ECE 3318
Applied Electricity and Magnetism

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Notes 23
Boundary Value Problems

Boundary Value Problem

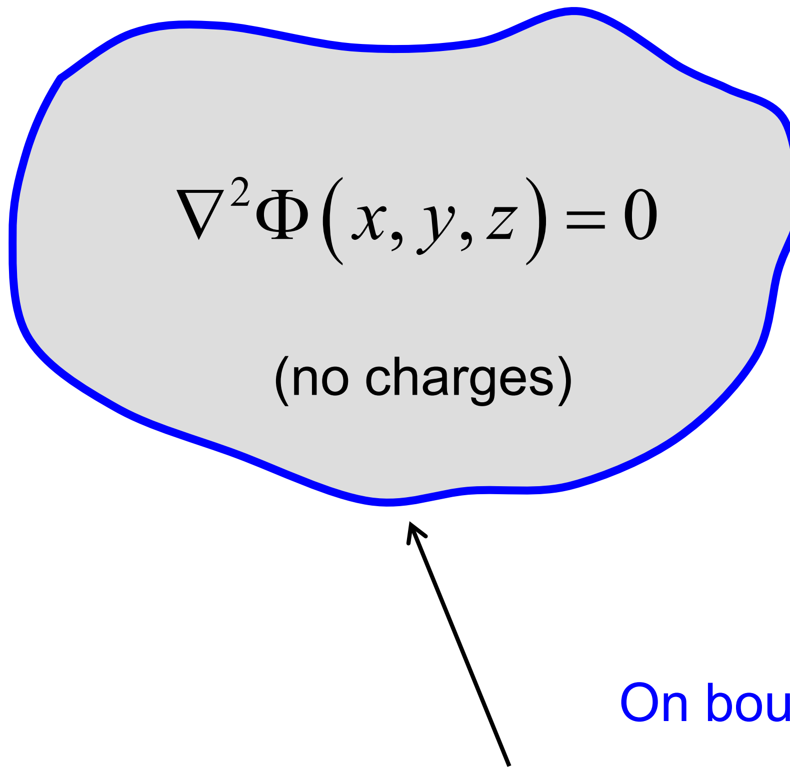
Goal:
Solve for the potential function inside of a hollow region (no charges inside), given the value of the potential function on the boundary.

Uniqueness theorem:

$\Phi(x, y, z)$ is unique.

⇩

As long as our solution satisfies the Laplace equation and the B.C.s, it must be correct!



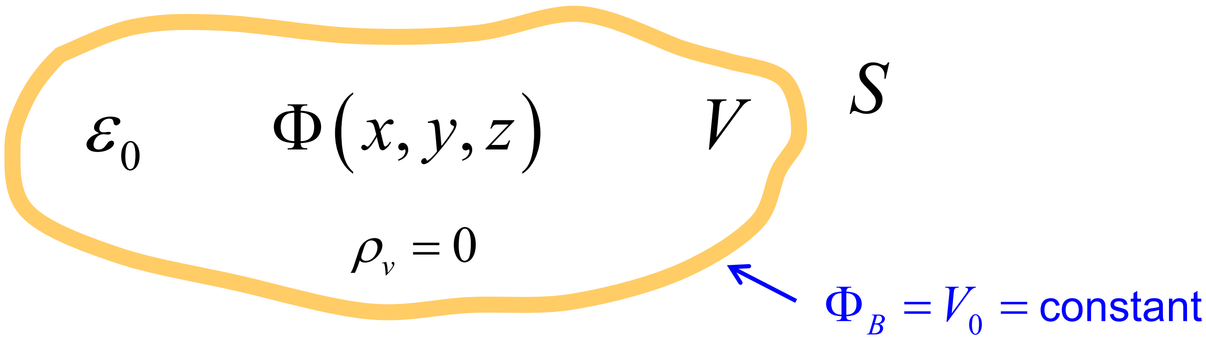
On boundary:

$$\Phi = \Phi_B(x, y, z)$$

(Please see the textbooks for a proof of the uniqueness theorem.)

Example: Faraday Cage Effect

Hollow PEC shell



Prove that $\underline{E} = \underline{0}$ inside a hollow PEC shell (Faraday cage effect).

