ECE 3318
Applied Electricity and Magnetism

Spring 2023

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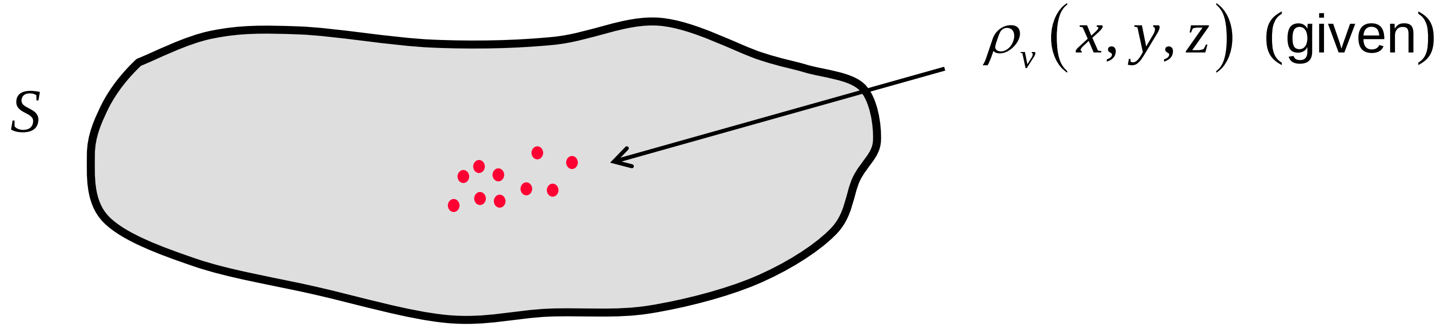
Dept. of ECE

Notes 24
Image Theory

Uniqueness Theorem



$\Phi = \Phi_B$ on boundary (given)

Given:

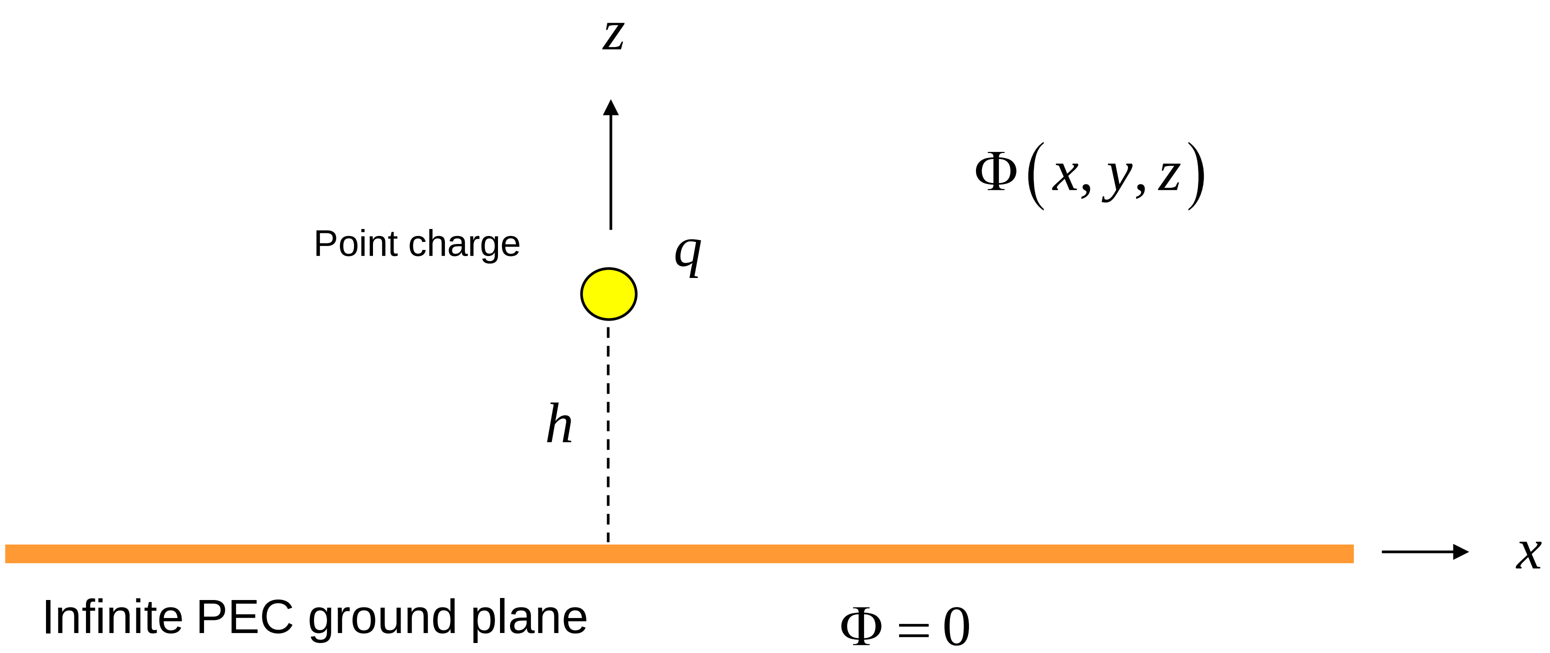
$$\nabla^2 \Phi = -\frac{\rho_v}{\varepsilon}$$ Inside region (known charge density)

$$\Phi = \Phi_B$$ on boundary (known B.C.)

⇨ $\Phi(x, y, z)$ is unique

Note: We can guess the solution, as long as we verify that the Poisson equation and the B.C.'s are correctly satisfied! (in other words, we have the correct boundary condition and the correct charge inside).

Image Theory



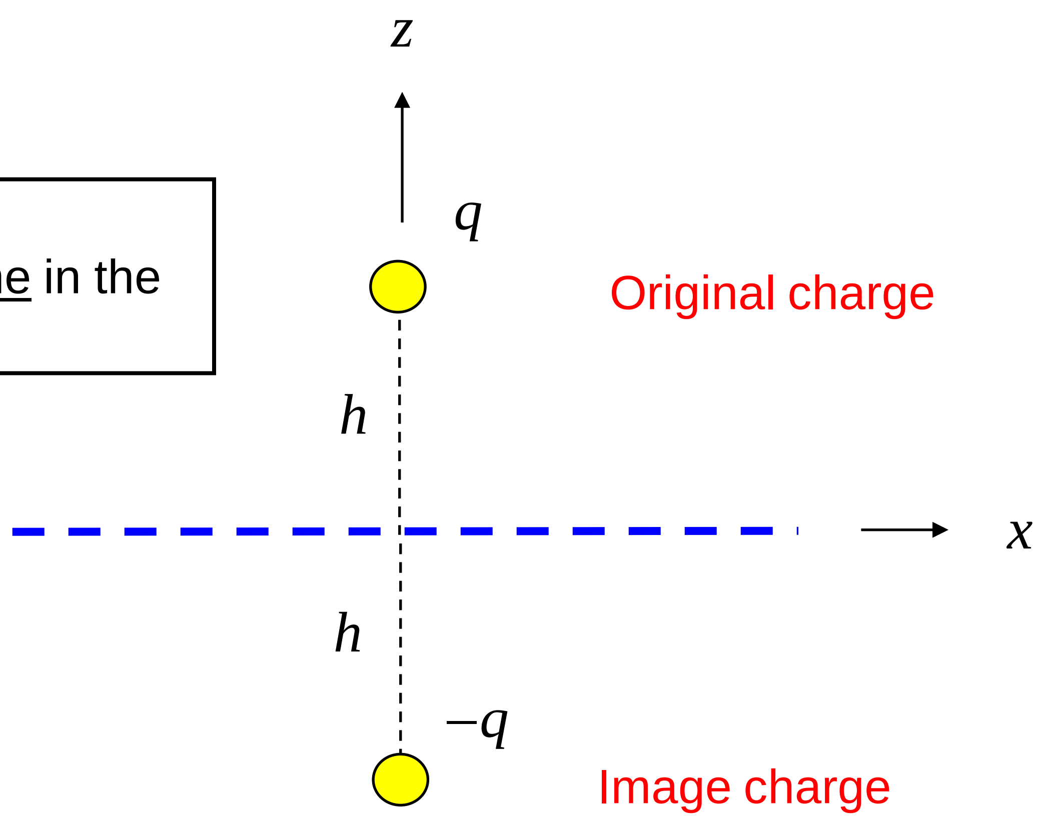
Note: The electric field is zero below the ground plane ($z < 0$).

This can be justified by the uniqueness theorem, just as we did for the Faraday cage discussion. (Make a closed surface by adding a large hemisphere in the lower region, and use the fact that the potential is zero on the hemisphere.)

Image Theory (cont.)

Image picture:

Note:
There is no ground plane in the image picture!

