right)\Delta z \quad (2)$$

Pair

$$+V_y\left(0, 0, -\frac{\Delta z}{2}\right)\Delta y \quad (3)$$

$$-V_y\left(0, 0, \frac{\Delta z}{2}\right)\Delta y \quad (4)$$

Pair

$$\underline{\hat{x}}\cdot curl\,\underline{V} \equiv \lim_{\Delta s_x \to 0} \frac{1}{\Delta S_x}\oint_{C_x} \underline{V}\cdot\underline{dr}$$

$$\Delta S_x = \Delta y \Delta z$$

Curl Calculation (cont.)

Hence, we have:

$$\frac{1}{\Delta y\,\Delta z}\oint_{C_x}\underline{V}\cdot\underline{dr}\approx\left(\cancel{\Delta}z\right)\left[\frac{V_z\left(0,\frac{\Delta y}{2},0\right)-V_z\left(0,-\frac{\Delta y}{2},0\right)}{\Delta y\cancel{\Delta}z}\right]$$

$$-\left(\cancel{\Delta}y\right)\left[\frac{V_y\left(0,0,\frac{\Delta z}{2}\right)-V_y\left(0,0,-\frac{\Delta z}{2}\right)}{\cancel{\Delta}y\Delta z}\right]$$

$$\text{Recall: }\ \underline{\hat{x}}\cdot curl\,\underline{V}\equiv\lim_{\Delta s_x\to 0}\frac{1}{\Delta S_x}\oint_{C_x}\underline{V}\cdot\underline{dr},\quad \Delta S_x=\Delta y\Delta z$$

Curl Calculation (cont.)

This gives us:

$$\frac{1}{\Delta S_x}\oint_{C_x} \underline{V}\cdot \underline{dr} \approx \left[\frac{V_z\left(0, \frac{\Delta y}{2}, 0\right) - V_z\left(0, -\frac{\Delta y}{2}, 0\right)}{\Delta y} \right]$$

$$-\left[\frac{V_y\left(0, 0, \frac{\Delta z}{2}\right) - V_y\left(0, 0, -\frac{\Delta z}{2}\right)}{\Delta z} \right]$$

$$\approx \frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}$$

Hence,

$$\underline{\hat{x}}\cdot curl\ \underline{V} = \frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}$$

Recall: $\underline{\hat{x}}\cdot curl\ \underline{V} \equiv \lim_{\Delta s_x \to 0} \frac{1}{\Delta S_x}\oint_{C_x} \underline{V}\cdot \underline{dr}$

Curl Calculation (cont.)

Similarly,

$$\frac{1}{\Delta S_y}\oint_{C_y} \underline{V}\cdot \underline{dr} \approx \frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x} \quad \Rightarrow \quad \underline{\hat{y}}\cdot curl\ \underline{V} = \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right)$$

$$\frac{1}{\Delta S_z}\oint_{C_z} \underline{V}\cdot \underline{dr} \approx \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \quad \Rightarrow \quad \underline{\hat{z}}\cdot curl\ \underline{V} = \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)$$

Hence, we have:

$$curl\ \underline{V} = \underline{\hat{x}}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) + \underline{\hat{y}}\left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right) + \underline{\hat{z}}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)$$

See Appendix A.2 in the Hayt & Buck book for a general derivation of curl that holds in <u>any</u> coordinate system.

Del Operator

Recall: $$\nabla \equiv \left(\underline{\hat{x}} \frac{\partial}{\partial x} + \underline{\hat{y}} \frac{\partial}{\partial y} + \underline{\hat{z}} \frac{\partial}{\partial z} \right)$$

$$\nabla \times \underline{V} = \left(\underline{\hat{x}} \frac{\partial}{\partial x} + \underline{\hat{y}} \frac{\partial}{\partial y} + \underline{\hat{z}} \frac{\partial}{\partial z} \right) \times \left(\underline{\hat{x}} V_x + \underline{\hat{y}} V_y + \underline{\hat{z}} V_z \right)$$

$$= \begin{vmatrix} \underline{\hat{x}} & \underline{\hat{y}} & \underline{\hat{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

$$= \underline{\hat{x}} \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} \right) - \underline{\hat{y}} \left(\frac{\partial V_z}{\partial x} - \frac{\partial V_x}{\partial z} \right) + \underline{\hat{z}} \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right)$$

Del Operator (cont.)