Guess: $\Phi(x, y, z) = V_0 \qquad (\underline{r} \in V)$

Check: $\nabla^2 \Phi = \nabla^2 (V_0) = 0 \quad \text{in } V$ ✓

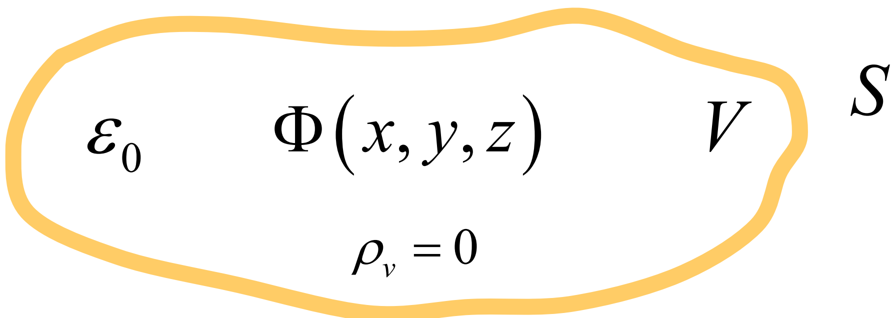
$\Phi = V_0 \text{ on } S$ ✓

Note:
We can make any guess that we wish, as long as our final solution satisfies Laplace's equation and the boundary conditions.

Conclusion: The correct solution is $\Phi(x, y, z) = V_0$

Example (cont.)

Hollow PEC shell



$$\Phi(x, y, z) = V_0$$

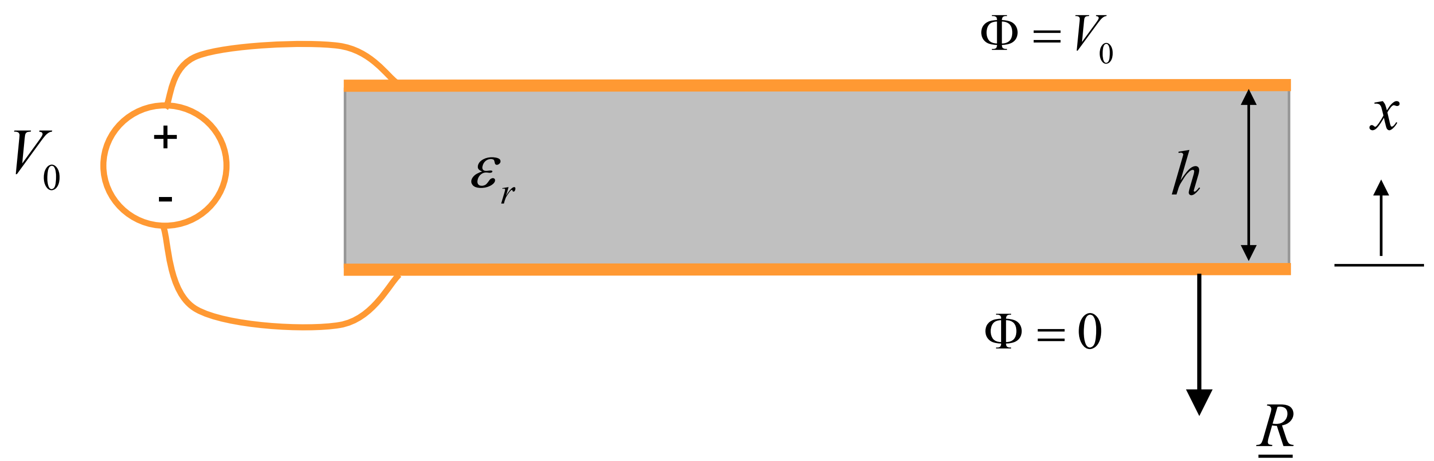
$$\Rightarrow \quad \underline{E} = -\nabla\Phi(x, y, z) = -\nabla V_0 = \underline{0}$$

Hence $\underline{E} = \underline{0}$ everywhere inside the hollow cavity.

(Faraday cage effect)

Example

Ideal parallel-plate capacitor



Solve for $\Phi\,(x, y, z)$

Choose the reference point to be on the bottom plate, and assign zero volts there.