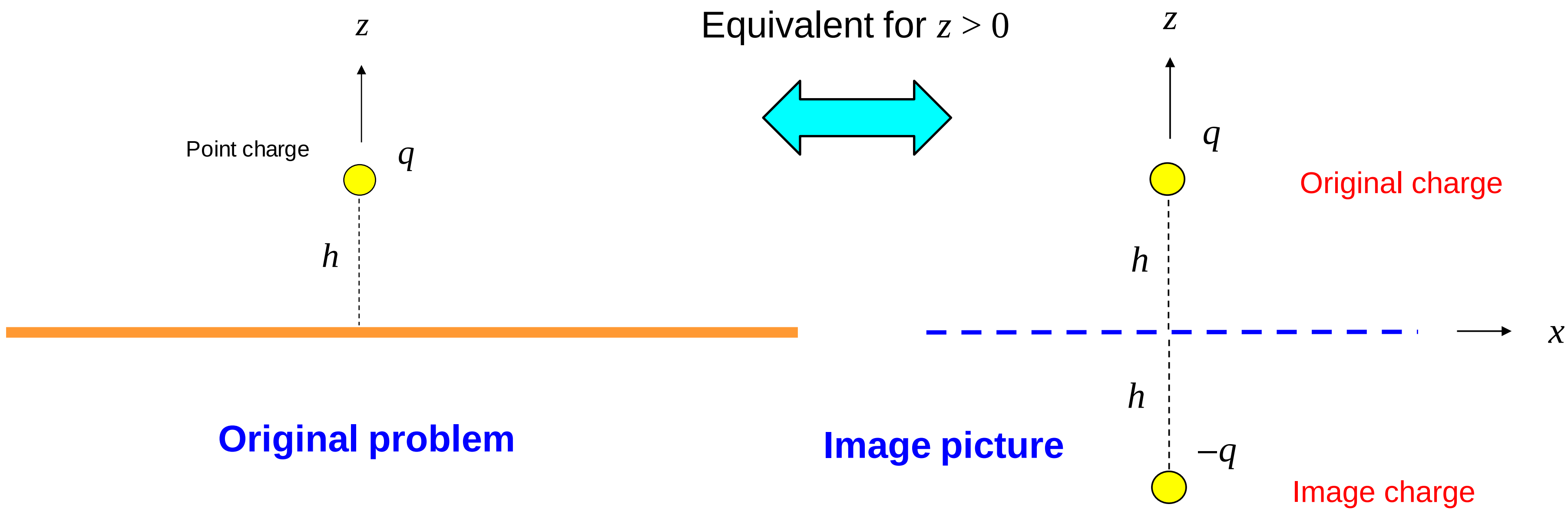


The original charge and the image charge together give the correct electric field in the region $z > 0$. They do NOT give the correct solution in the region $z < 0$.

(A proof that this is a valid solution is given on slides 7-8.)

Image Theory (cont.)

Summary of Image Method



Original problem

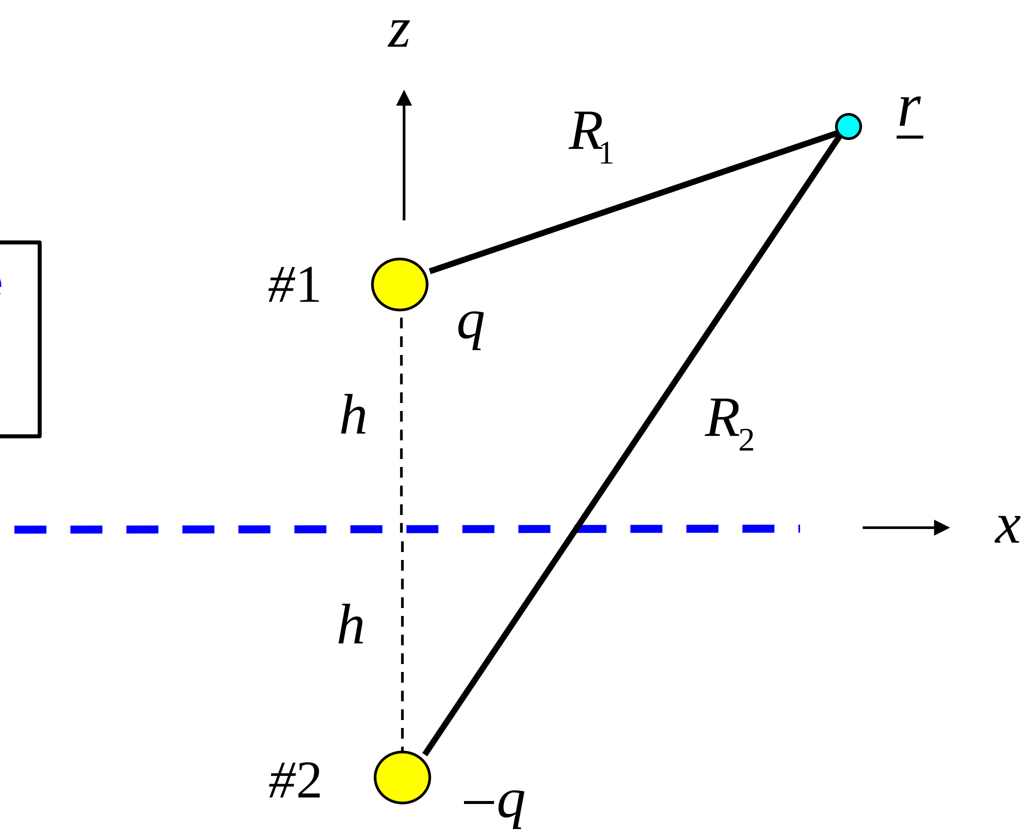
Image picture

- These two problems have the same potential and electric field in the upper region ($z > 0$).
- The image picture gives the wrong answer in the lower region ($z < 0$).

Image Theory (cont.)

$z > 0$

Here is what the image solution looks like (for the potential).

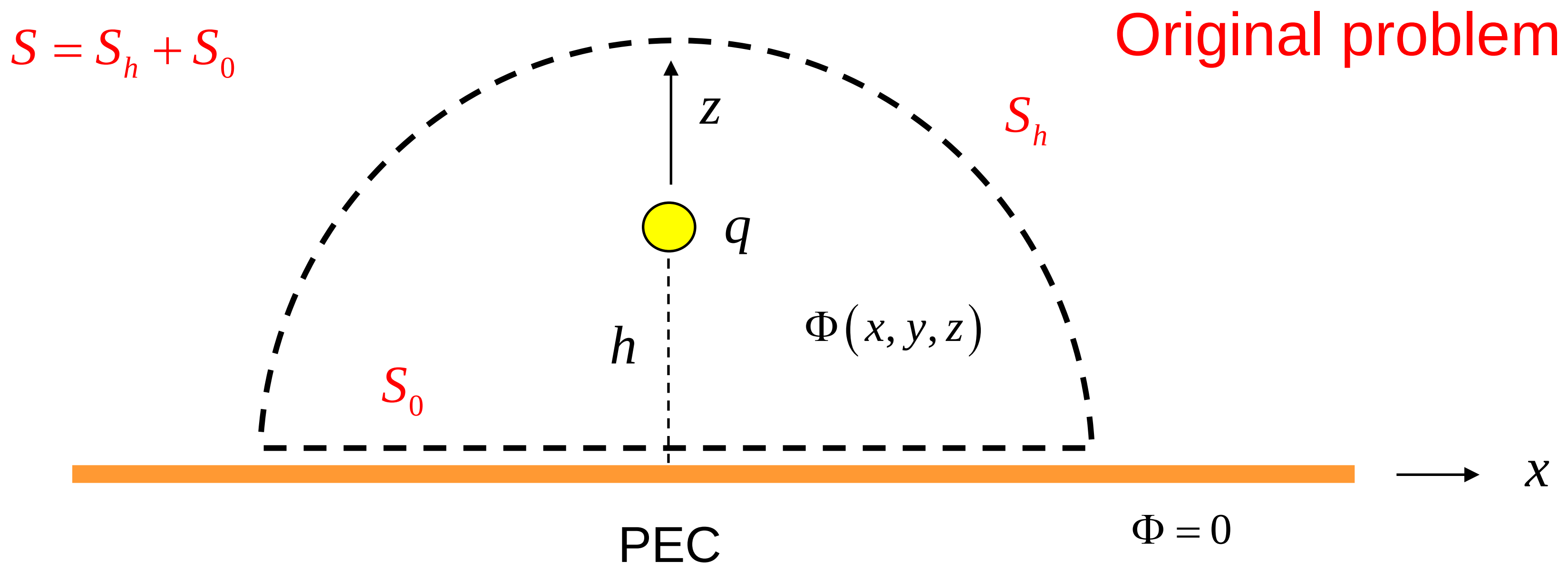


$$\Phi = \Phi_1 + \Phi_2$$

$$\Phi = \frac{q}{4\pi\varepsilon_0 R_1} + \frac{-q}{4\pi\varepsilon_0 R_2}$$

Image Theory: Proof

To see why image theory works, we construct a closed surface S that has a large hemispherical cap (the radius goes to infinity).