Hence, in rectangular coordinates, we have

$$curl\ \underline{V} = \nabla \times \underline{V}$$

Summary of Curl Formulas

Rectangular

$$\nabla \times \underline{V} = \underline{\hat{x}}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) + \underline{\hat{y}}\left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right) + \underline{\hat{z}}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)$$

Cylindrical

$$\nabla \times \underline{V} = \underline{\hat{\rho}}\left(\frac{1}{\rho}\frac{\partial V_z}{\partial \phi} - \frac{\partial V_\phi}{\partial z}\right) + \underline{\hat{\phi}}\left(\frac{\partial V_\rho}{\partial z} - \frac{\partial V_z}{\partial \rho}\right) + \underline{\hat{z}}\,\frac{1}{\rho}\left(\frac{\partial\left(\rho V_\phi\right)}{\partial \rho} - \frac{\partial V_\rho}{\partial \phi}\right)$$

Spherical

$$\nabla \times \underline{V} = \underline{\hat{r}}\,\frac{1}{r\sin\theta}\left[\frac{\partial\left(V_\phi \sin\theta\right)}{\partial \theta} - \frac{\partial V_\theta}{\partial \phi}\right] + \underline{\hat{\theta}}\,\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial V_r}{\partial \phi} - \frac{\partial\left(rV_\phi\right)}{\partial r}\right] + \underline{\hat{\phi}}\,\frac{1}{r}\left[\frac{\partial\left(rV_\theta\right)}{\partial r} - \frac{\partial V_r}{\partial \theta}\right]$$

Summary of Curl Formulas (cont.)

Determinant Forms

Rectangular

$$\nabla \times \underline{A} = \begin{vmatrix} \hat{\underline{x}} & \hat{\underline{y}} & \hat{\underline{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

Cylindrical

$$\nabla \times \underline{A} = \frac{1}{\rho} \begin{vmatrix} \hat{\underline{\rho}} & \rho\,\hat{\underline{\phi}} & \hat{\underline{z}} \\ \dfrac{\partial}{\partial \rho} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix}$$

Spherical

$$\nabla \times \underline{A} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{\underline{r}} & r\hat{\underline{\theta}} & r\sin\theta\,\hat{\underline{\phi}} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix}$$

Note on ∇ Operator

Curl can be calculated in any coordinated system, but the ∇ operator is only defined in rectangular coordinates:

$$\nabla \equiv \underline{\hat{x}}\frac{\partial}{\partial x} + \underline{\hat{y}}\frac{\partial}{\partial y} + \underline{\hat{z}}\frac{\partial}{\partial z}$$

$$\nabla \neq \underline{\hat{\rho}}\frac{\partial}{\partial \rho} + \underline{\hat{\phi}}\frac{\partial}{\partial \phi} + \underline{\hat{z}}\frac{\partial}{\partial z}$$

$$\nabla \neq \underline{\hat{r}}\frac{\partial}{\partial r} + \underline{\hat{\theta}}\frac{\partial}{\partial \theta} + \underline{\hat{\phi}}\frac{\partial}{\partial \phi}$$

For example, in spherical coordinates:

$$curl\,\underline{V} \neq \left(\underline{\hat{r}}\frac{\partial}{\partial r} + \underline{\hat{\theta}}\frac{\partial}{\partial \theta} + \underline{\hat{\phi}}\frac{\partial}{\partial \phi}\right) \times \left(\underline{\hat{r}}V_r + \underline{\hat{\theta}}V_\theta + \underline{\hat{\phi}}V_\phi\right)$$

Example

Calculate the curl of the following vector function:

$$\underline{V} = \underline{\hat{x}}\left(3xy^2z\right) + \underline{\hat{y}}\left(2x^2 - z^3\right) + \underline{\hat{z}}\left(2xz\right)$$

$$\nabla \times \underline{V} = \underline{\hat{x}}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) + \underline{\hat{y}}\left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right) + \underline{\hat{z}}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)$$

