ECE 3318
Applied Electricity and Magnetism

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Notes 19
Gradient and Laplacian

Gradient

$$\Phi\left(x, y, z\right) = \text{scalar function}$$

$$\text{grad}\ \Phi \equiv \underline{\hat{x}}\,\frac{\partial \Phi}{\partial x} + \underline{\hat{y}}\,\frac{\partial \Phi}{\partial y} + \underline{\hat{z}}\,\frac{\partial \Phi}{\partial z}$$

We can write this as

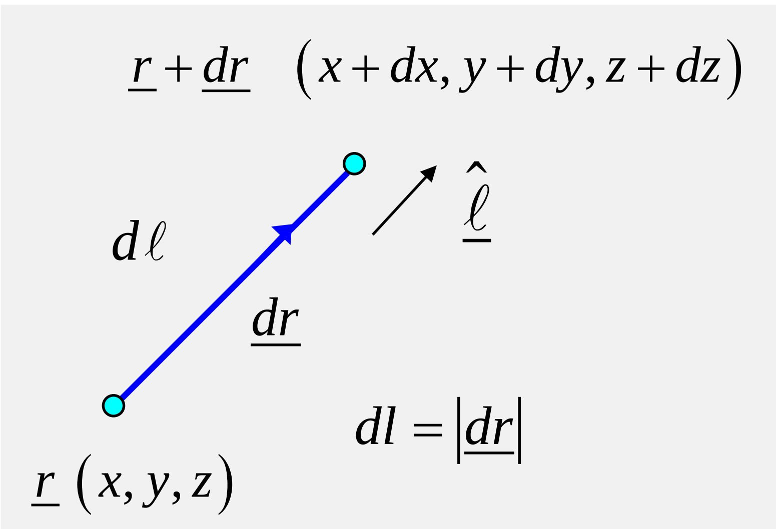
$$\text{grad}\ \Phi = \left(\underline{\hat{x}}\frac{\partial}{\partial x} + \underline{\hat{y}}\frac{\partial}{\partial y} + \underline{\hat{z}}\frac{\partial}{\partial z}\right)\Phi$$

Hence

$$\text{grad}\ \Phi = \nabla \Phi$$

Directional Derivative Property

We look at how a function changes from one point to a nearby point.



$$d\Phi = \Phi\left(\underline{r}+\underline{dr}\right) - \Phi\left(\underline{r}\right)$$

$$= \frac{\partial \Phi}{\partial x}dx + \frac{\partial \Phi}{\partial y}dy + \frac{\partial \Phi}{\partial z}dz$$

(from calculus)

Recall:

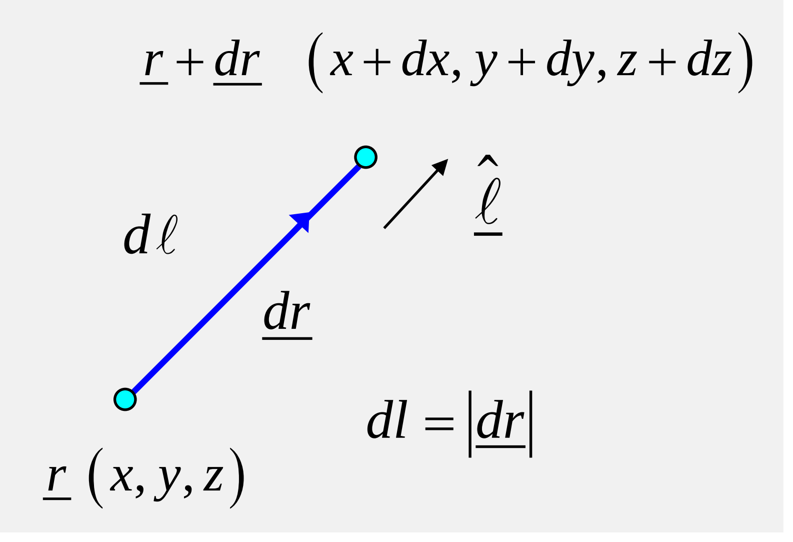
$$\nabla \Phi \equiv \underline{\hat{x}}\frac{\partial \Phi}{\partial x} + \underline{\hat{y}}\frac{\partial \Phi}{\partial y} + \underline{\hat{z}}\frac{\partial \Phi}{\partial z}$$

$$d\underline{r} = \underline{\hat{x}}\left(dx\right) + \underline{\hat{y}}\left(dy\right) + \underline{\hat{z}}\left(dz\right)$$