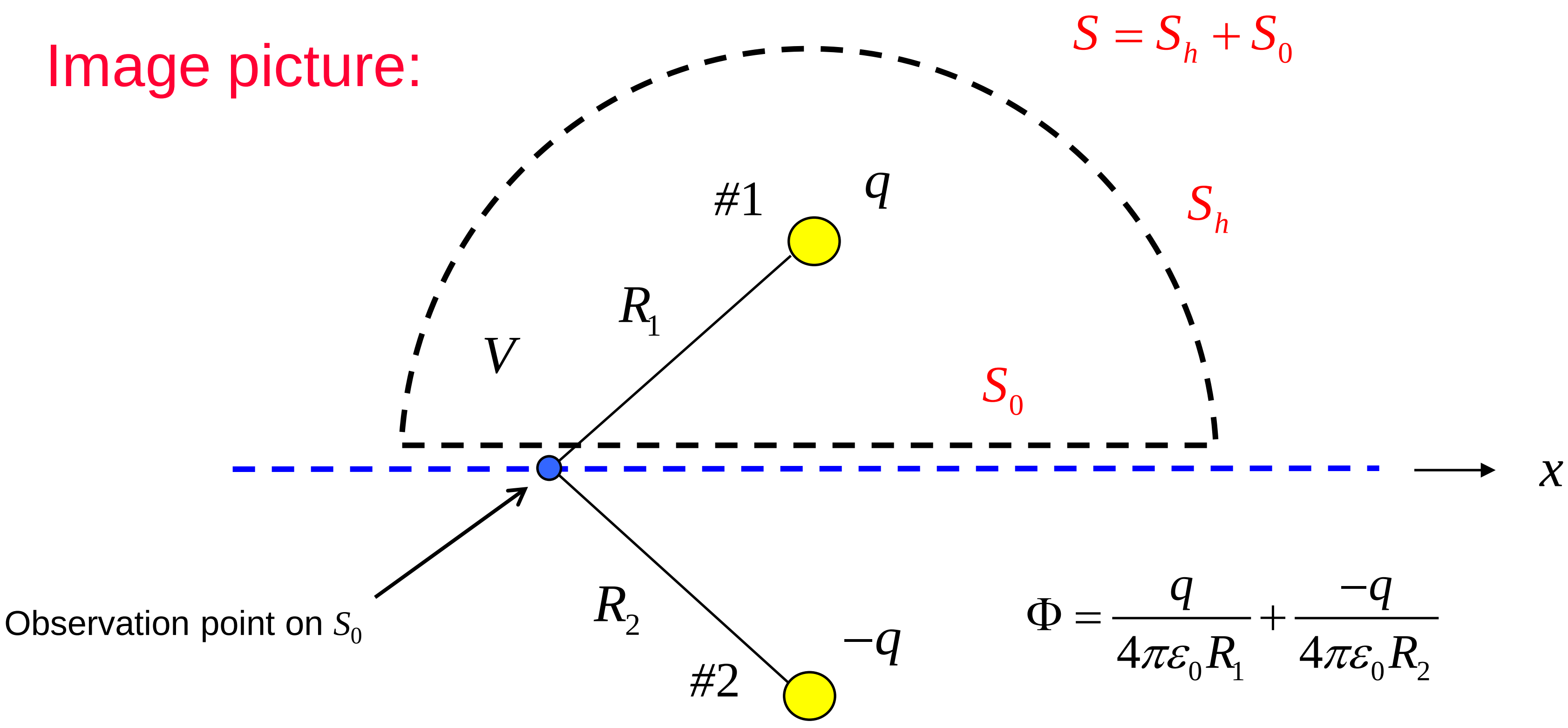
On S_0: $\Phi = 0$ (the surface is on a perfect electric conductor at zero volts).

On S_h: $\Phi = 0$ (the surface is at infinity where $\underline{E} = 0$, and $\Phi = 0$ on the bottom).

$\Rightarrow$ $\Phi = 0$ on S

Image Theory Proof (cont.)

Image picture:



$$\Phi = \frac{q}{4\pi\varepsilon_0 R_1} + \frac{-q}{4\pi\varepsilon_0 R_2}$$

✓ correct charge in V (charge #1)

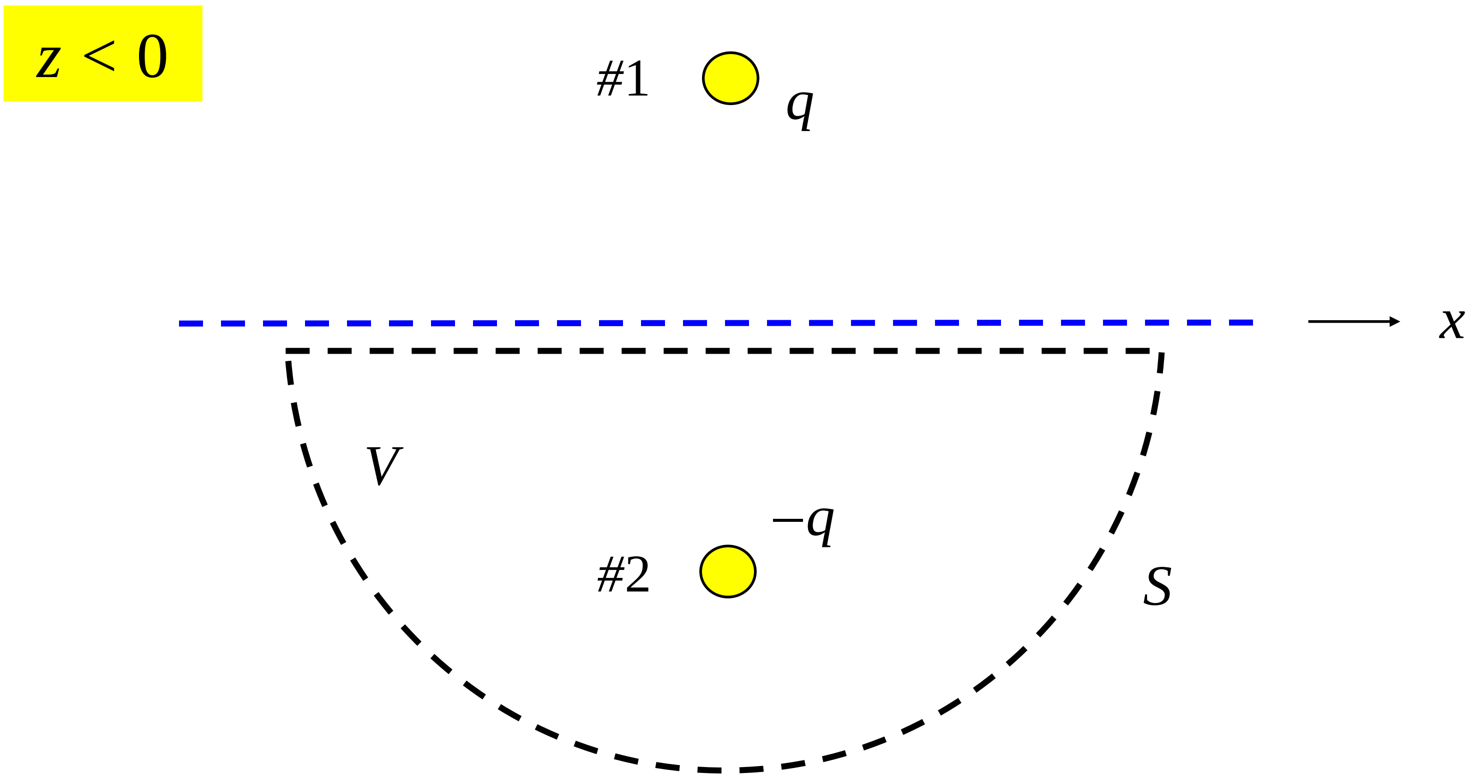
✓

$\underline{r} \in S_0 : \quad \Phi = 0$ since $R_1 = R_2$

$\underline{r} \in S_h : \quad \Phi = 0$ since $R_1 = R_2 = \infty$

(correct B.C. on S)

Image Theory Proof (cont.)



Wrong charge inside!

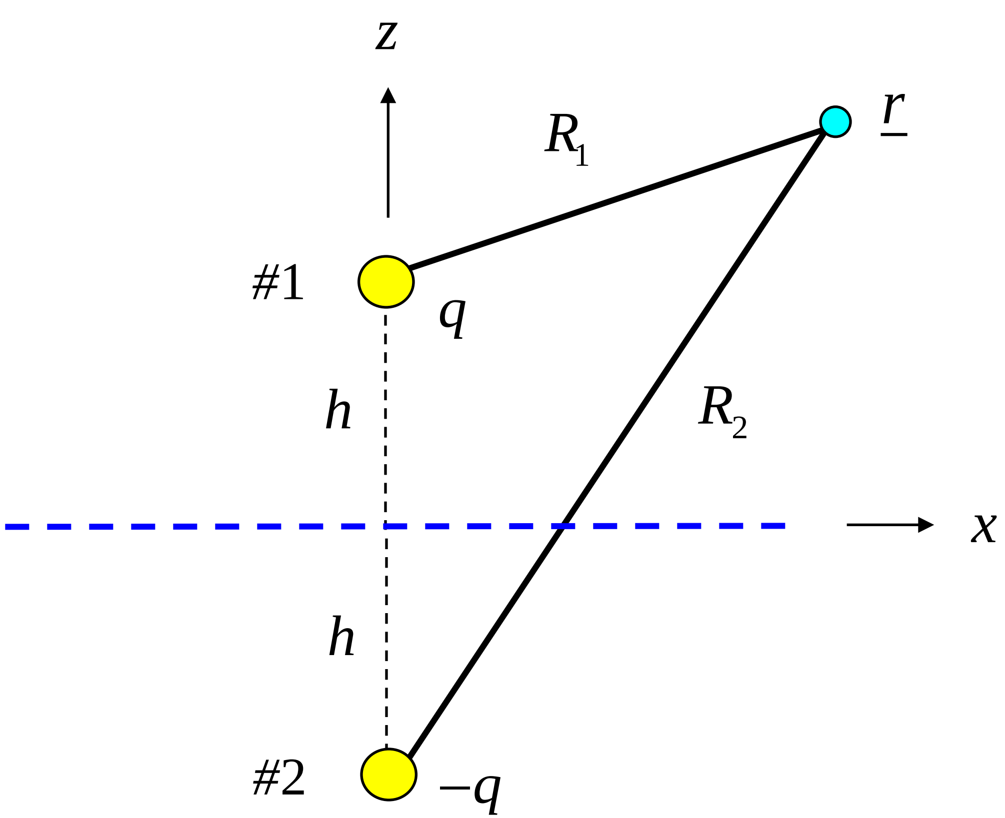
(The original problem does not have a charge in region #2.)