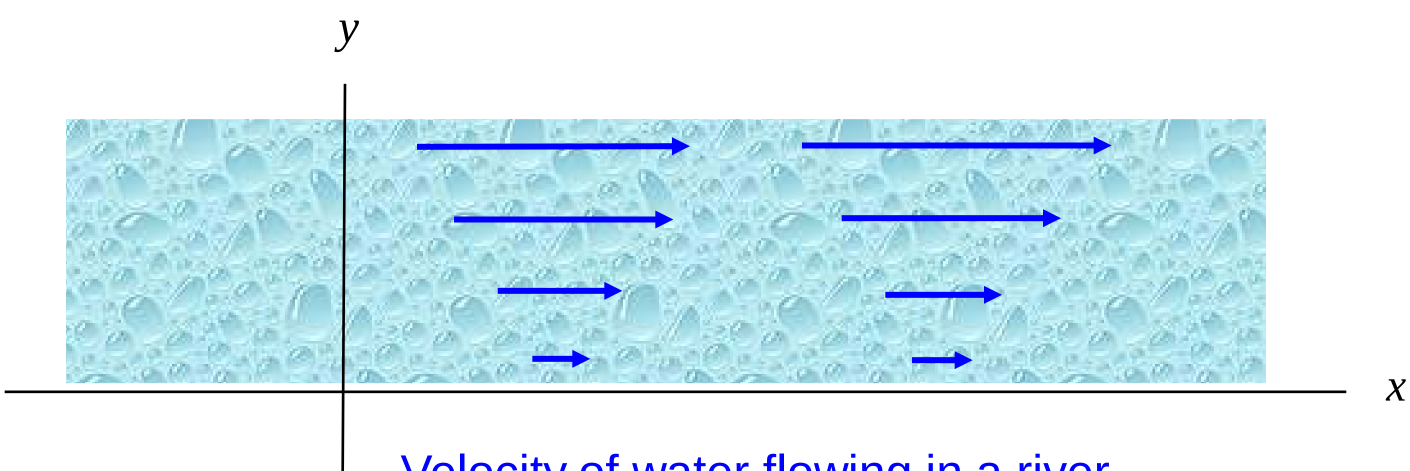
$$\nabla \times \underline{V} = \underline{\hat{x}}\left(0 + 3z^2\right) + \underline{\hat{y}}\left(3xy^2 - 2z\right) + \underline{\hat{z}}\left(4x - 6xyz\right)$$

Example

Calculate the curl: $\boxed{\underline{V} = \underline{\hat{x}}\left(y\right)}$ $\qquad V_x = y,\;\; V_y = 0,\;\; V_z = 0$

$$\nabla \times \underline{V} = \underline{\hat{x}}\left(\frac{\partial \cancel{V_z}}{\partial y} - \frac{\partial \cancel{V_y}}{\partial z}\right) + \underline{\hat{y}}\left(\cancel{\frac{\partial V_x}{\partial z}} - \frac{\partial \cancel{V_z}}{\partial x}\right) + \underline{\hat{z}}\left(\frac{\partial \cancel{V_y}}{\partial x} - \frac{\partial V_x}{\partial y}\right)$$

Hence, $\boxed{\nabla \times \underline{V} = \underline{\hat{z}}\left(-1\right)}$



Velocity of water flowing in a river

Example (cont.)

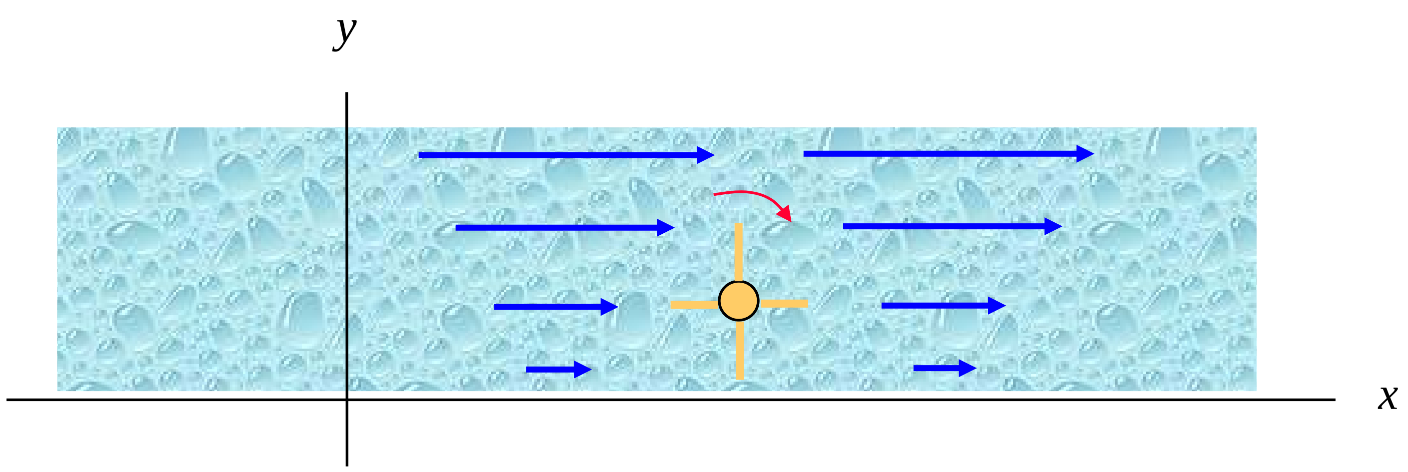
$$\nabla \times \underline{V} = \underline{\hat{z}}(-1)$$

Note:
The paddle wheel will not spin if the axis is pointed in the x or y directions
(the x and y components of the curl are zero).

