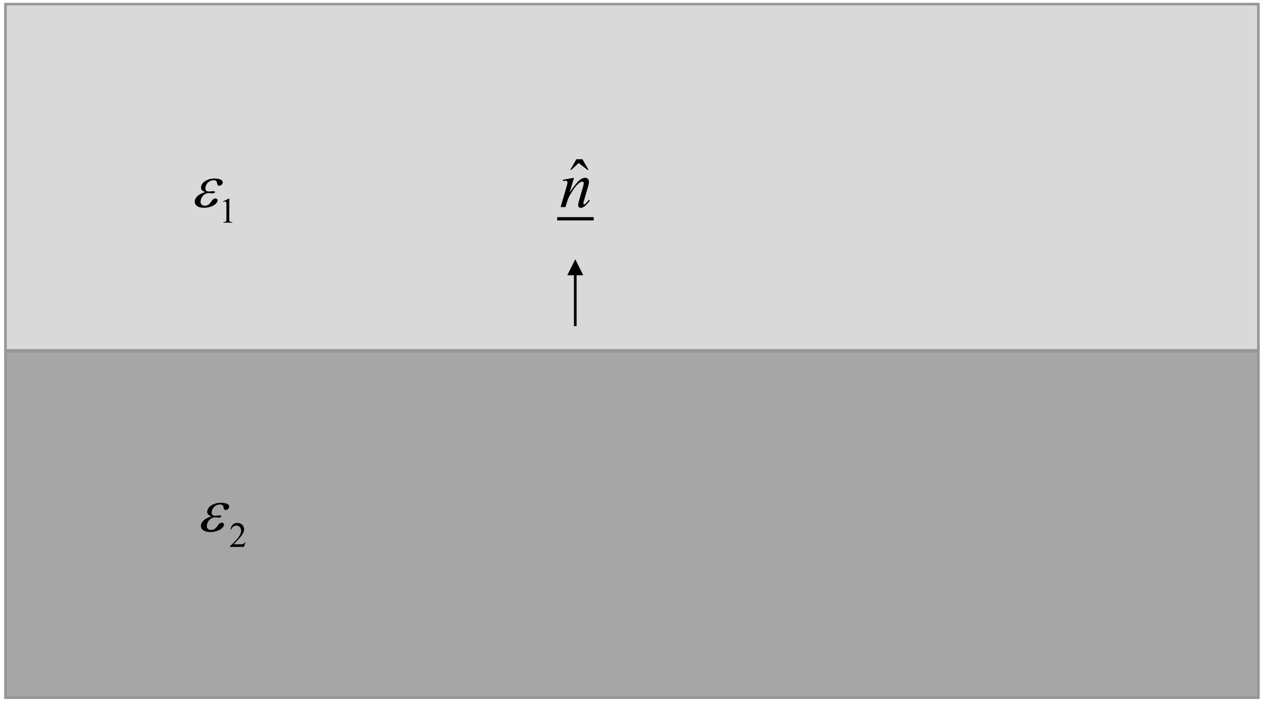
ECE 3318
Applied Electricity and Magnetism

Spring 2023

Prof. David R. Jackson
Dept. of ECE

Notes 21
Boundary Conditions

Boundary Conditions (cont.)

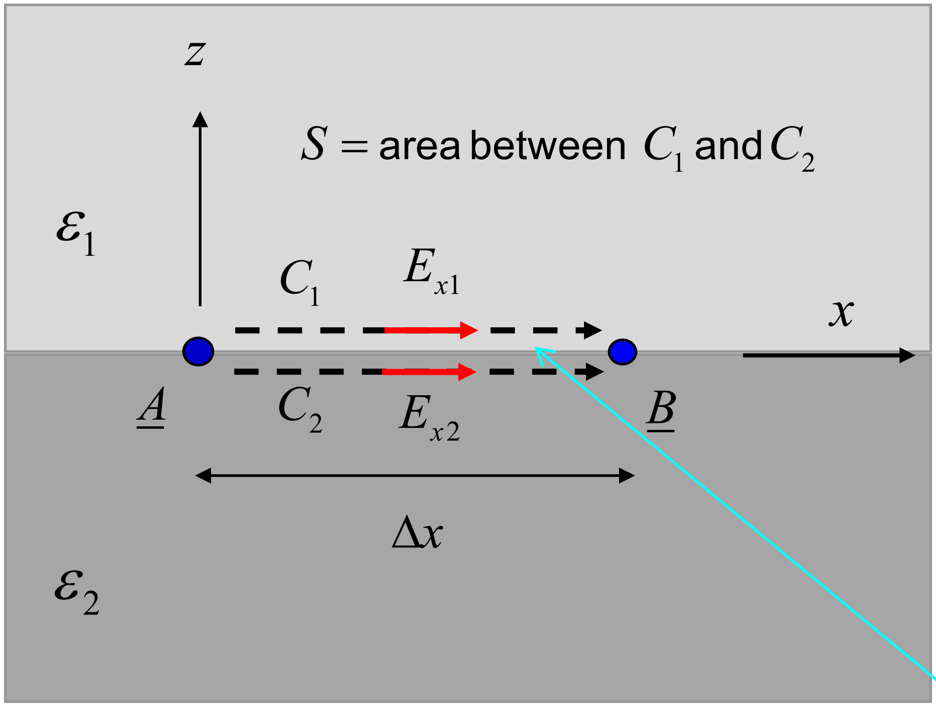


There is a boundary between two different materials.