Image theory does <u>not</u> give the correct result in the lower region (the region where the image charge is).

Final Solution for $z > 0$

$z > 0$:

$$\Phi = \frac{q}{4\pi\varepsilon_0\sqrt{x^2 + y^2 + (z-h)^2}} + \frac{-q}{4\pi\varepsilon_0\sqrt{x^2 + y^2 + (z+h)^2}}$$



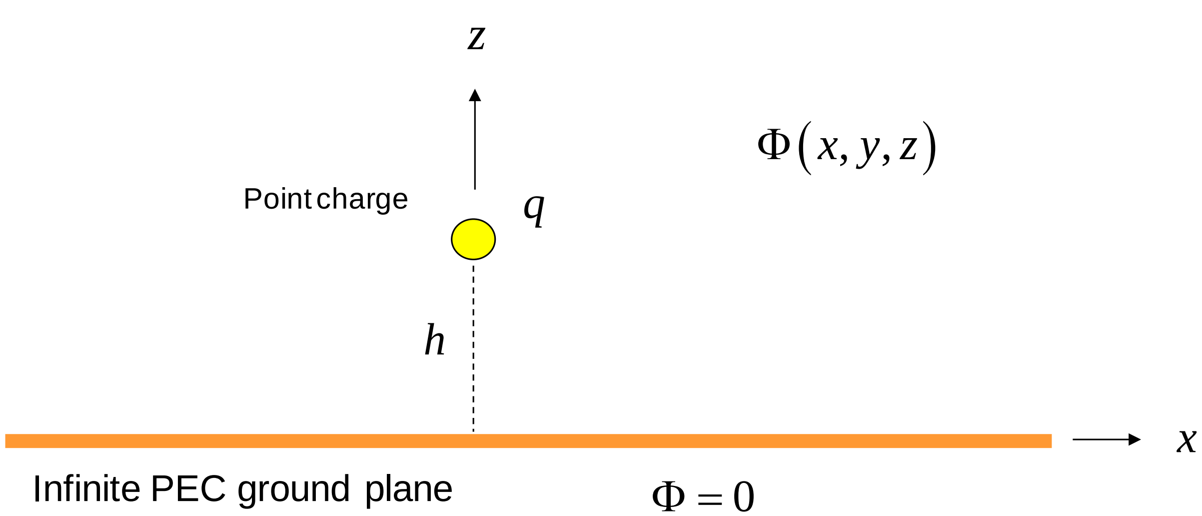
Final Solution (All Regions)

$z > 0$:

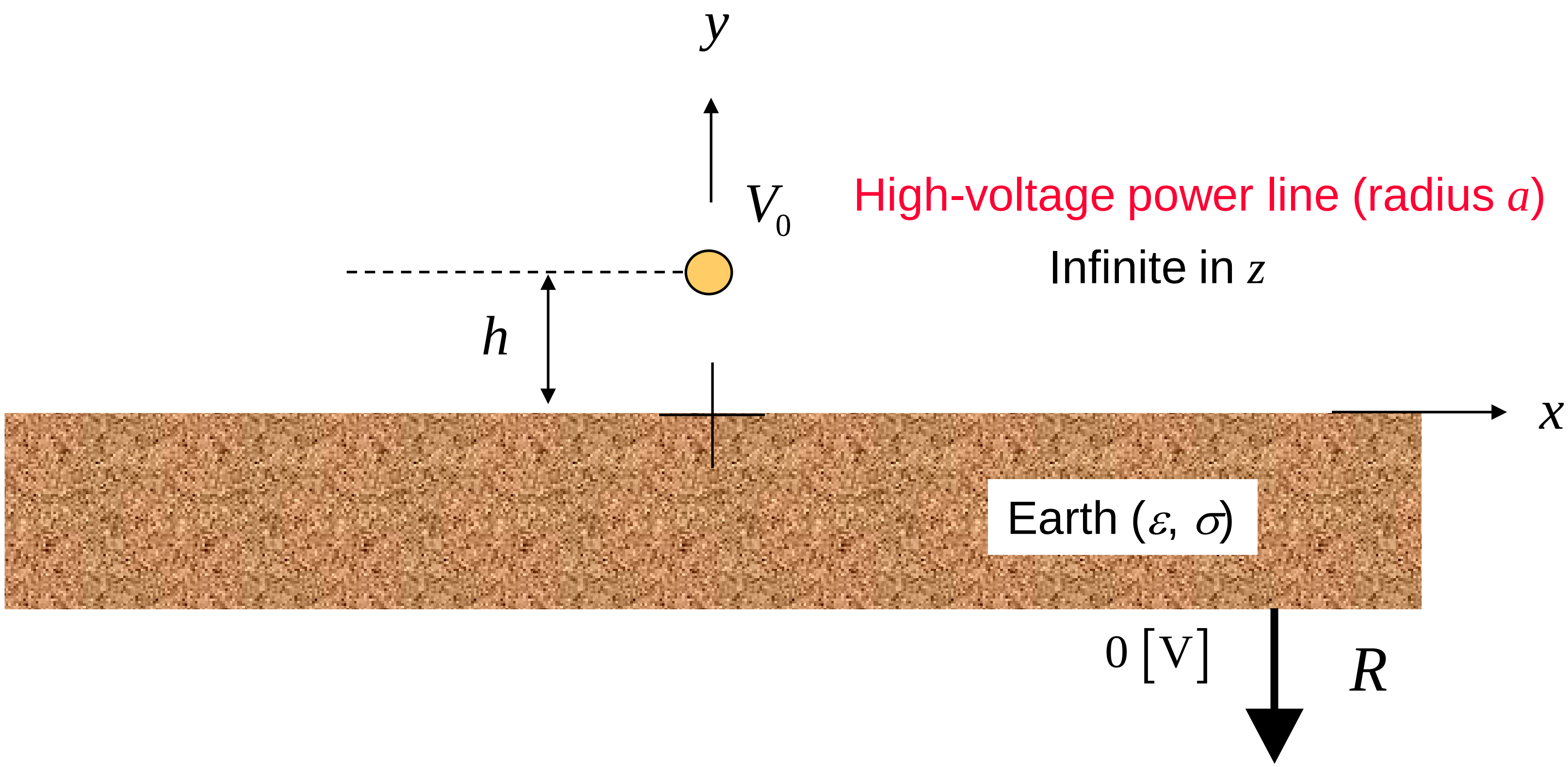
$$\Phi = \frac{q}{4\pi\varepsilon_0\sqrt{x^2 + y^2 + (z-h)^2}} + \frac{-q}{4\pi\varepsilon_0\sqrt{x^2 + y^2 + (z+h)^2}}$$

$z < 0$:

$$\Phi = 0$$



High-Voltage Power Line



The earth is modeled as a perfect conductor ($\underline{E} = \underline{0}$).

(The next slide discusses this approximation.)

High-Voltage Power Line (cont.)

The loss tangent of a material shows how good of a conductor it is at any frequency.
(This is discussed in ECE 3317.)

$$\omega = 2\pi f$$

$$\varepsilon = \varepsilon_0 \varepsilon_r$$

$$\varepsilon_0 = 8.854 \times 10^{-12} \text{ [F/m]}$$

$$\tan\delta \equiv \frac{\sigma}{\omega\varepsilon}$$

$$\tan\delta >> 1 \quad (\text{good conductor})$$
$$\tan\delta << 1 \quad (\text{good dielectric})$$