Hence $$(\nabla \times \underline{V}) \cdot \underline{\hat{z}} = -1 < 0$$

Point your thumb in the z direction:

The paddle wheel spins <u>opposite</u> to the fingers of the right hand.



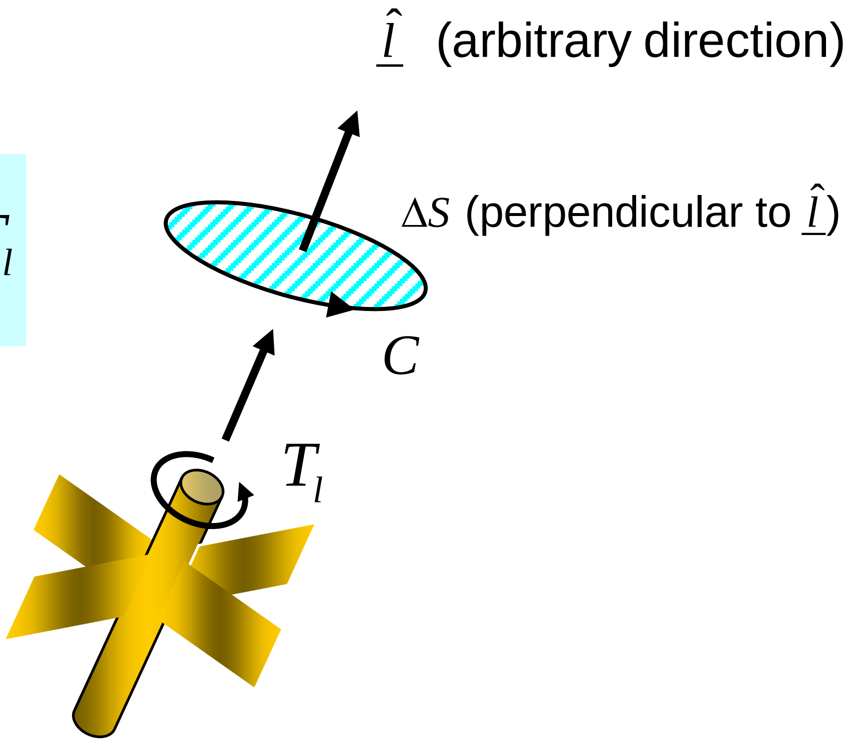
Arbitrary Component of Curl Vector

Consider taking a component of the curl vector in an arbitrary direction.

We have the following property:

$$\left(\nabla\times\underline{V}\right)\cdot\underline{\hat{l}} = \lim_{\Delta S\to 0}\frac{1}{\Delta S}\oint_C \underline{V}\cdot\underline{dr} \propto T_l$$

(The proof is in Appendix B.)