Assumption: *The unit normal points towards region* 1.

Boundary Conditions for Dielectrics

Tangential component of electric field



Consider the field E_x:

Establish paths C_1 and C_2 (infinitesimally close, above and below the boundary).

$$C \equiv C_1 - C_2$$

(clockwise closed path)

Faraday's law:

$$\oint_C \underline{E} \cdot d\underline{r} = \int_S -\frac{\partial \underline{B}}{\partial t} \cdot \underline{\hat{n}}\, dS$$

$$\underline{\hat{n}} = \underline{\hat{y}}$$

The rectangular surface S (between the two paths) has <u>zero</u> area.

$$\Rightarrow \quad \int_{C_1} \underline{E} \cdot d\underline{r} = \int_{C_2} \underline{E} \cdot d\underline{r}$$

Boundary Conditions for Dielectrics

$$\text{Path } C_1 : \ z = 0^+ : \quad V_{AB} = \int_0^{\Delta x} E_x dx \approx E_{x1}\Delta x$$

$$\text{Path } C_2 : z = 0^- : \quad V_{AB} = \int_0^{\Delta x} E_x dx \approx E_{x2}\Delta x$$

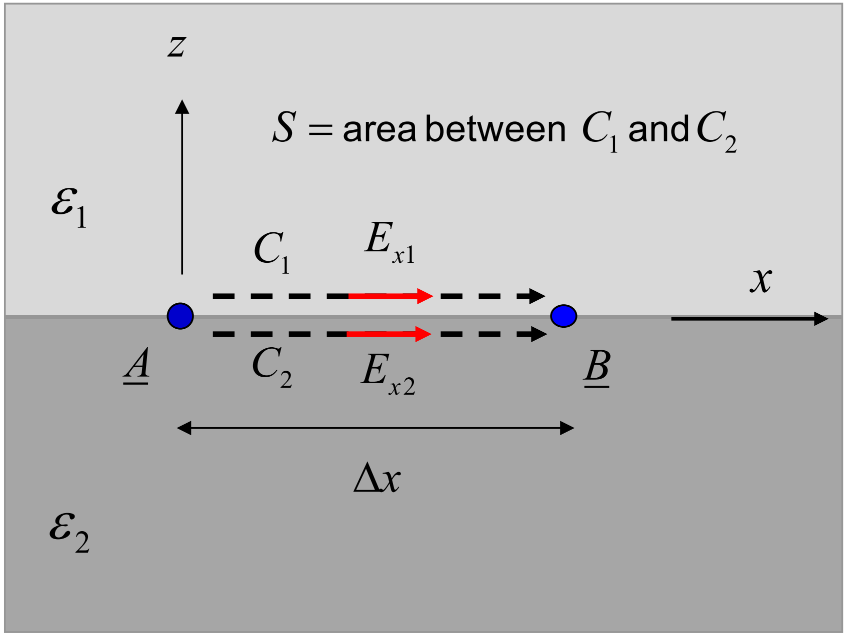
$$\Rightarrow E_{x1} = E_{x2}$$

Similarly,

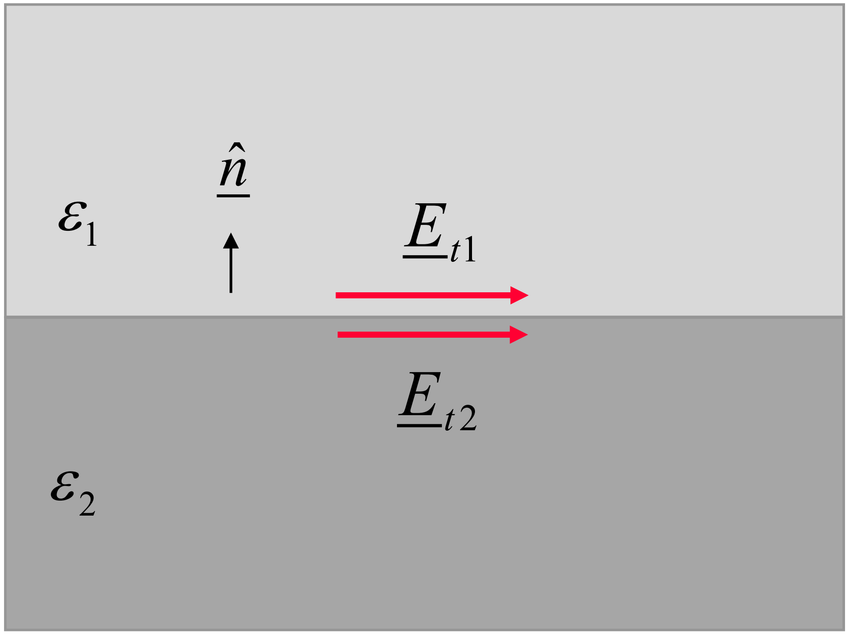
$$E_{y1} = E_{y2}$$

In general,

$$\underline{E}_{t1} = \underline{E}_{t2}$$



Boundary Conditions (cont.)