Assume the following parameters for earth:

$$\sigma = 0.1 \text{ [S/m]}$$

$$\varepsilon_r = 8$$

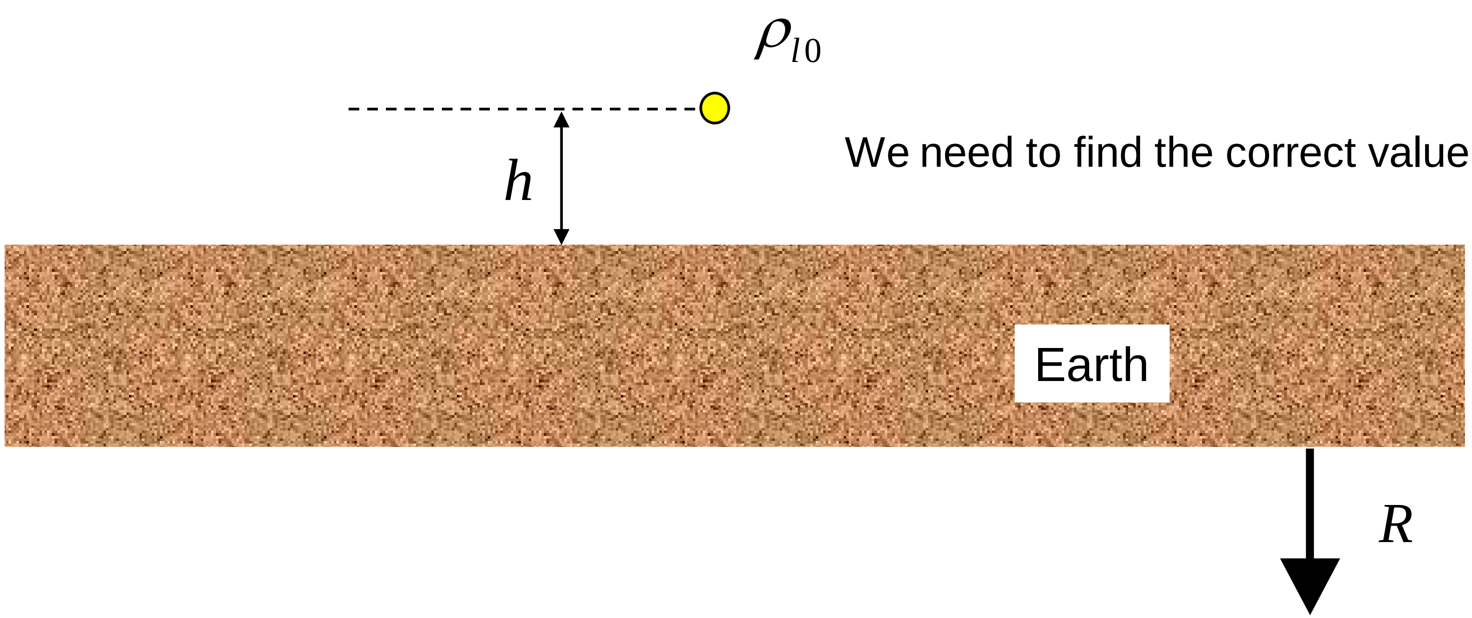
Table showing $\tan\delta$ for earth

10 [Hz]	2.25×10^7
100 [Hz]	2.25×10^6
1 [kHz]	2.25×10^5
10 [kHz]	2.25×10^4
100 [kHz]	2.25×10^3
1 [MHz]	225
10 [MHz]	22.5
100 [MHz]	2.25
1.0 [GHz]	0.225
10.0 [GHz]	0.0225

High-Voltage Power Line (cont.)

Line-charge approximation:

Note: $\rho_{l0} = 2\pi a \rho_{s0}$

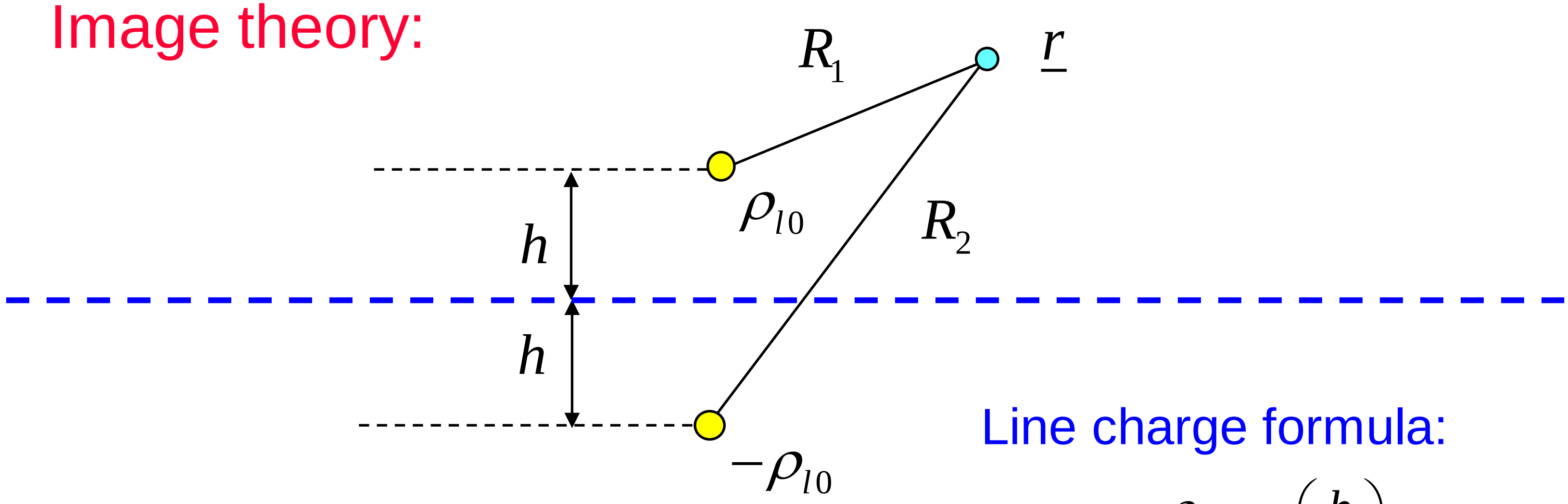


We need to find the correct value of ρ_{l0}.

The power line is modeled as an <u>effective line charge</u> at the <u>center</u> of the power line. This line-charge model is valid outside the power line (from Gauss's law) as long as the charge density on the surface of the wire is approximately uniform and we are outside the wire.

High-Voltage Power Line (cont.)

Image theory:



Line charge formula:

$$\Phi = \frac{\rho_{l0}}{2\pi\varepsilon_0}\ln\left(\frac{b}{\rho}\right)$$

$$\Phi = \frac{\rho_{l0}}{2\pi\varepsilon_0}\ln\left(\frac{b}{R_1}\right) + \frac{-\rho_{l0}}{2\pi\varepsilon_0}\ln\left(\frac{b}{R_2}\right)$$

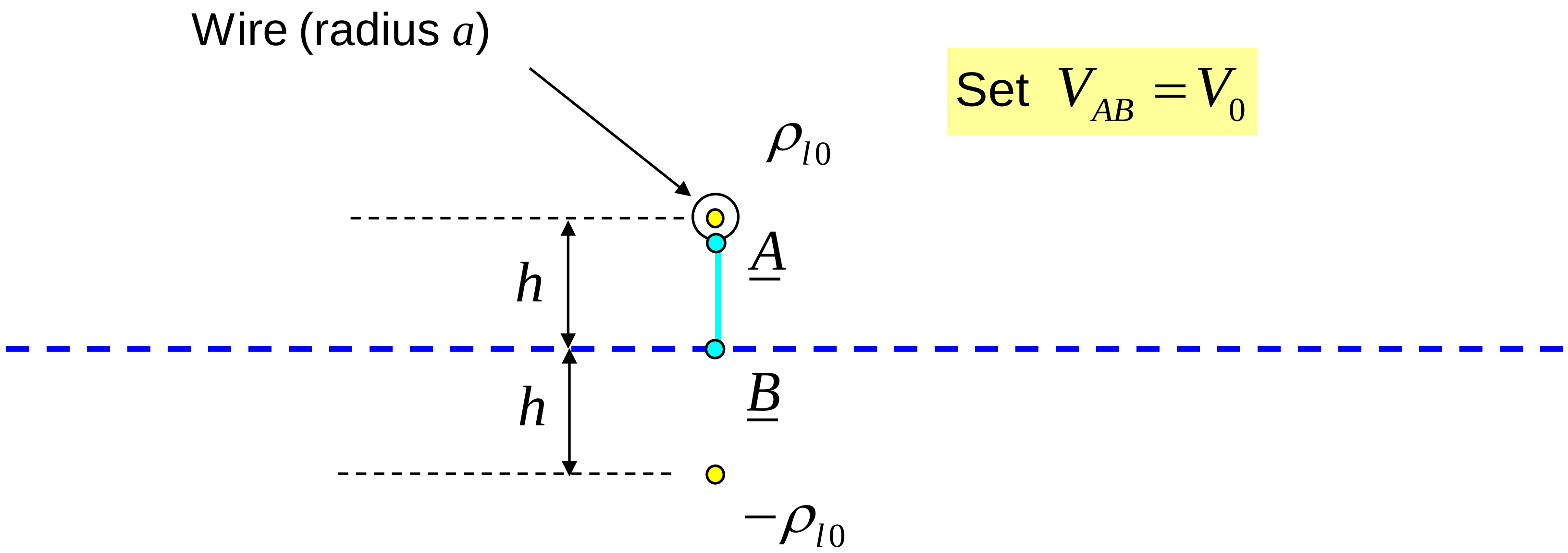
Note:
In the formula b is the distance to the arbitrary reference "point" for the single line-charge solution.

or

$$\Phi = \frac{\rho_{l0}}{2\pi\varepsilon_0}\ln\left(\frac{R_2}{R_1}\right)$$

The line-charge density ρ_{l0} can be found by forcing the voltage to be V_0 at the surface of the wire (with respect to the earth).