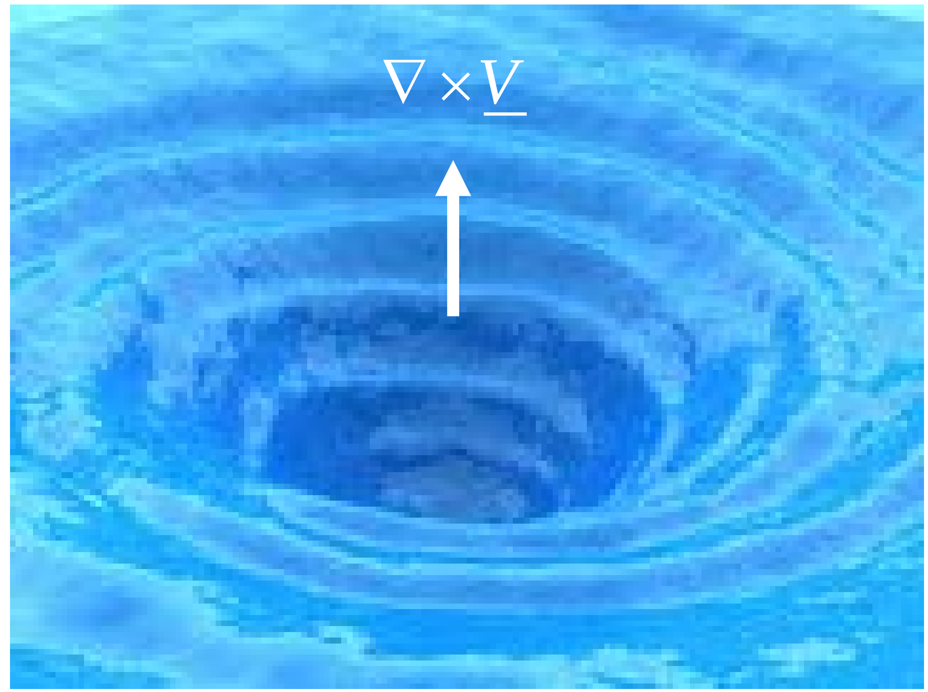


Note:
This property is obviously true for the x, y, and z directions, due to the definition of the curl vector. This theorem now says that the property is true for any direction in space.

Component of Curl Vector (cont.)

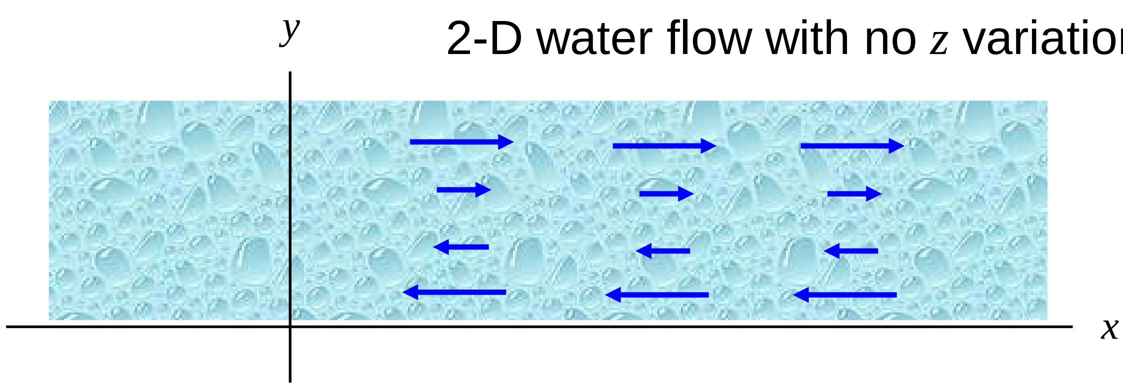
Physical interpretation of curl component (water flow)

The curl vector points in the direction of the "whirlpool" effect.



- If we call the axis of the whirlpool the z direction, then the curl of the velocity vector $\underline{V}$ has a z component but no x or y components (visual a paddle wheel in the water being aligned in the x or y directions in the figure below).

2-D water flow with no z variation



Summary of Curl Properties

- The x, y, z components of the curl vector are defined by the circulation (per area) about the corresponding axis. This translates into torque on the paddle wheel when pointed in these directions.
- The component of the curl vector in an arbitrary direction gives the circulation (per area) about the corresponding axis. This translates into torque on the paddle wheel when pointed in this direction.
- Physically, the curl vector points in the direction of the "whirlpool" of the vector function.