$$\underline{E}_{t1} = \underline{E}_{t2}$$

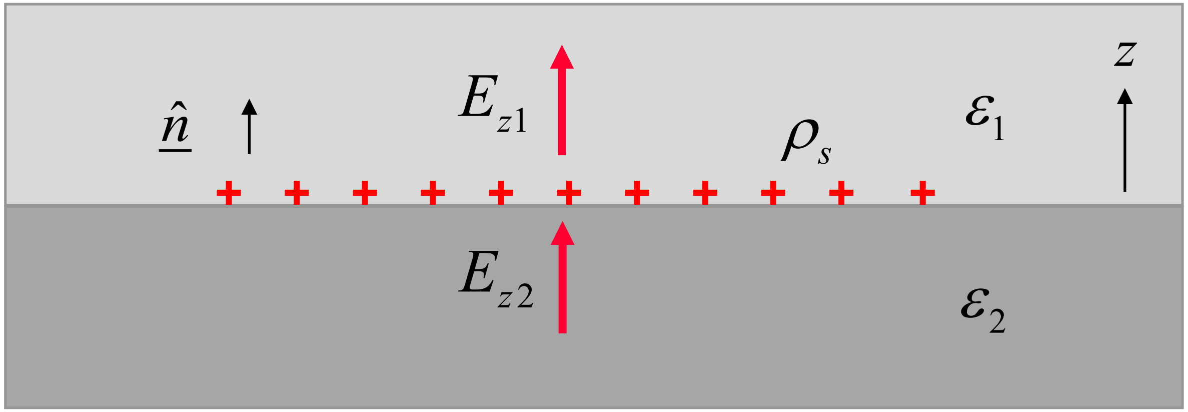
Another form of the boundary condition:

$$\underline{\hat{n}} \times \underline{E}_1 = \underline{\hat{n}} \times \underline{E}_2$$

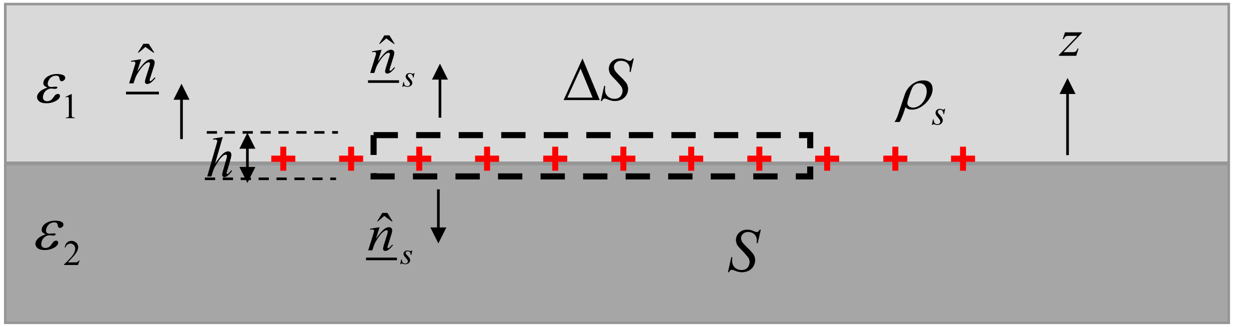
The cross product removes the normal component of the electric field and rotates the tangential part by 90^{o}.

Boundary Conditions (cont.)

Normal component of electric field



Gauss's Law applied to a "pillbox" surface (height $h \to 0$):



$$\oint_S \underline{D} \cdot \underline{\hat{n}}_s \, ds = Q_{encl} \quad \Rightarrow \quad D_{z1}\Delta S - D_{z2}\Delta S \approx \rho_s \Delta S$$

Boundary Conditions (cont.)

$$D_{z1}\Delta S - D_{z2}\Delta S \approx \rho_s \Delta S$$

Let $\Delta S \to 0$: $\quad D_{z1} - D_{z2} = \rho_s$

In general, $D_{n1} - D_{n2} = \rho_s$

The direction $\underline{\hat{z}}$ is now denoted as $\underline{\hat{n}}$.

This direction points towards region 1.

Hence, we have

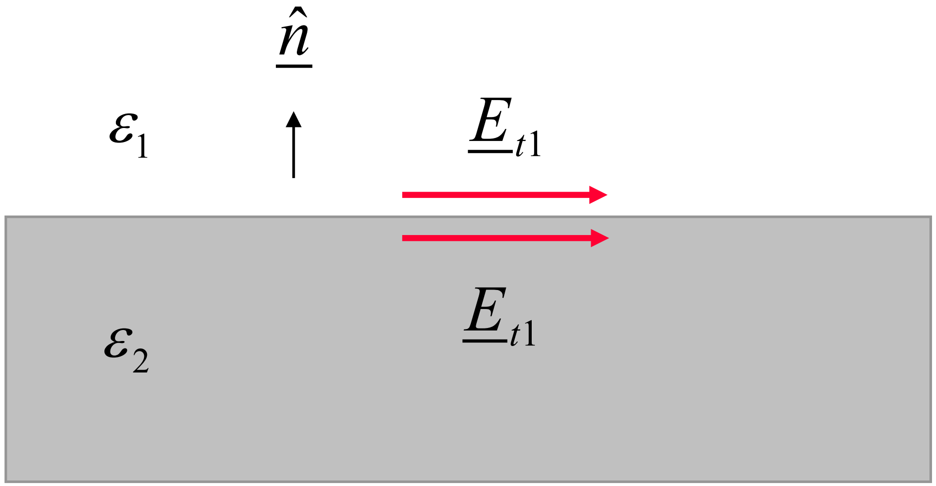
$$\underline{\hat{n}} \cdot \left(\underline{D}_1 - \underline{D}_2 \right) = \rho_s$$