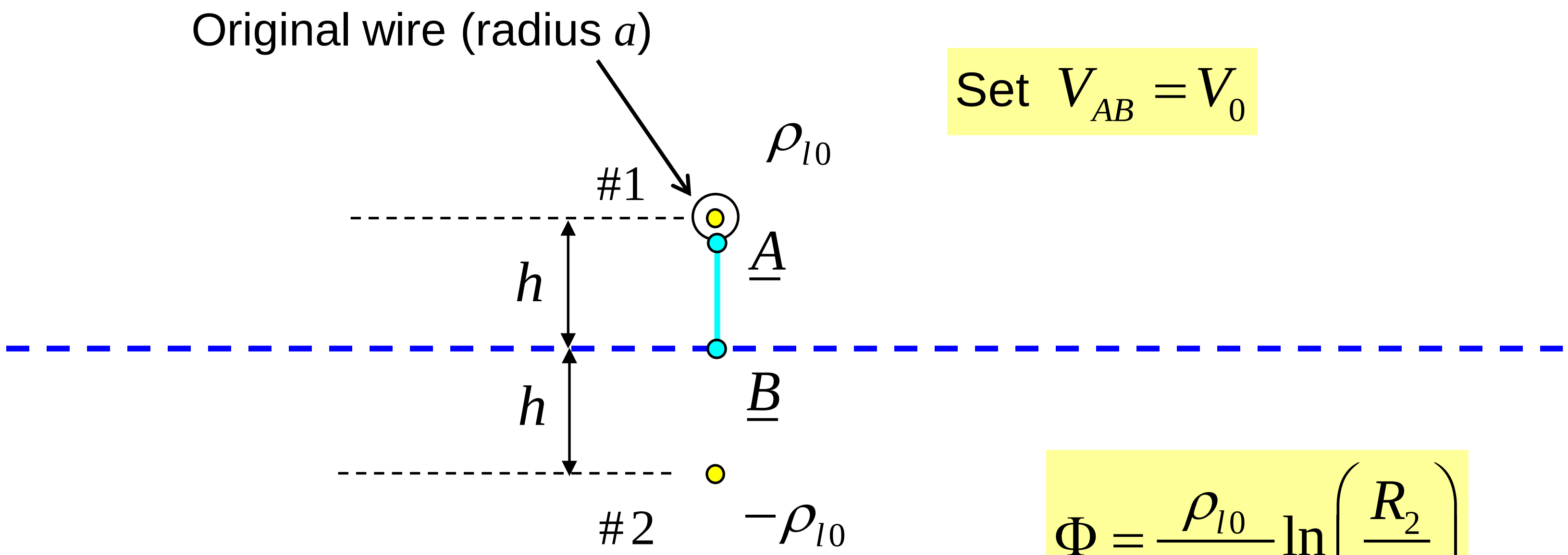
High-Voltage Power Line (cont.)



Set $V_{AB} = V_0$

The point $\underline{A}$ is selected as the point on the bottom surface of the power line.

High-Voltage Power Line (cont.)



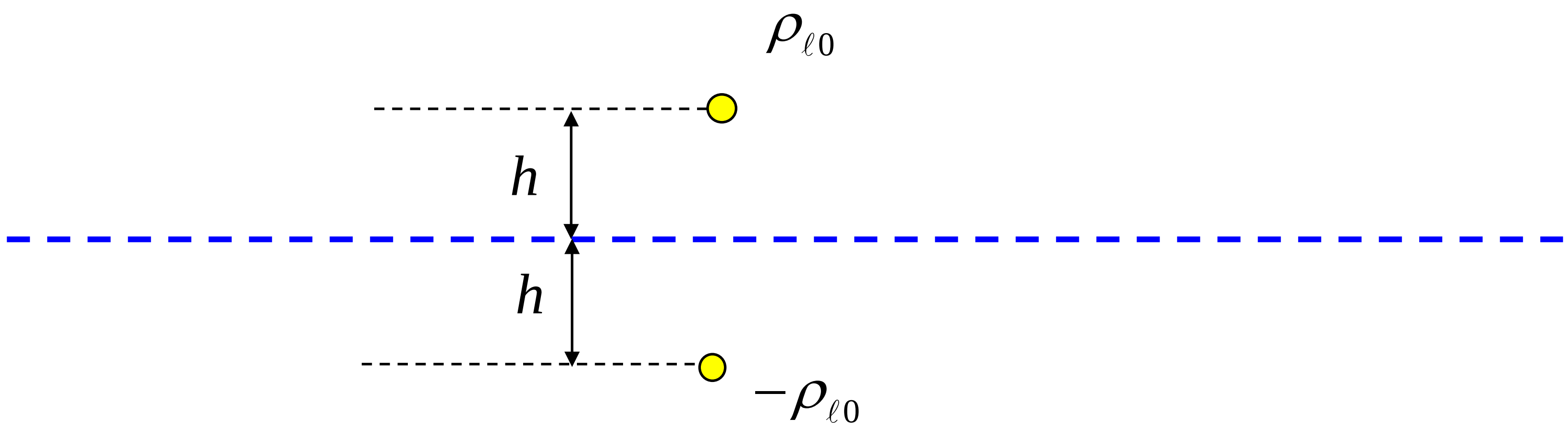
Set $V_{AB} = V_0$

$$\Phi = \frac{\rho_{l0}}{2\pi\varepsilon_0}\ln\left(\frac{R_2}{R_1}\right)$$

$$\left.\begin{aligned}\Phi(\underline{A}) &= \frac{\rho_{l0}}{2\pi\varepsilon_0}\ln\left(\frac{2h-a}{a}\right)\\ \Phi(\underline{B}) &= \frac{\rho_{l0}}{2\pi\varepsilon_0}\ln\left(\frac{h}{h}\right)=0\end{aligned}\right\}$$

$$V_0 = V_{AB} = \Phi(\underline{A}) - \Phi(\underline{B})$$
$$= \frac{\rho_{l0}}{2\pi\varepsilon_0}\ln\left(\frac{2h-a}{a}\right)$$

High-Voltage Power Line (cont.)

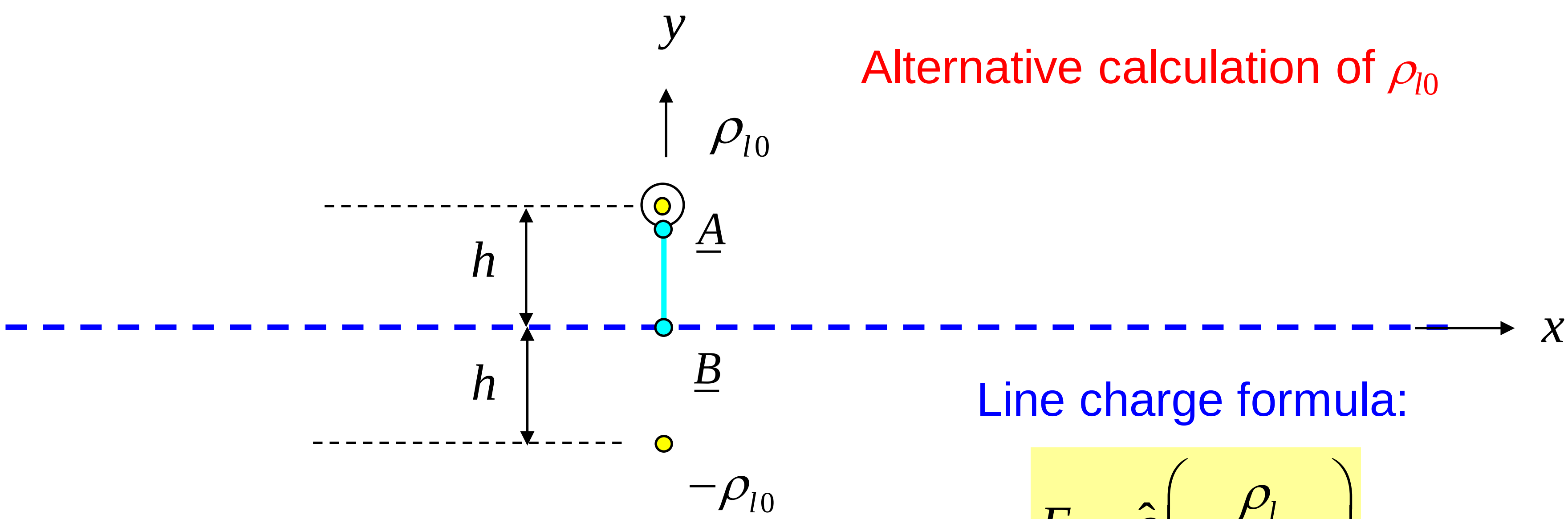


Hence

$$\rho_{\ell 0} = \frac{2\pi\varepsilon_0 V_0}{\ln\left(\dfrac{2h-a}{a}\right)}$$

High-Voltage Power Line (cont.)