Hence

$$d\Phi = \nabla \Phi \cdot \underline{dr}$$

Directional Derivative Property (cont.)



$$d\Phi = \nabla\Phi \cdot \underline{dr}$$

Use $\underline{dr} = \hat{\underline{\ell}}\, d\ell$

Then $d\Phi = \left(\nabla\Phi \cdot \hat{\underline{\ell}}\right) d\ell$

This gives us the *directional derivative*:

$$\frac{d\Phi}{d\ell} = \nabla\Phi \cdot \hat{\underline{\ell}}$$

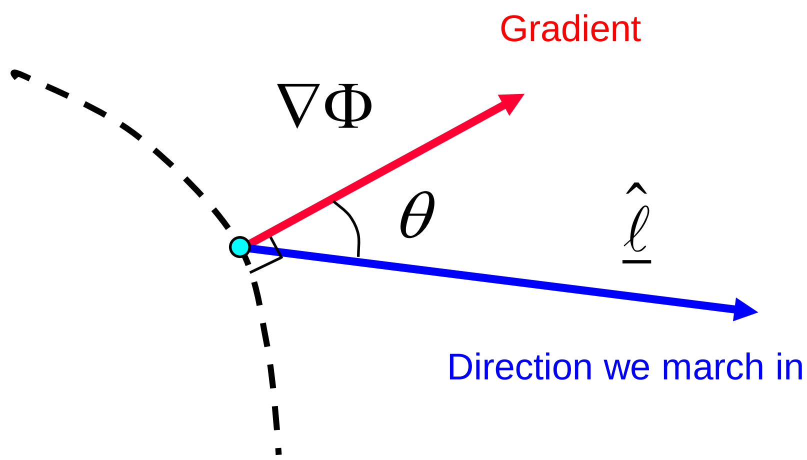
"Directional derivative"

Physical Interpretation

$$\frac{d\Phi}{d\ell} = \nabla\Phi \cdot \hat{\underline{\ell}}$$

We can also write:

$$\frac{d\Phi}{d\ell} = \left|\nabla\Phi\right| \cos\theta$$



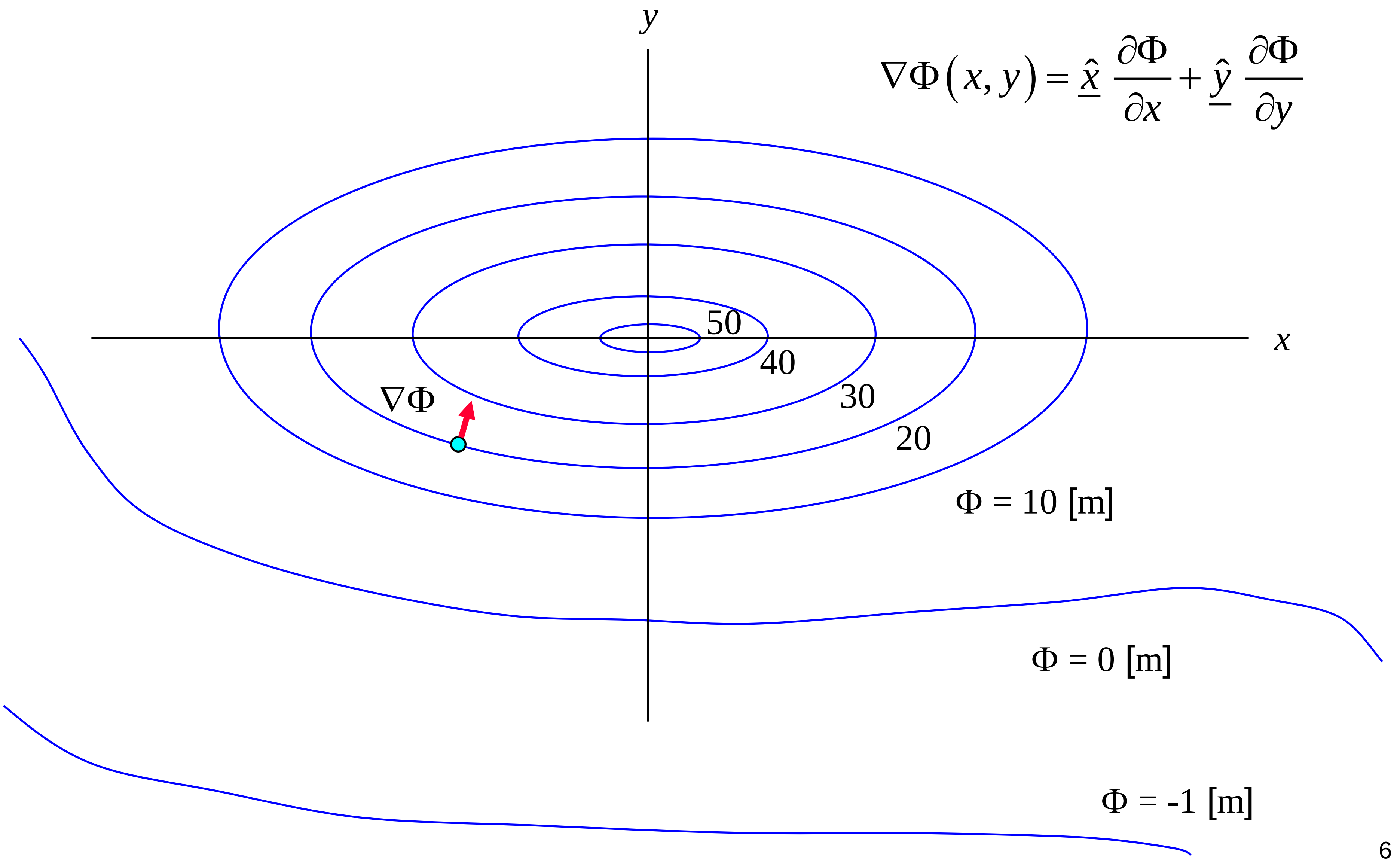
$\Phi =$ constant $\left(\frac{d\Phi}{d\ell} = 0 \text{ along path} \right)$

- The gradient is perpendicular to a level curve of the function ($\cos\theta = 0 \rightarrow \theta = \pi/2$).
- We maximize the directional derivative when we march along in the direction of the gradient ($\theta = 0$).
- The magnitude of the gradient vector gives us the directional derivative when we go in the direction of the gradient.

Mountain Example

Topographic map: $\Phi(x, y)$ = height of the landscape at any point.

$$\nabla\Phi(x,y) = \underline{\hat{x}}\,\frac{\partial\Phi}{\partial x} + \underline{\hat{y}}\,\frac{\partial\Phi}{\partial y}$$



Summary of Gradient Formulas

Rectangular

$$\nabla\Phi = \underline{\hat{x}}\,\frac{\partial\Phi}{\partial x} + \underline{\hat{y}}\,\frac{\partial\Phi}{\partial y} + \underline{\hat{z}}\,\frac{\partial\Phi}{\partial z}$$

See Appendix A.2 of the Hayt & Buck book for a derivation that holds in any coordinate system.