Assume: $\Phi(x, y, z) = \Phi(x)$

Note:
We can make any assumptions that we wish,
as long as our final solution satisfies the boundary conditions.

Example (cont.)

$$\nabla^2 \Phi = 0$$

$$\frac{\partial^2 \Phi}{\partial x^2} + \cancel{\frac{\partial^2 \Phi}{\partial y^2}} + \cancel{\frac{\partial^2 \Phi}{\partial z^2}} = 0$$

Hence $$\frac{\partial^2 \Phi(x)}{\partial x^2} = 0$$

Solution: $$\Phi'(x) = C_1$$

$$\Phi(x) = C_1 x + C_2$$

Example (cont.)

$x = 0:$ $\quad \Phi(0) = 0 \;\Rightarrow\; C_1(0) + C_2 = 0 \;\Rightarrow\; C_2 = 0$

$x = h:$ $\quad \Phi(h) = V_0 \;\Rightarrow\; C_1 h + \cancel{C_2} = V_0 \;\Rightarrow\; C_1 = V_0 / h$

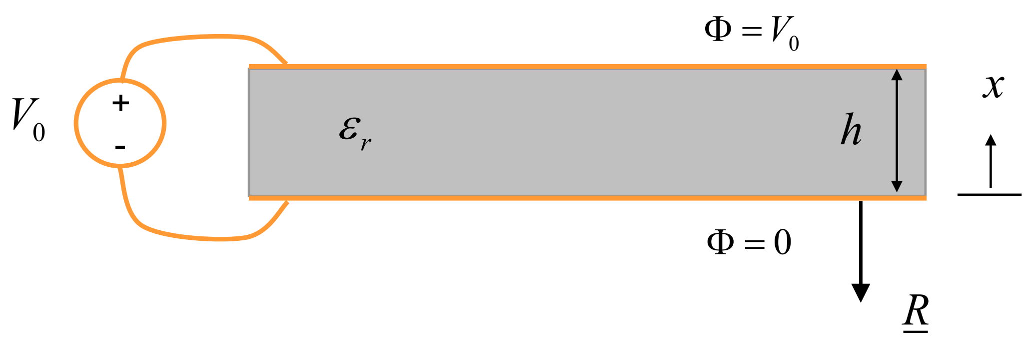
Hence we have

$$\begin{cases} C_1 = \dfrac{V_0}{h} \\ C_2 = 0 \end{cases}$$

$$\Phi(x) = C_1 x + C_2$$

The solution is then

$$\Phi(x, y, z) = \left(\frac{V_0}{h}\right) x \quad [\mathrm{V}]$$



Example (cont.)