Alternative calculation of ρ_{l0}



Line charge formula:

$$\underline{E} = \hat{\underline{\rho}}\left(\frac{\rho_l}{2\pi\varepsilon_0 \rho}\right)$$

$$V_0 = V_{AB} = \int_{\underline{A}}^{\underline{B}} \underline{E}\cdot d\underline{r} = \int_{h-a}^{0} E_y\, dy$$

Along the vertical line we have:

$$\underline{E} = -\hat{\underline{y}}\left(\frac{\rho_{l0}}{2\pi\varepsilon_0 (h-y)}\right) + \hat{\underline{y}}\left(\frac{-\rho_{l0}}{2\pi\varepsilon_0 (h+y)}\right)$$

From top charge   From bottom charge

Hence

$$V_0 = \frac{-\rho_{l0}}{2\pi\varepsilon_0}\int_{h-a}^{0}\left(\frac{1}{h-y} + \frac{1}{h+y}\right)dy$$

High-Voltage Power Line (cont.)

Performing the integration, we have:

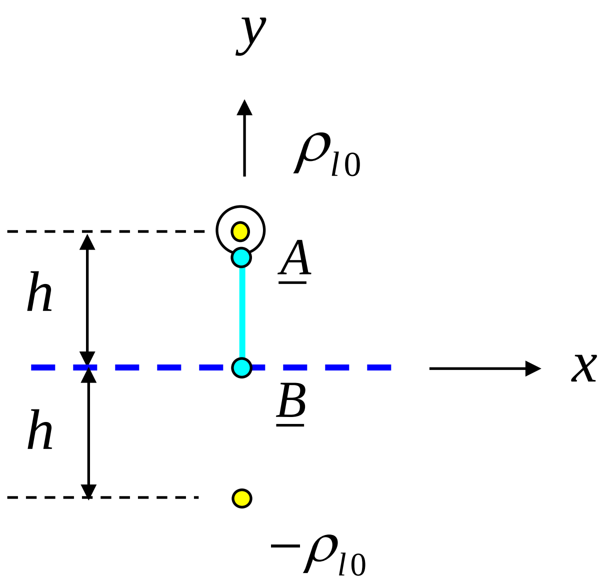
$$V_0 = \frac{-\rho_{l0}}{2\pi\varepsilon_0}\int_{h-a}^{0}\left(\frac{1}{h-y}+\frac{1}{h+y}\right)dy$$

$$= \frac{-\rho_{l0}}{2\pi\varepsilon_0}\left(-\ln(h-y)+\ln(h+y)\right)\Big|_{h-a}^{0}$$

$$= \frac{-\rho_{l0}}{2\pi\varepsilon_0}\left(-\ln(h)+\ln(a)+\ln(h)-\ln(2h-a)\right)$$

$$= \frac{\rho_{l0}}{2\pi\varepsilon_0}\left(\ln(2h-a)-\ln(a)\right)$$

$$= \frac{\rho_{l0}}{2\pi\varepsilon_0}\ln\left(\frac{2h-a}{a}\right)$$

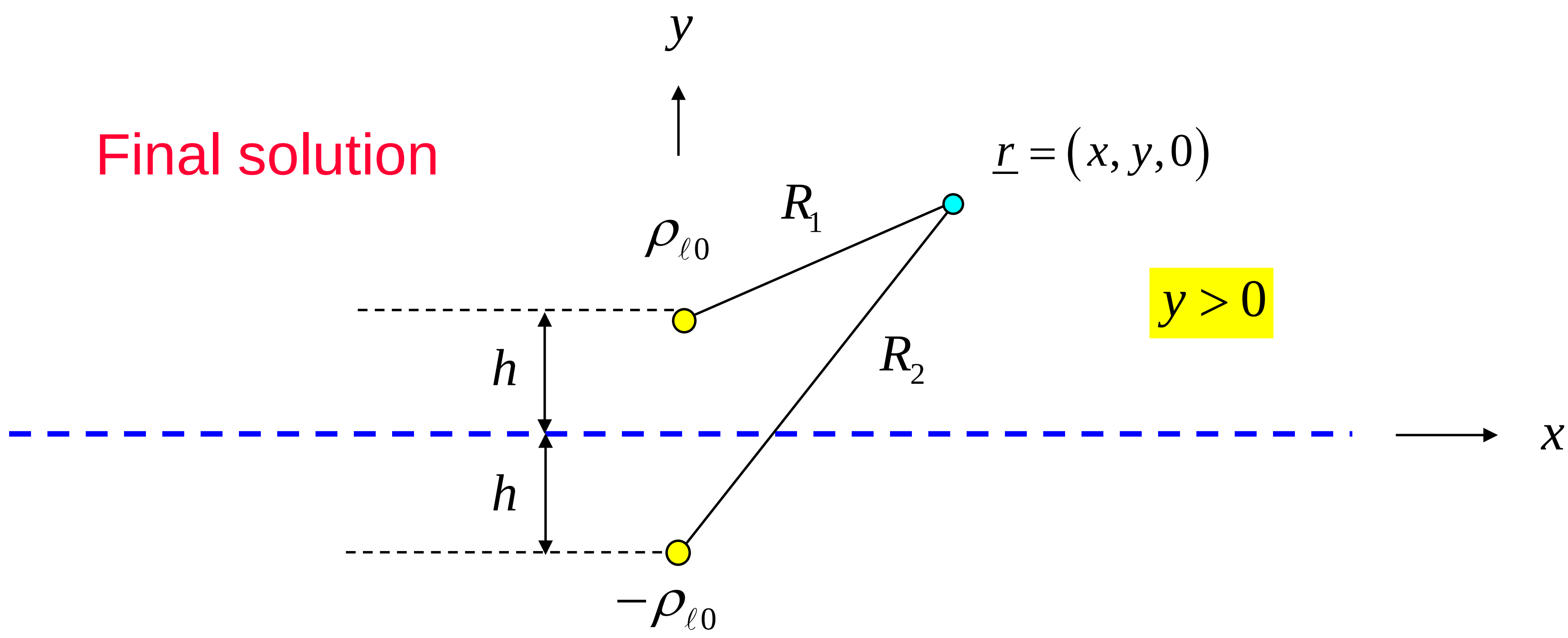


Hence

$$V_0 = \frac{\rho_{l0}}{2\pi\varepsilon_0}\ln\left(\frac{2h-a}{a}\right)$$

High-Voltage Power Line (cont.)

Final solution



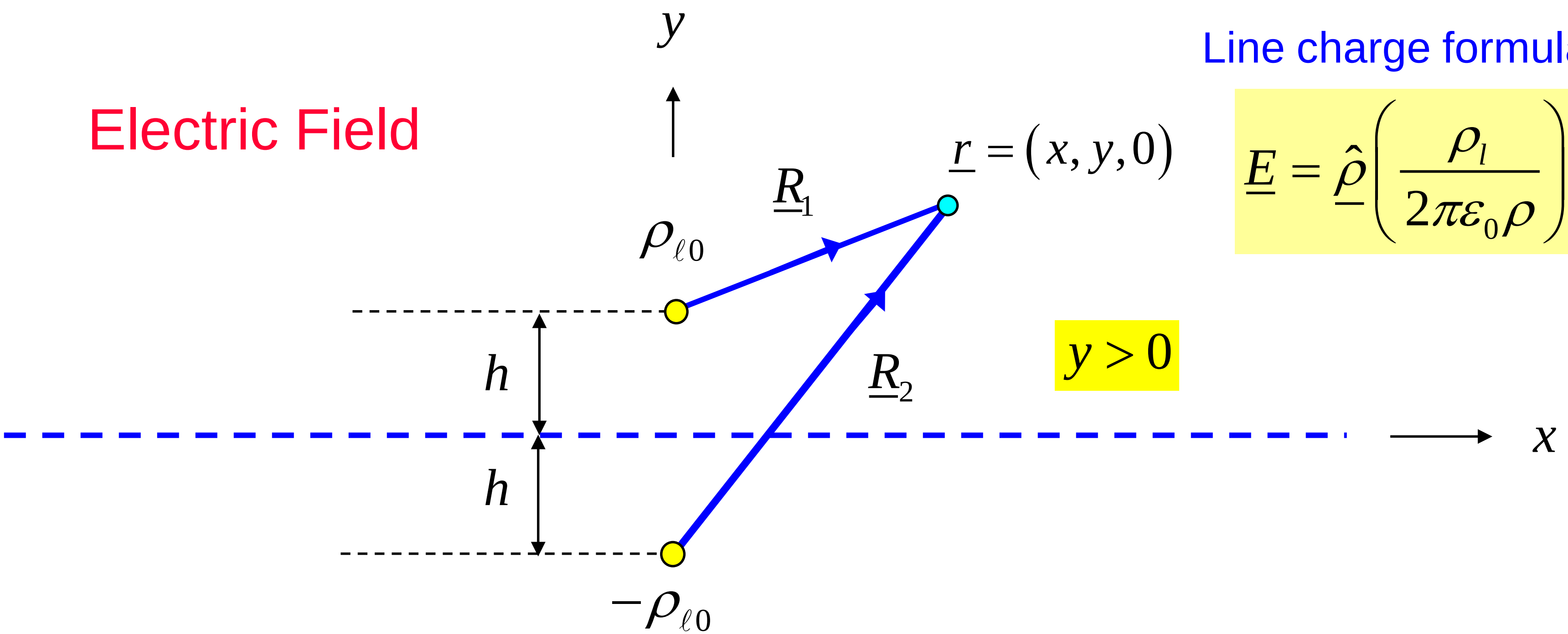
$$\Phi(x, y) = \frac{\rho_{\ell 0}}{2\pi\varepsilon_0} \ln\left(\frac{R_2}{R_1}\right)$$

$$\rho_{\ell 0} = \frac{2\pi\varepsilon_0 V_0}{\ln\left(\dfrac{2h-a}{a}\right)}$$

$$R_1 = \sqrt{x^2 + (y-h)^2} \qquad R_2 = \sqrt{x^2 + (y+h)^2}$$

High-Voltage Power Line (cont.)