Cylindrical

$$\nabla\Phi = \underline{\hat{\rho}}\,\frac{\partial\Phi}{\partial\rho} + \underline{\hat{\phi}}\,\frac{1}{\rho}\frac{\partial\Phi}{\partial\phi} + \underline{\hat{z}}\,\frac{\partial\Phi}{\partial z}$$

Spherical

$$\nabla\Phi = \underline{\hat{r}}\,\frac{\partial\Phi}{\partial r} + \underline{\hat{\theta}}\,\frac{1}{r}\frac{\partial\Phi}{\partial\theta} + \underline{\hat{\phi}}\,\frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}$$

Relation Between $\underline{E}$ and Φ

Recall:

$$V_{AB} = \Phi(\underline{A}) - \Phi(\underline{B}) \equiv \int_{\underline{A}}^{\underline{B}} \underline{E} \cdot \underline{dr}$$

Also, from calculus

$$\Phi(\underline{A}) - \Phi(\underline{B}) = \int_{\underline{B}}^{\underline{A}} d\Phi = \int_{\underline{B}}^{\underline{A}} \nabla\Phi \cdot \underline{dr}$$

Hence, from the above two results we have

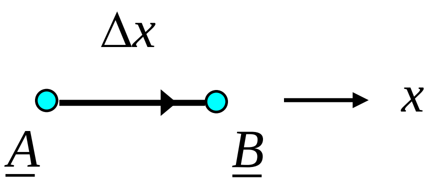
$$\int_{\underline{A}}^{\underline{B}} \underline{E} \cdot \underline{dr} = \int_{\underline{B}}^{\underline{A}} \nabla\Phi \cdot \underline{dr} = \int_{\underline{A}}^{\underline{B}} -\nabla\Phi \cdot \underline{dr}$$

Relation Between $\underline{E}$ and Φ (cont.)

$$\int_{\underline{A}}^{\underline{B}} \underline{E} \cdot \underline{dr} = \int_{\underline{A}}^{\underline{B}} -\nabla\Phi \cdot \underline{dr}$$

This must be true for <u>any</u> path.

Assume a small path in the x direction:



$$\int_{\underline{A}}^{\underline{B}} \underline{E} \cdot \underline{dr} = \int_{\underline{A}}^{\underline{B}} \underline{E} \cdot \left(\underline{\hat{x}}\, dx \right) = \int_{x_A}^{x_B} E_x dx \approx E_x \int_{x_A}^{x_B} dx = E_x \Delta x$$

Similarly, for the second integral:

$$\int_{\underline{A}}^{\underline{B}} -\nabla\Phi \cdot \underline{dr} \approx \left(-\nabla\Phi \right)_x \Delta x$$

Relation Between $\underline{E}$ and Φ (cont.)

Hence:

$$E_x = -\left(\nabla\Phi\right)_x$$

Similarly, using paths in the y and z directions, we have

$$E_y = -\left(\nabla\Phi\right)_y$$

$$E_z = -\left(\nabla\Phi\right)_z$$

Hence, we have

$$\underline{E} = -\nabla\Phi$$

Relation Between $\underline{E}$ and Φ (cont.)

Summary:

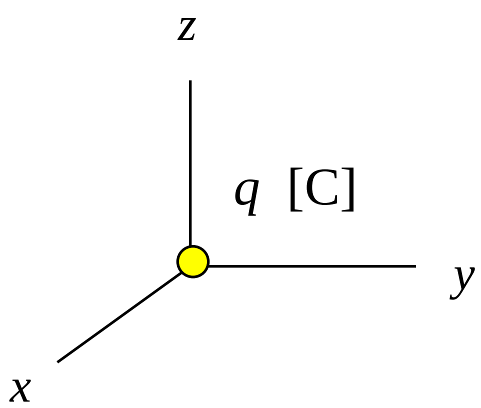
$$\underline{E} = -\nabla\Phi$$

This gives us a new way to find the electric field, by first calculating the potential and then taking the gradient (illustrated next with examples).

Note:
The choice of $\underline{R}$ (the reference point) does not affect $\underline{E}$
(the gradient of a constant is zero).