$\underline{\hat{n}}$ points toward region 1

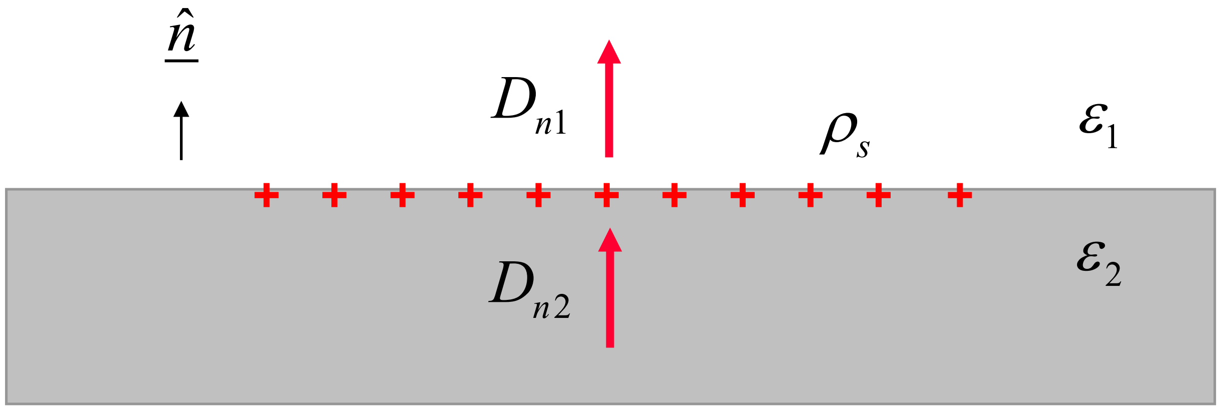
Boundary Conditions: Summary

Summary of Boundary Conditions

$\underline{\hat{n}}$ points towards regon 1



$$\underline{E}_{t1} = \underline{E}_{t2}$$

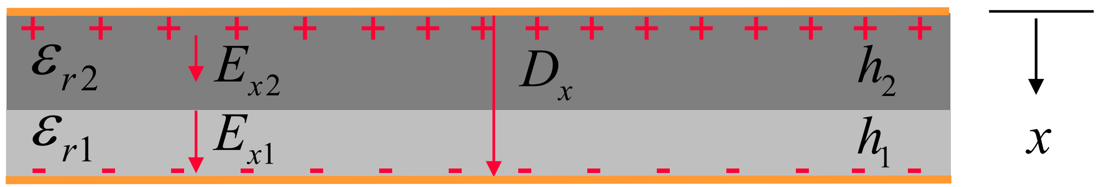


$$\underline{\hat{n}} \cdot \left(\underline{D}_1 - \underline{D}_2 \right) = \rho_s$$

Example

Examples of capacitors with different layers

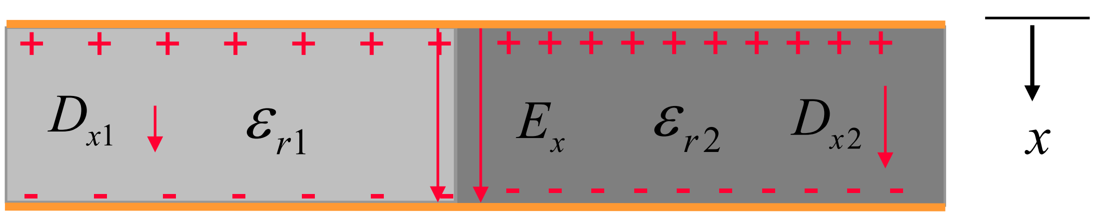
Assume : $\varepsilon_{r2} > \varepsilon_{r1}$