$$\Phi(x, y, z) = \left(\frac{V_0}{h}\right)x$$

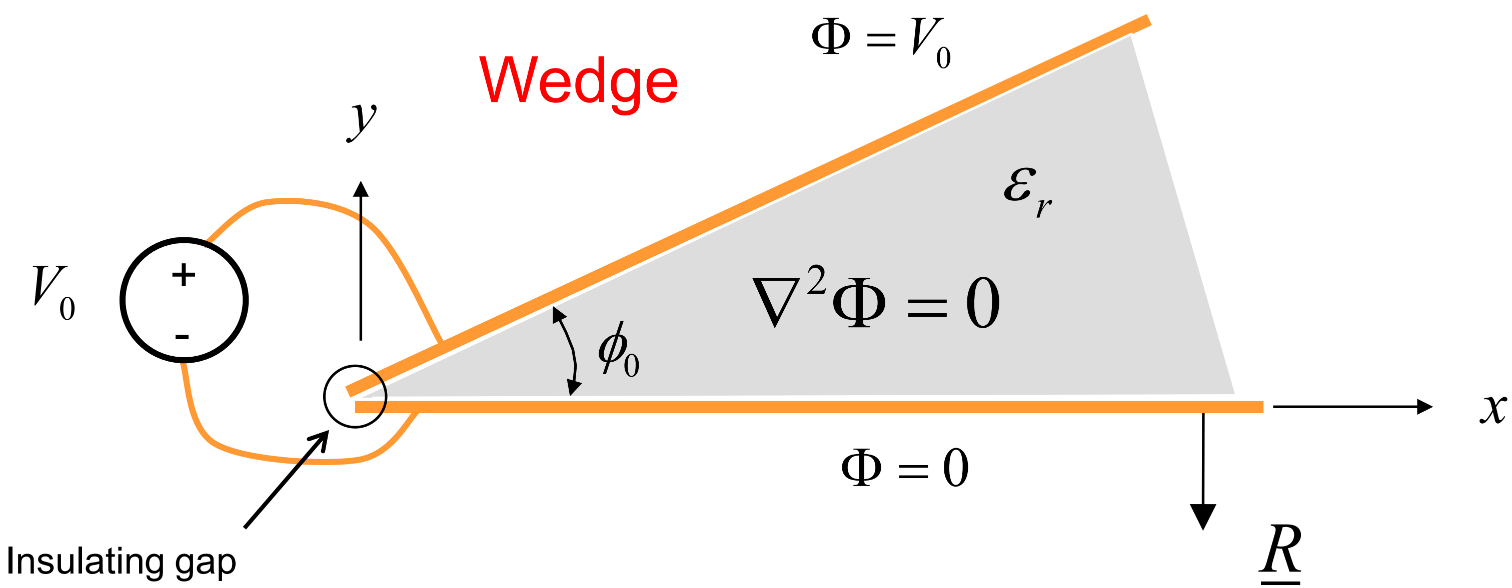
Calculate the electric field:

$$\underline{E} = -\nabla\Phi = -\underline{\hat{x}}\frac{\partial\Phi(x)}{\partial x} = -\underline{\hat{x}}\left(\frac{V_0}{h}\right)$$

Hence we have

$$\underline{E} = -\underline{\hat{x}}\left(\frac{V_0}{h}\right) \quad [\mathrm{V/m}]$$

Example



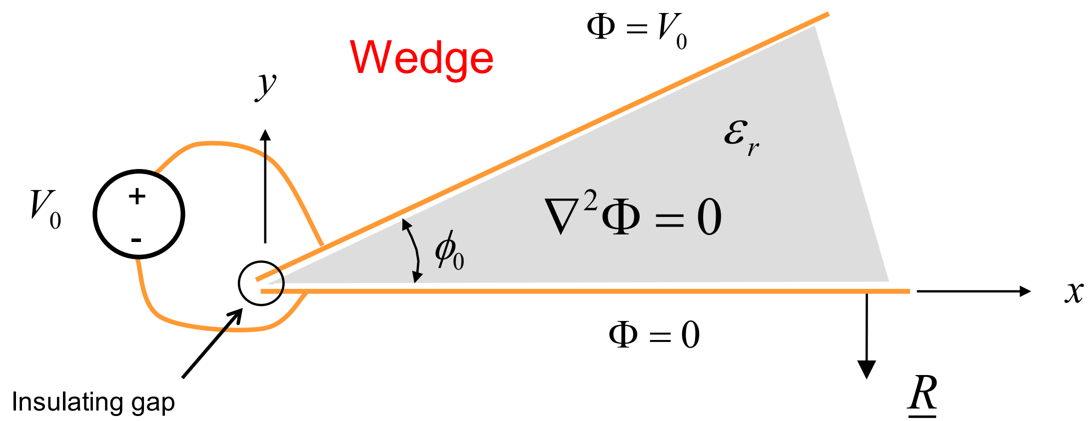
Choose the reference point to be on the bottom plate, and assign zero volts there.

Assume $\Phi(\rho, \phi, z) = \Phi(\phi)$

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \Phi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \Phi}{\partial \phi^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

Example (cont.)