Example

Find $\underline{E}$ from the point charge



$$\underline{E} = -\nabla\Phi = -\left[\underline{\hat{r}}\frac{\partial\Phi}{\partial r} + \underline{\hat{\theta}}\frac{1}{r}\frac{\partial\Phi}{\partial\theta} + \underline{\hat{\phi}}\frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}\right]$$

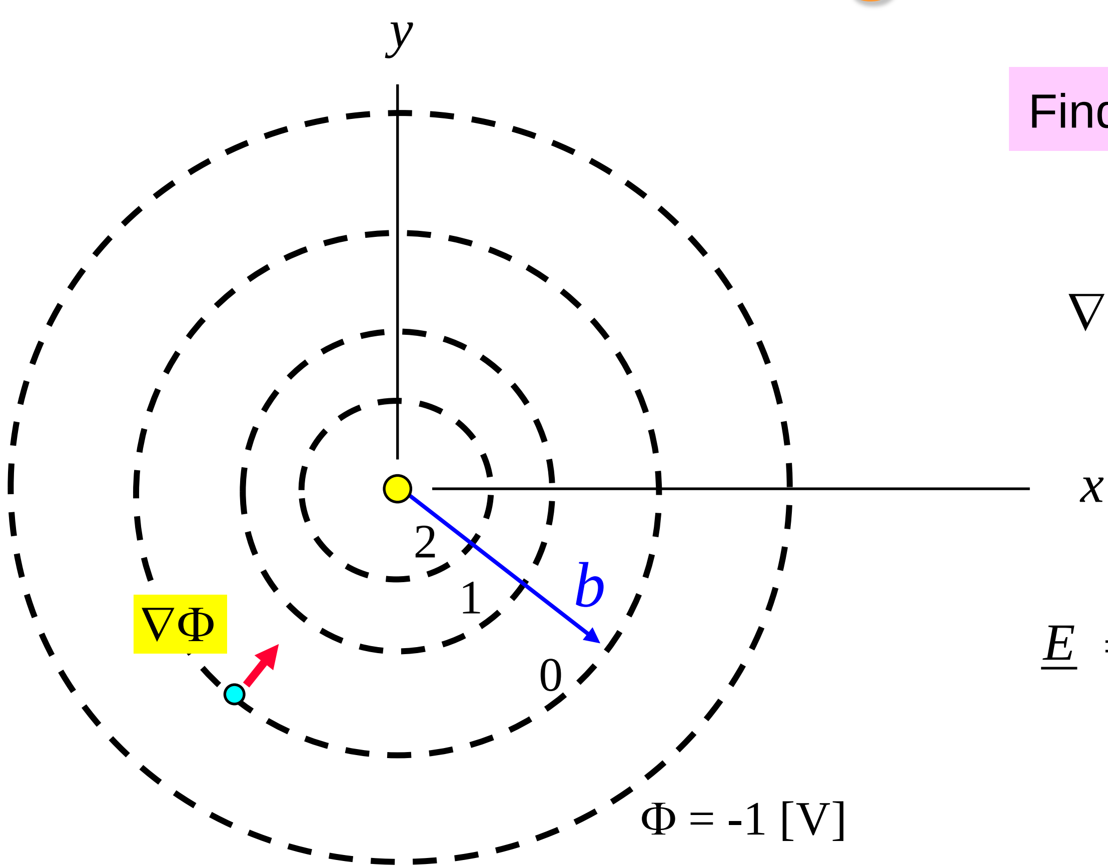
$$= -\underline{\hat{r}}\frac{\partial\Phi}{\partial r}$$

$$= -\underline{\hat{r}}\left(\frac{-q}{4\pi\varepsilon_0 r^2}\right)$$

$$\Phi = \frac{q}{4\pi\varepsilon_0 r} \quad [\mathrm{V}]$$

$$\underline{E} = \underline{\hat{r}}\left(\frac{q}{4\pi\varepsilon_0 r^2}\right) \qquad [\mathrm{V/m}]$$