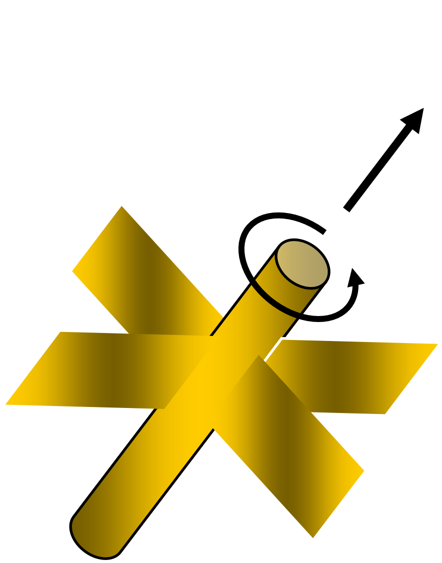


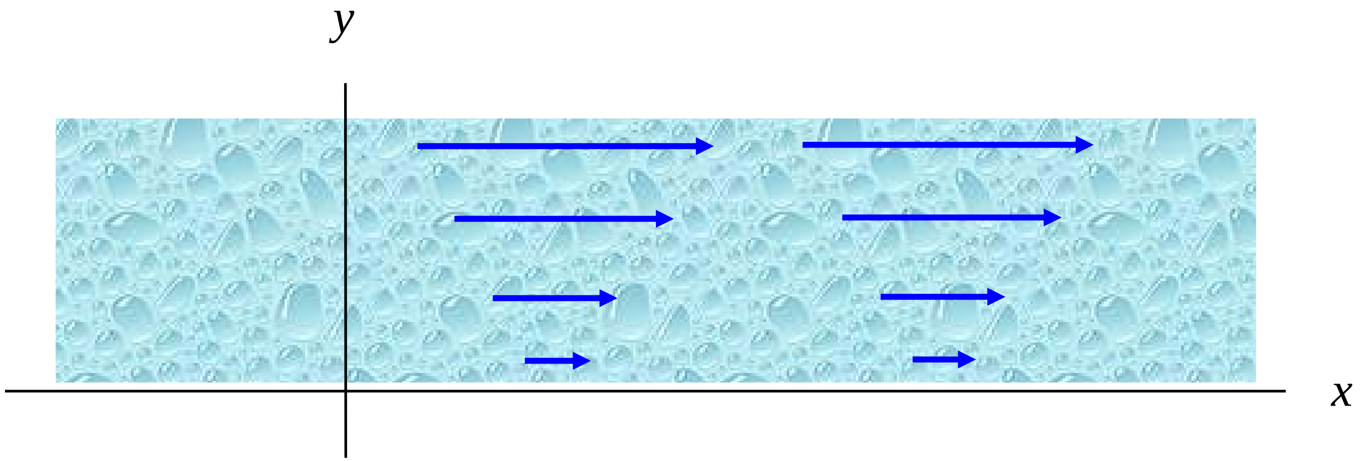
Illustration of Curl Properties

Example: $\underline{V} = \underline{\hat{x}}\left(y\right)$

From calculations: $\nabla \times \underline{V} = \underline{\hat{z}}\left(-1\right)$

Hence:

- The paddle wheel spins the fastest when the axis is along the z axis.
- The z axis is the axis of the “whirlpool” in the water.



Vector Identity

$$\nabla \cdot \left(\nabla \times \underline{V} \right) = 0$$

Proof:

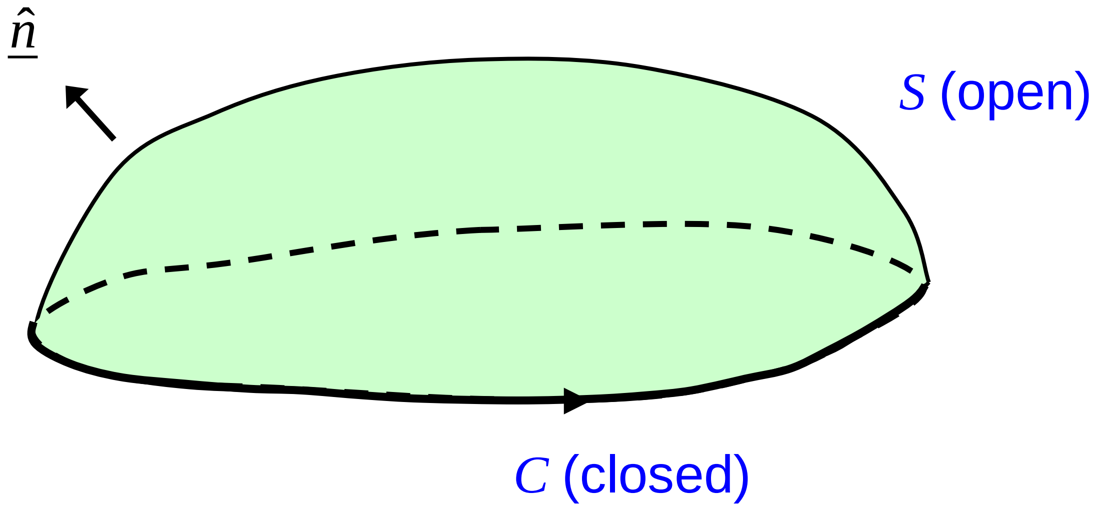
$$\nabla \times \underline{V} = \underline{\hat{x}} \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} \right) - \underline{\hat{y}} \left(\frac{\partial V_z}{\partial x} - \frac{\partial V_x}{\partial z} \right) + \underline{\hat{z}} \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right)$$

$$\nabla \cdot \underbrace{\left(\nabla \times \underline{V} \right)}_{\underline{A}} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$= \left(\cancel{\frac{\partial^2 V_z}{\partial x \, \partial y}} - \cancel{\frac{\partial^2 V_y}{\partial x \, \partial z}} \right) - \left(\cancel{\frac{\partial^2 V_z}{\partial y \, \partial x}} - \cancel{\frac{\partial^2 V_x}{\partial y \, \partial z}} \right) + \left(\cancel{\frac{\partial^2 V_y}{\partial z \, \partial x}} - \cancel{\frac{\partial^2 V_x}{\partial z \, \partial y}} \right)$$

$$= 0$$

Stokes's Theorem



The unit normal is chosen from a "right-hand rule" according to the direction along C.
(An outward normal corresponds to a counter clockwise path.)