$$D_{x1} = D_{x2}$$

$$\varepsilon_1 E_{x1} = \varepsilon_2 E_{x2}$$

$$E_{x2} < E_{x1}$$

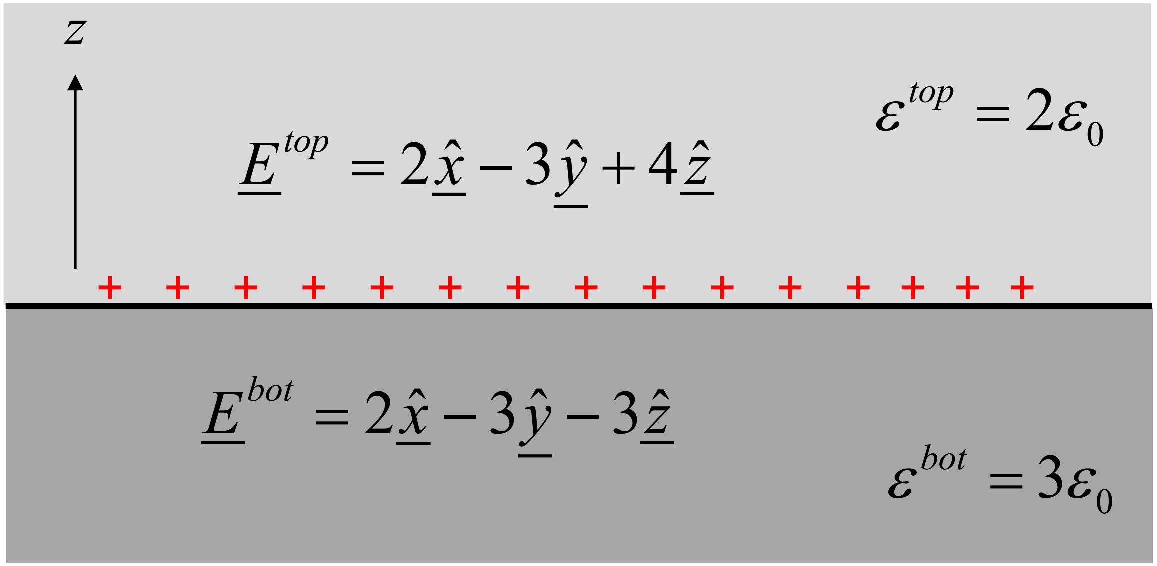


$$E_{x1} = E_{x2}$$

$$\frac{D_{x1}}{\varepsilon_1} = \frac{D_{x2}}{\varepsilon_2}$$

$$D_{x2} > D_{x1}$$

Example



Note that the tangential electric field is continuous (as it must be).

Find ρ_s

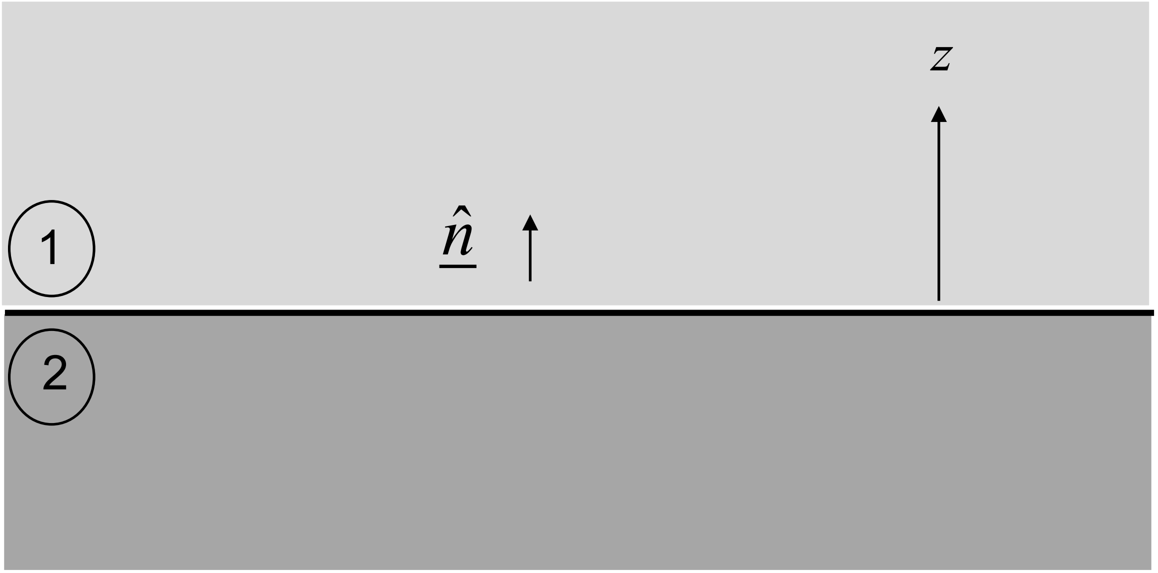
Note:
In this example the electric field vectors are assumed to be constant (uniform) in each region.

Example

Choose $\underline{\hat{n}} = \underline{\hat{z}}$

(This establishes which region is region 1.)

Region 1 is on top.
Region 2 is on bottom.



Example (cont.)

$$\left(\underline{D}_1 - \underline{D}_2\right) \cdot \underline{\hat{n}} = \rho_s$$

$$\Rightarrow \quad \left(\underline{D}^{top} - \underline{D}^{bot}\right) \cdot \underline{\hat{z}} = \rho_s$$

Hence

$$\rho_s = \varepsilon_0 \varepsilon_r^{top} E_z^{top} - \varepsilon_0 \varepsilon_r^{bot} E_z^{bot}$$
$$= \varepsilon_0 (2)(4) - \varepsilon_0 (3)(-3)$$