$$\frac{\partial^2 \Phi}{\partial \phi^2} = 0$$



Hence

$$\Phi = C_1 \phi + C_2$$

$\phi = 0:$

$$C_1(0) + C_2 = 0$$

$$C_2 = 0$$

$\phi = \phi_0:$

$$C_1 \phi_0 + C_2 = V_0$$

$$C_1 = \frac{V_0}{\phi_0}$$

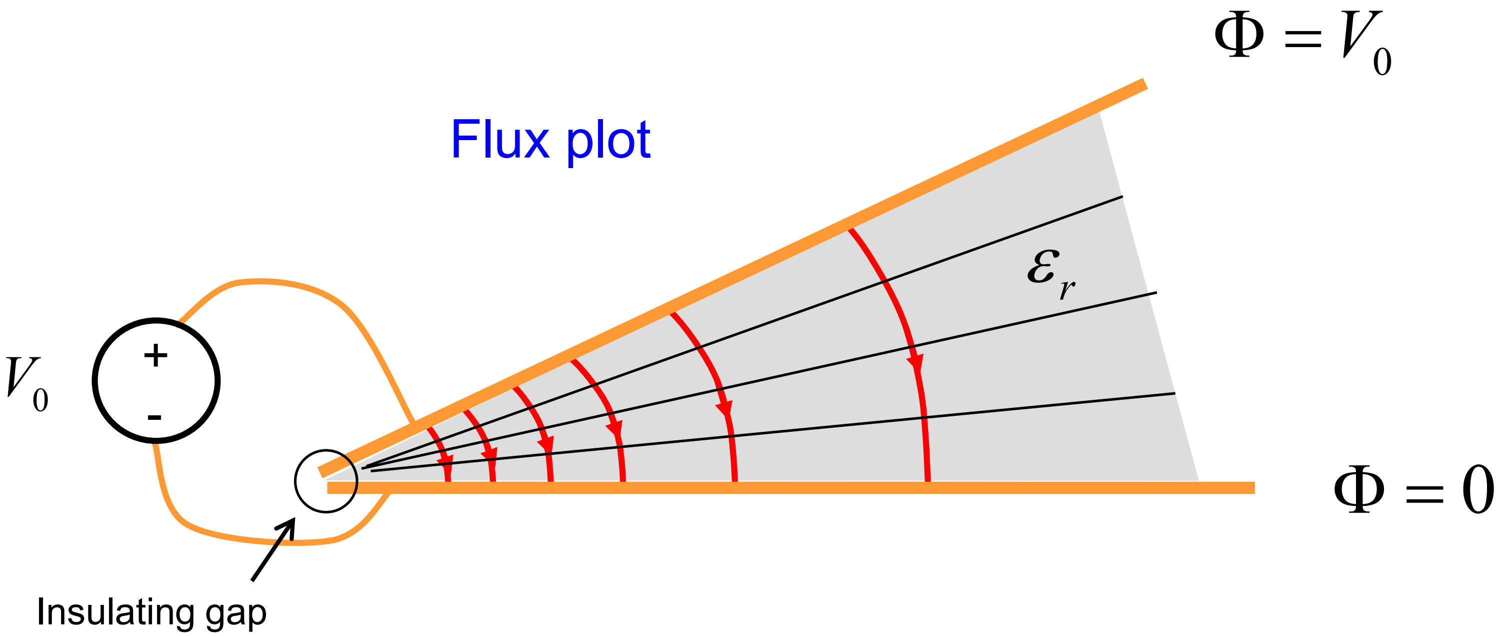
Example (cont.)

Hence $$\Phi(\rho, \phi, z) = \left(\frac{V_0}{\phi_0}\right)\phi \quad [\mathrm{V}]$$

Find the electric field: $$\underline{E} = -\nabla\Phi$$
$$= -\left[\hat{\underline{\rho}}\frac{\partial\Phi}{\partial\rho} + \hat{\underline{\phi}}\frac{1}{\rho}\frac{\partial\Phi}{\partial\phi} + \hat{\underline{z}}\frac{\partial\Phi}{\partial z}\right]$$
$$= -\hat{\underline{\phi}}\frac{1}{\rho}\frac{\partial\Phi}{\partial\phi}$$

We then have $$\underline{E} = -\hat{\underline{\phi}}\left(\frac{1}{\rho}\right)\left(\frac{V_0}{\phi_0}\right) \quad [\mathrm{V/m}]$$

Example (cont.)



$$\underline{E} = -\underline{\hat{\phi}}\left(\frac{1}{\rho}\right)\left(\frac{V_0}{\phi_0}\right) \quad [\text{V/m}]$$