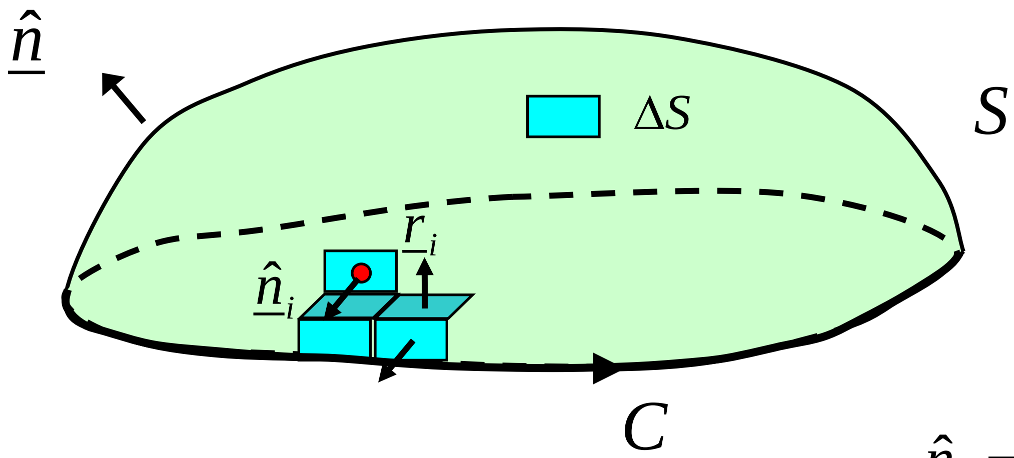
$$\int_S \left(\nabla \times \underline{V}\right) \cdot \underline{\hat{n}}\, dS = \oint_C \underline{V} \cdot \underline{dr}$$

(A proof is in Appendix A.)

"The surface integral of circulation per unit area equals the total circulation."

Appendix A: Proof of Stokes's Theorem

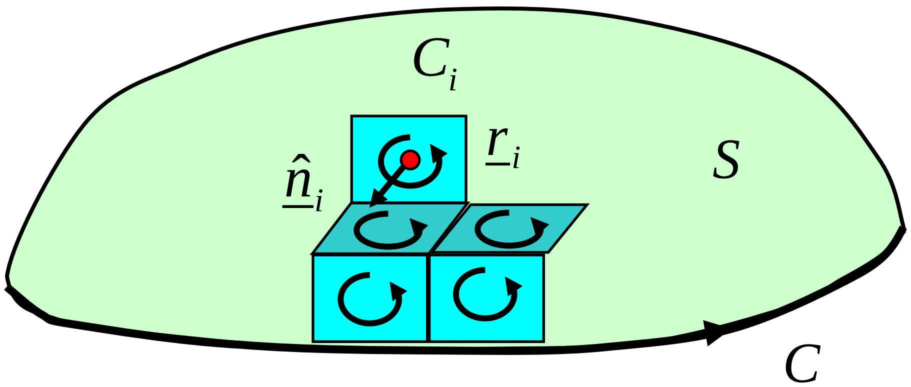
Divide S into rectangular patches that are normal to x, y, or z axes (all with the same area ΔS for simplicity).



$\underline{\hat{n}}_i = \underline{\hat{x}}, \underline{\hat{y}}$, or $\underline{\hat{z}}$

$$\int_S \left(\nabla \times \underline{V}\right) \cdot \underline{\hat{n}}\, dS \approx \sum_n \left(\nabla \times \underline{V}\right)_{\underline{r}_i} \cdot \underline{\hat{n}}_i\, \Delta S$$

Proof (cont.)