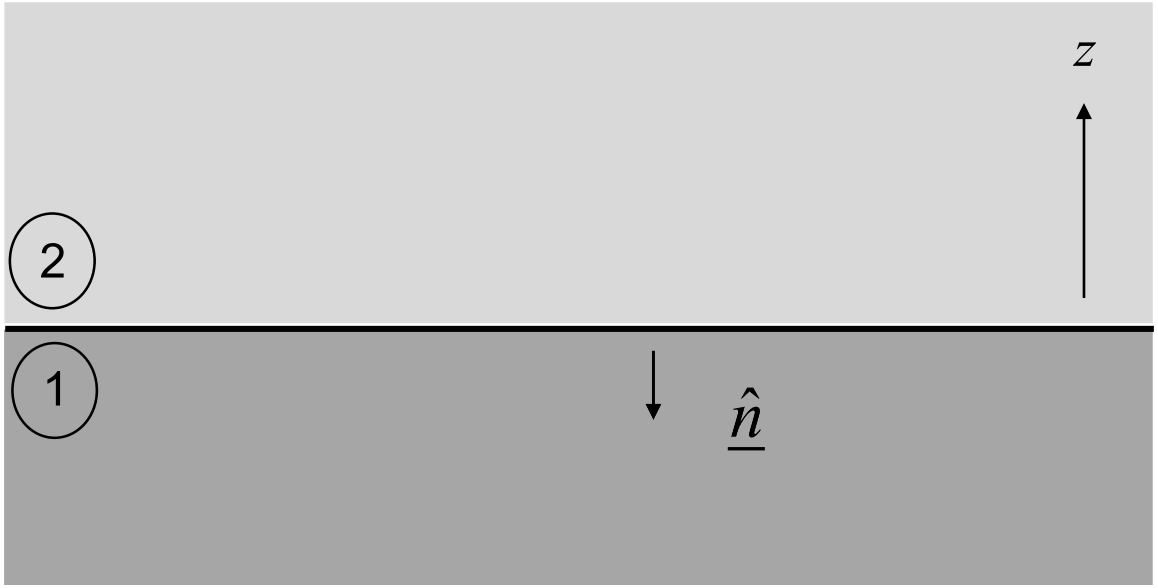
so

$$\rho_s = 17\varepsilon_0 \quad \left[\mathrm{C/m^2}\right]$$

Example (cont.)

Alternative: Choose $\underline{\hat{n}} = -\underline{\hat{z}}$

Region 1 is on bottom.
Region 2 is on top.



$$\left(\underline{D}_1 - \underline{D}_2\right) \cdot \underline{\hat{n}} = \rho_s$$

$$\Rightarrow \left(\underline{D}^{bot} - \underline{D}^{top}\right) \cdot \left(-\underline{\hat{z}}\right) = \rho_s$$

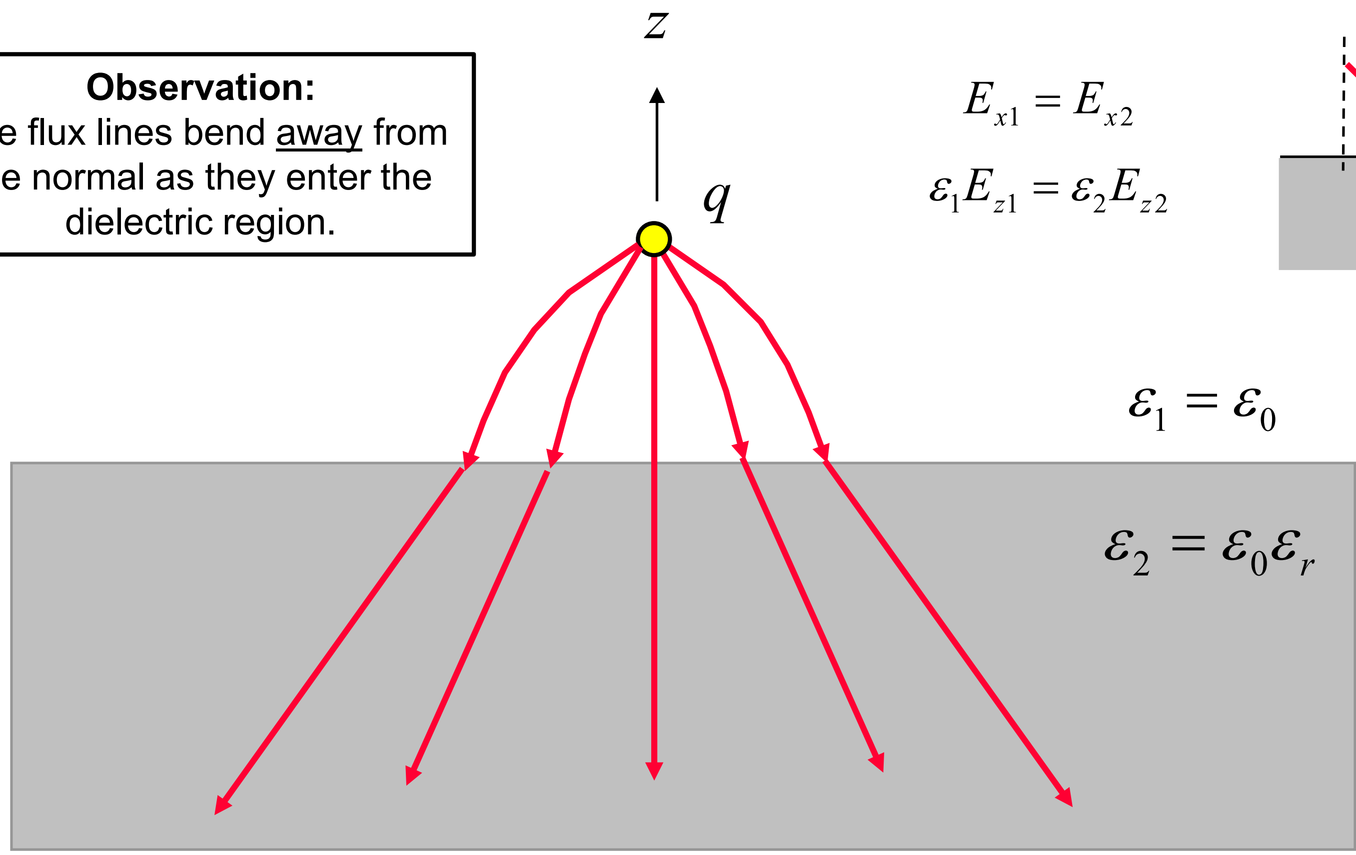
(The same result is obtained.)