Line Charge Example



Find $\underline{E}$ from the line charge

$$\nabla\Phi = \underline{\hat{\rho}}\frac{\partial\Phi}{\partial\rho} + \underline{\hat{\phi}}\frac{1}{\rho}\frac{\partial\Phi}{\partial\phi} + \underline{\hat{z}}\frac{\partial\Phi}{\partial z}$$

$$\underline{E} = -\nabla\Phi = -\underline{\hat{\rho}}\frac{\partial}{\partial\rho}\left[\frac{\rho_{l0}}{2\pi\varepsilon_0}\ln\left(\frac{b}{\rho}\right)\right]$$

$$= -\underline{\hat{\rho}}\frac{\rho_{l0}}{2\pi\varepsilon_0}\frac{\partial}{\partial\rho}\left(\ln b - \ln\rho\right)$$

$$= \underline{\hat{\rho}}\frac{\rho_{l0}}{2\pi\varepsilon_0\rho}$$

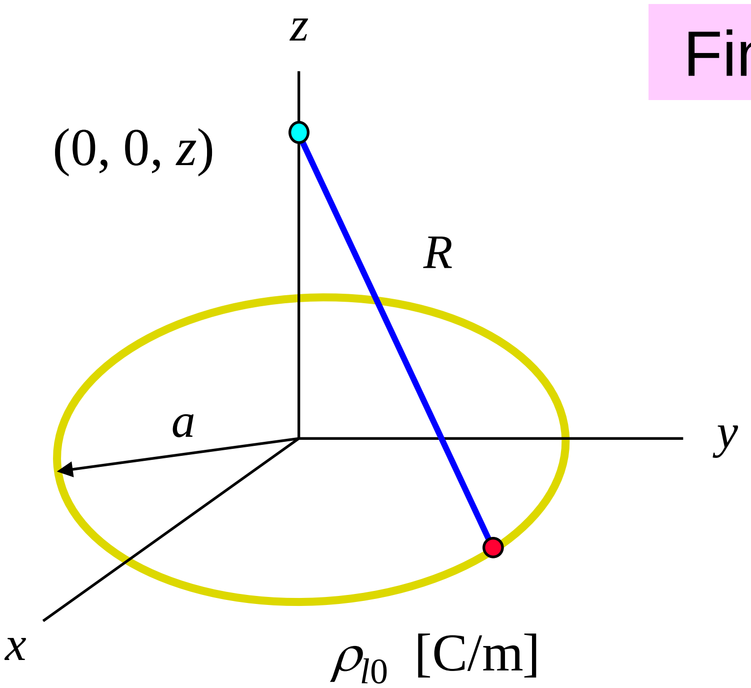
From previous calculation:

$$\Phi = \frac{\rho_{l0}}{2\pi\varepsilon_0}\ln\left(\frac{b}{\rho}\right)$$

Arbitrary reference point

$$\underline{E} = \underline{\hat{\rho}}\frac{\rho_{l0}}{2\pi\varepsilon_0\rho} \quad [\text{V/m}]$$

Example



Find: $\underline{E}\,(0, 0, z)$

On the z axis:

$$\underline{E}(0, 0, z) = \underline{\hat{z}}\, E_z(0, 0, z)$$

$$\underline{E} = -\nabla\Phi$$

$$\Rightarrow \quad E_z = -\frac{\partial \Phi}{\partial z}$$

From previous calculation:

$$\Phi(0, 0, z) = \frac{\rho_{\ell 0}\, a}{2\varepsilon_0 \sqrt{z^2 + a^2}}$$

Example (cont.)

$$E_z(0,0,z) = -\frac{d\Phi(0,0,z)}{dz} = -\frac{d}{dz}\left(\frac{\rho_{\ell 0} a}{2\varepsilon_0\sqrt{z^2+a^2}}\right)$$

so

$$E_z(0,0,z) = -\frac{\rho_{\ell 0} a}{2\varepsilon_0}\left(-\frac{1}{2}\right)\left(z^2+a^2\right)^{-3/2}(2z)$$