$\underline{\hat{n}}_i = \underline{\hat{x}}, \underline{\hat{y}}$, or $\underline{\hat{z}}$

$$\int_S \left(\nabla \times \underline{V}\right) \cdot \underline{\hat{n}}\, dS \approx \sum_n \left(\nabla \times \underline{V}\right)_{\underline{r}_i} \cdot \underline{\hat{n}}_i\, \Delta S$$

Curl definition:

$$\underline{\hat{n}}_i \cdot \left(\nabla \times \underline{V}\right)_{\underline{r}_i} = \lim_{\Delta s \to 0} \frac{1}{\Delta S} \oint_{C_i} \underline{V} \cdot \underline{dr}$$

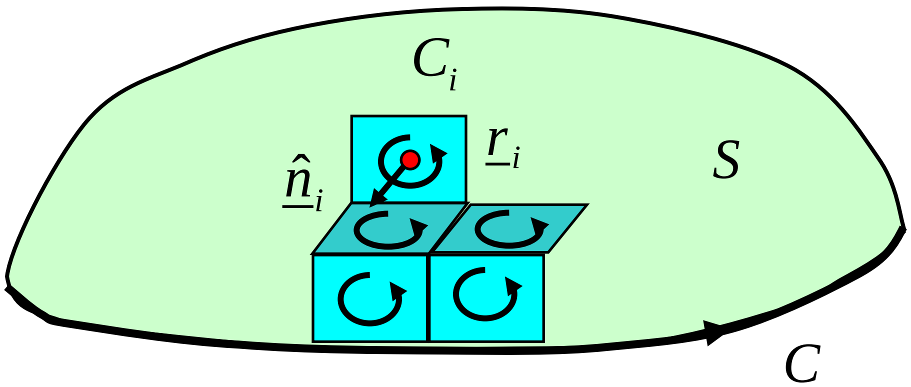
$$\approx \frac{1}{\Delta S} \oint_{C_i} \underline{V} \cdot \underline{dr}$$

Substitute

$$\Rightarrow \quad \left(\nabla \times \underline{V}\right)_{\underline{r}_i} \cdot \underline{\hat{n}}_i\, \Delta S \approx \oint_{C_i} \underline{V} \cdot \underline{dr}$$

so $$\int_S \left(\nabla \times \underline{V}\right) \cdot \underline{\hat{n}}\, ds \approx \sum_n \oint_{C_i} \underline{V} \cdot \underline{dr}$$

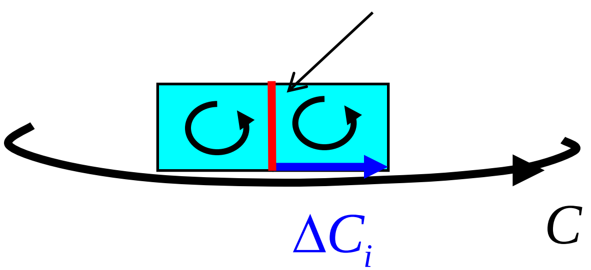
Proof (cont.)



$$\int_S \left(\nabla \times \underline{V}\right) \cdot \underline{\hat{n}}\, ds \approx \sum_n \oint_{C_i} \underline{V} \cdot \underline{dr}$$

$$= \sum_{\text{exterior edges}} \int_{\Delta C_i} \underline{V} \cdot \underline{dr}$$

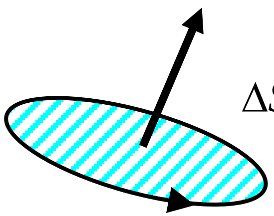
$$\rightarrow \oint_C \underline{V} \cdot \underline{dr}$$



Interior edges cancel,
leaving only exterior edges.

Proof complete

Appendix B: Component of Curl Vector



Proof:

Stokes' Theorem: $$\int_{\Delta S} \left(\nabla \times \underline{V}\right) \cdot \underline{\hat{n}}\, ds = \oint_C \underline{V} \cdot \underline{dr}$$

For the LHS: $$\int_{\Delta S} \left(\nabla \times \underline{V}\right) \cdot \underline{\hat{n}}\, ds = \int_{\Delta S} \left(\nabla \times \underline{V}\right) \cdot \underline{\hat{l}}\, ds \approx \left(\nabla \times \underline{V}\right) \cdot \underline{\hat{l}}\ \Delta S$$

Hence, $$\left(\nabla \times \underline{V}\right) \cdot \underline{\hat{l}}\ \Delta S \approx \oint_C \underline{V} \cdot \underline{dr}$$

Taking the limit: $$\left(\nabla \times \underline{V}\right) \cdot \underline{\hat{l}} = \lim_{\Delta S \to 0} \frac{1}{\Delta S} \oint_C \underline{V} \cdot \underline{dr}$$