Example

Draw flux lines going into a dielectric (no surface charge density)

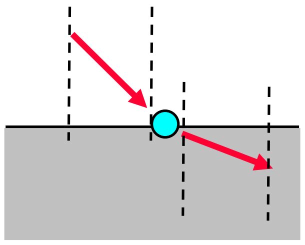
Note: The electric field below the interface has the same horizontal component, but a smaller vertical component.

Observation:
The flux lines bend away from the normal as they enter the dielectric region.



$$E_{x1} = E_{x2}$$

$$\varepsilon_1 E_{z1} = \varepsilon_2 E_{z2}$$

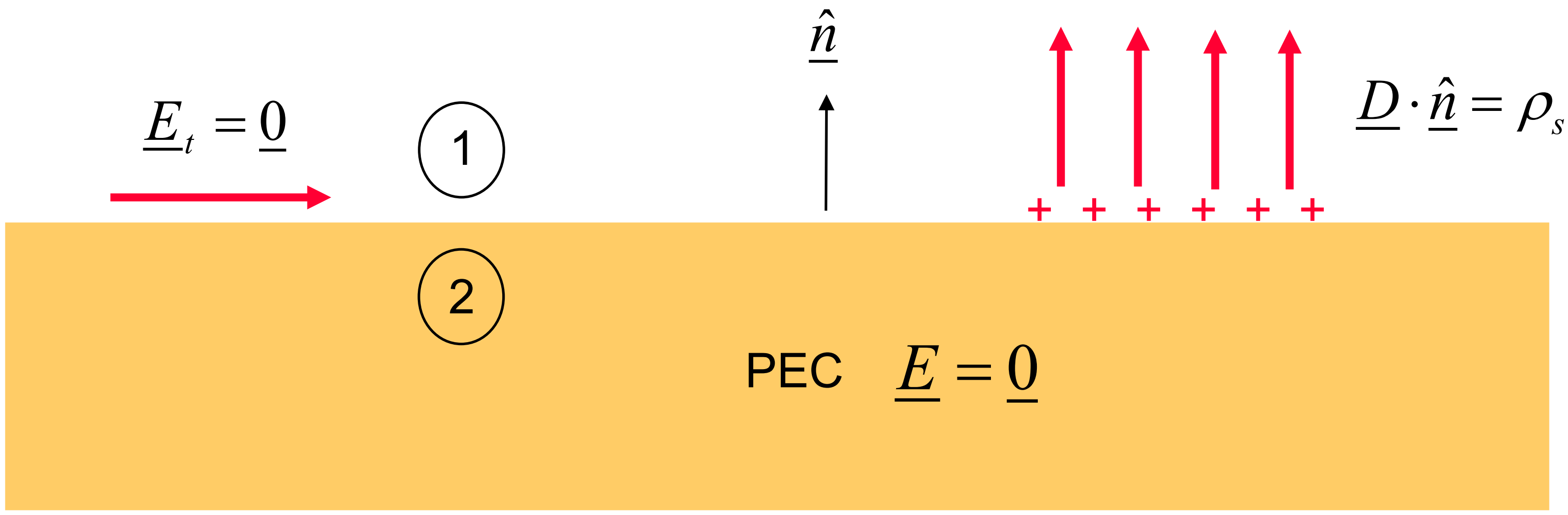


$$\varepsilon_2 > \varepsilon_1$$

$$E_{z2} < E_{z1}$$

Boundary Conditions for PEC

Region 2 is PEC



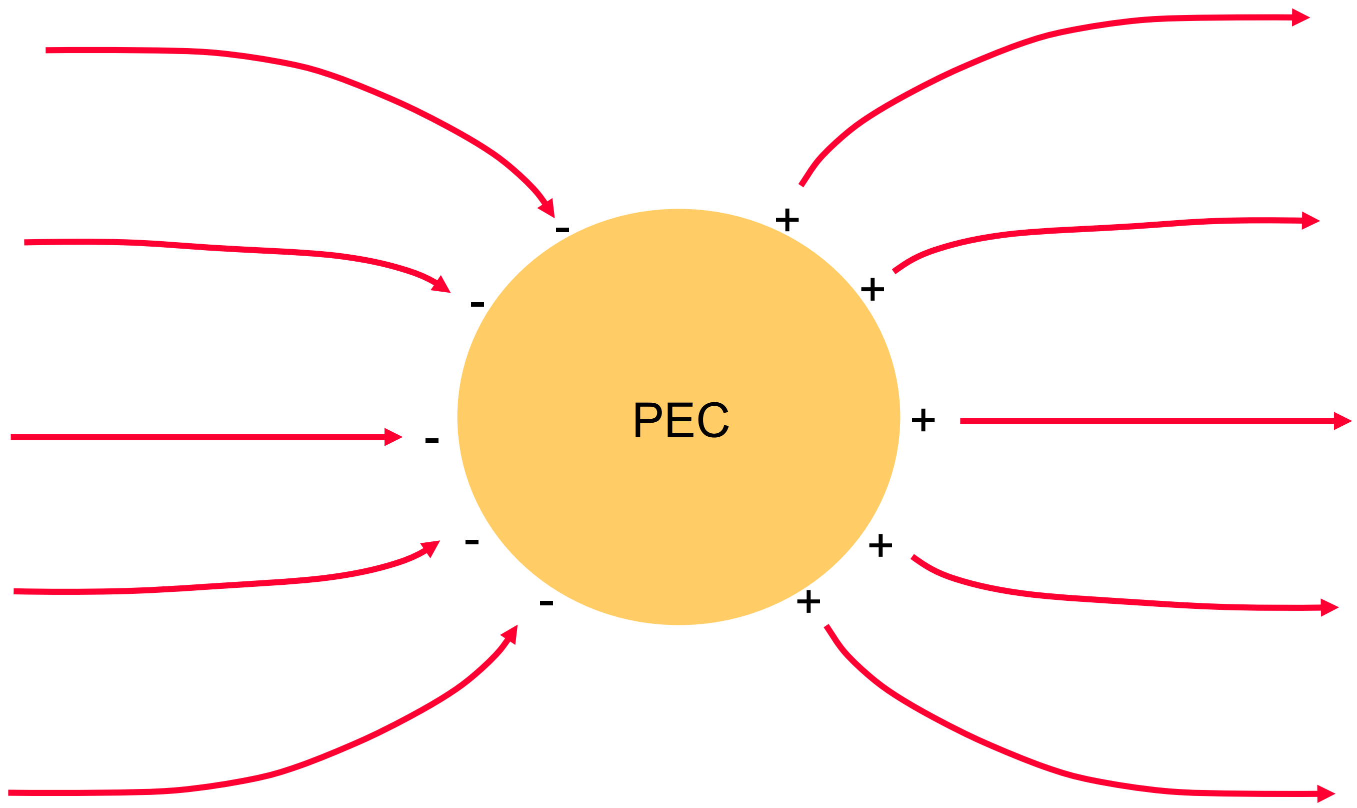
This is a special case of the previous boundary conditions, for which $\underline{E}_2 = 0$.

$$\underline{E}_t = \underline{0}$$

$$\underline{D} \cdot \underline{\hat{n}} = \rho_s$$

(The normal points outward from the PEC.)

Boundary Conditions for PEC (cont.)



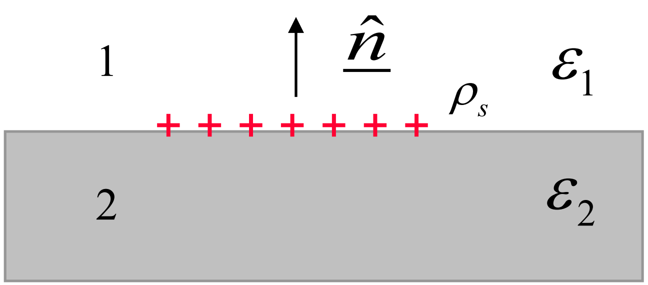
Electric lines of flux must start/end on a PEC perpendicular to the boundary.

Summary of Boundary Conditions

Dielectric boundary

$$\underline{E}_{t1} = \underline{E}_{t2}$$

$$\underline{\hat{n}} \cdot \left(\underline{D}_1 - \underline{D}_2 \right) = \rho_s$$