We thus have

$$E_z(0,0,z) = \frac{\rho_{\ell 0} a}{2\varepsilon_0}\left(\frac{z}{\left(z^2+a^2\right)^{3/2}}\right) \qquad [\mathrm{V/m}]$$

Vector Identity

$$\nabla \times \left(\nabla \psi \right) = \underline{0}$$

Proof:

$$\nabla \times \left(\nabla \psi \right) = \begin{vmatrix} \underline{\hat{x}} & \underline{\hat{y}} & \underline{\hat{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial \psi}{\partial x} & \dfrac{\partial \psi}{\partial y} & \dfrac{\partial \psi}{\partial z} \end{vmatrix}$$

$$= \underline{\hat{x}} \left(\frac{\partial^2 \psi}{\partial y \partial z} - \frac{\partial^2 \psi}{\partial z \partial y} \right) - \underline{\hat{y}} \left(\frac{\partial^2 \psi}{\partial x \partial z} - \frac{\partial^2 \psi}{\partial z \partial x} \right) + \underline{\hat{z}} \left(\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} \right)$$

$$= \underline{0}$$

Curl Property in Electrostatics (revisited)

$$\underline{E} = -\nabla\Phi$$ (in statics)

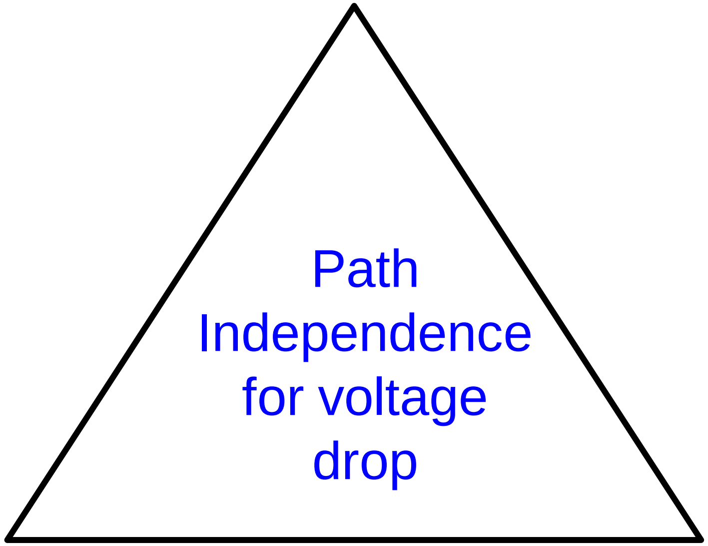
$$\nabla \times \underline{E} = \nabla \times \left(-\nabla\Phi\right)$$
$$= -\nabla \times \left(\nabla\Phi\right)$$
$$= \underline{0}$$

so

$$\nabla \times \underline{E} = \underline{0}$$

Equivalent Statements of Path Independence In Statics

$$\underline{E} = -\nabla\Phi$$



$$\nabla \times \underline{E} = \underline{0}$$

$$\oint_C \underline{E} \cdot d\underline{r} = 0$$

Poisson Equation

This is a differential equation that the potential satisfies.

(This is useful for solving “boundary value problems” that involve conductors or dielectrics.)

Start with the electric Gauss law: $\nabla \cdot \underline{D} = \rho_v$

$$\Rightarrow \quad \nabla \cdot \left(\varepsilon_0 \underline{E} \right) = \rho_v$$

$$\Rightarrow \quad \nabla \cdot \left(-\varepsilon_0 \nabla \Phi \right) = \rho_v$$

$$\Rightarrow \quad \nabla \cdot \left(\nabla \Phi \right) = -\frac{\rho_v}{\varepsilon_0}$$

Poisson Equation (cont.)

$$\nabla \cdot (\nabla \Phi) = \nabla \cdot \left[\underline{\hat{x}} \frac{\partial \Phi}{\partial x} + \underline{\hat{y}} \frac{\partial \Phi}{\partial y} + \underline{\hat{z}} \frac{\partial \Phi}{\partial z} \right]$$

$$= \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2}$$

Define the "Laplacian":

$$\operatorname{Lap} \Phi \equiv \nabla \cdot (\nabla \Phi) = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2}$$

Poisson's Eq.: $$\operatorname{Lap} \Phi = -\frac{\rho_v}{\varepsilon_0}$$

Poisson Equation (cont.)