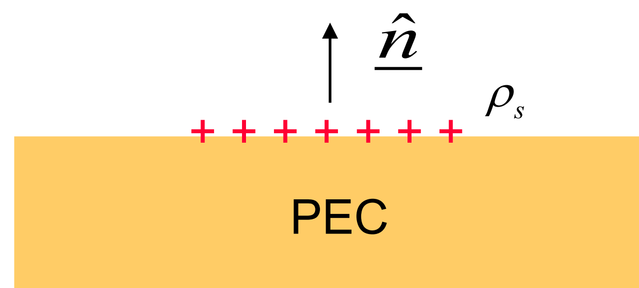


The normal points towards region 1.

PEC boundary

$$\underline{E}_t = \underline{0}$$

$$\underline{\hat{n}} \cdot \underline{D} = \rho_s$$



The normal points outward from the PEC.

Summary of Electrostatic Laws

	Point Form	Integral Form	Boundary Form*
Faraday's Law	$\nabla \times \underline{E} = \underline{0}$	$\oint_C \underline{E} \cdot d\underline{r} = 0$	$\underline{E}_{t1} - \underline{E}_{t2} = \underline{0}$
Gauss's Law	$\nabla \cdot \underline{D} = \rho_v$	$\oint_S \underline{D} \cdot \underline{\hat{n}}\, dS = Q_{encl}$	$\underline{\hat{n}} \cdot \left(\underline{D}_1 - \underline{D}_2 \right) = \rho_s$

Constitutive Equation: $\underline{D} = \varepsilon \underline{E} = \varepsilon_0 \varepsilon_r \underline{E}$

* The boundary form (i.e., boundary conditions) is actually valid for arbitrary time-varying fields, not only for statics.

Boundary Condition for Potential

At the boundary:

$$\Phi_1 = \Phi_2$$

Proof:

$$V_{AB} = \int_{\underline{A}}^{\underline{B}} \underline{E} \cdot d\underline{r} = 0$$

(The path has zero length.)

Two points on either side of the boundary