Del-operator notation for Laplacian:

$$\nabla = \underline{\hat{x}}\frac{\partial}{\partial x} + \underline{\hat{y}}\frac{\partial}{\partial y} + \underline{\hat{z}}\frac{\partial}{\partial z}$$

$$\nabla \cdot \nabla = \nabla^2 = \left(\underline{\hat{x}}\frac{\partial}{\partial x} + \underline{\hat{y}}\frac{\partial}{\partial y} + \underline{\hat{z}}\frac{\partial}{\partial z} \right) \cdot \left(\underline{\hat{x}}\frac{\partial}{\partial x} + \underline{\hat{y}}\frac{\partial}{\partial y} + \underline{\hat{z}}\frac{\partial}{\partial z} \right) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right)$$

so

$$\nabla^2 \Phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \Phi$$

∇^2 = “Laplacian operator”

Hence

$$\text{Lap}\,\Phi \equiv \nabla \cdot \left(\nabla \Phi \right) = \nabla^2 \Phi$$

Poisson Equation (cont.)

Hence, we have

$$\nabla^2 \Phi = -\frac{\rho_v}{\varepsilon_0}$$

"Poisson Equation"

If

$$\rho_v = 0$$

then

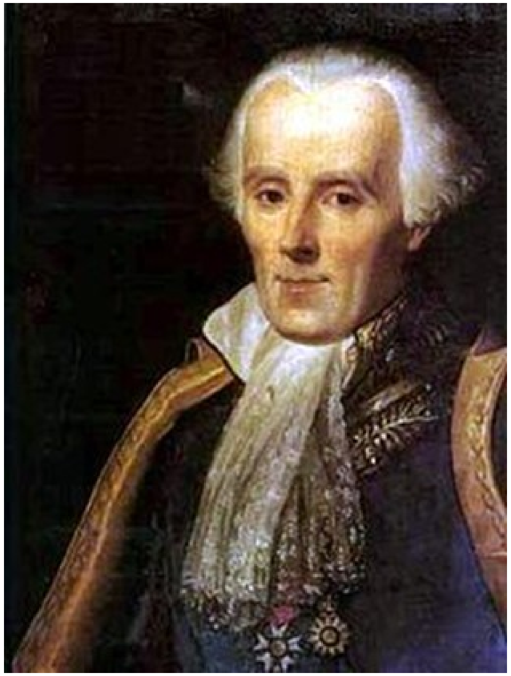
$$\nabla^2 \Phi = 0$$

"Laplace Equation"

Note: $\Phi = \Phi\left(x, y, z\right), \quad \rho_v = \rho_v\left(x, y, z\right)$

Poisson Equation (cont.)

Siméon Denis Poisson
1781 – 1840



Pierre-Simon Laplace
1749 – 1827

(from Wikipedia)

Laplacian

See Appendix A.2 of the Hayt & Buck book for a derivation that holds in any coordinate system.

Rectangular

$$\nabla^2 \Phi = \left(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} \right)$$

Cylindrical

$$\nabla^2 \Phi = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \Phi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \Phi}{\partial \phi^2} + \frac{\partial^2 \Phi}{\partial z^2}$$

Spherical

$$\nabla^2 \Phi = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \Phi}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial \Phi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 \Phi}{\partial \phi^2}$$

Summary of Formulas:
Electrostatic Triangle

One nice way to summarize all of the equations of electrostatics into one nice visual display is the "electrostatic triangle"

(courtesy of Prof. Donald R. Wilton).

Electrostatic Triangle