Electric Field



Line charge formula:

$$\underline{E} = \underline{\hat{\rho}}\left(\frac{\rho_l}{2\pi\varepsilon_0 \rho}\right)$$

$$\underline{E}(x,y) = \frac{\rho_{\ell 0}}{2\pi\varepsilon_0 R_1}\underline{\hat{R}}_1 + \frac{-\rho_{\ell 0}}{2\pi\varepsilon_0 R_2}\underline{\hat{R}}_2$$

$$\rho_{\ell 0} = \frac{2\pi\varepsilon_0 V_0}{\ln\left(\frac{2h-a}{a}\right)}$$

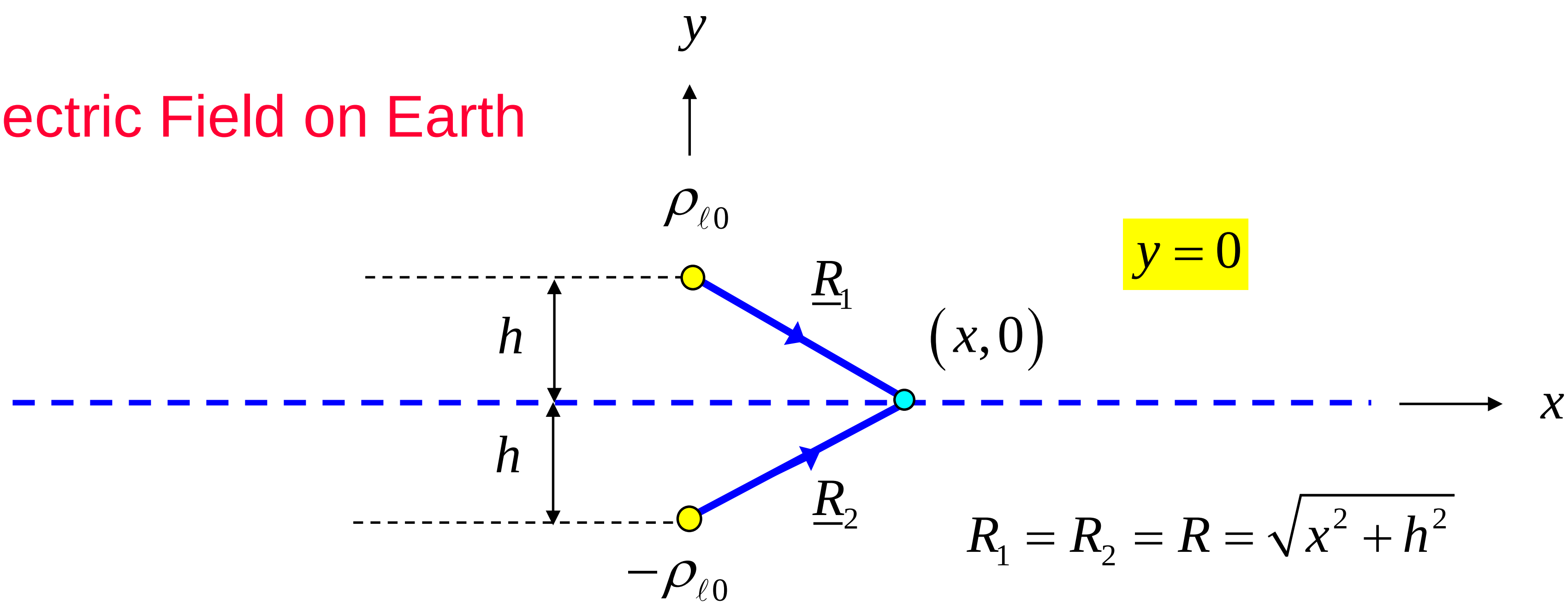
$$\underline{R}_1 = \underline{\hat{x}}(x) + \underline{\hat{y}}(y-h) \qquad \underline{R}_2 = \underline{\hat{x}}(x) + \underline{\hat{y}}(y+h)$$

$$R_1 = \sqrt{x^2 + (y-h)^2} \qquad R_2 = \sqrt{x^2 + (y+h)^2}$$

From these we get the unit vectors.

High-Voltage Power Line (cont.)

Electric Field on Earth



$$R_1 = R_2 = R = \sqrt{x^2 + h^2}$$

$$\underline{E}(x,0) = \frac{\rho_{\ell 0}}{2\pi\varepsilon_0 R_1}\underline{\hat{R}}_1 + \frac{-\rho_{\ell 0}}{2\pi\varepsilon_0 R_2}\underline{\hat{R}}_2 = \frac{\rho_{\ell 0}}{2\pi\varepsilon_0 R}\left(\underline{\hat{R}}_1 - \underline{\hat{R}}_2\right)$$

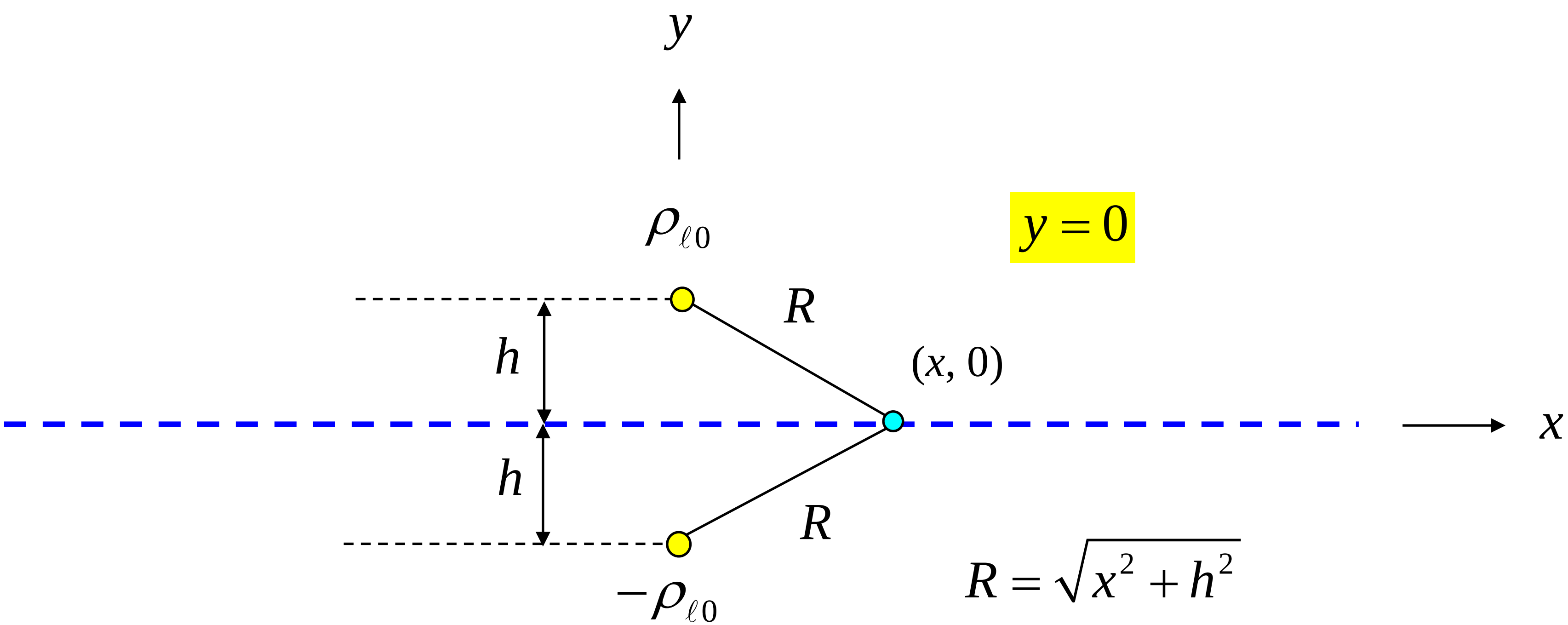
$$\underline{R}_1 = \underline{\hat{x}}(x) + \underline{\hat{y}}(0-h) \qquad \underline{R}_2 = \underline{\hat{x}}(x) + \underline{\hat{y}}(0+h)$$

$$\underline{\hat{R}}_1 - \underline{\hat{R}}_2 = \frac{1}{R}\left(\underline{R}_1 - \underline{R}_2\right) = \frac{1}{R}\left(-\underline{\hat{y}}(2h)\right)$$

High-Voltage Power Line (cont.)

Hence, we have on the surface on the Earth:

$$\underline{E}(x,0) = -\underline{\hat{y}}\frac{\rho_{\ell 0} h}{\pi \varepsilon_0 R^2} \qquad \rho_{\ell 0} = \frac{2\pi\varepsilon_0 V_0}{\ln\left(\frac{2h-a}{a}\right)}$$

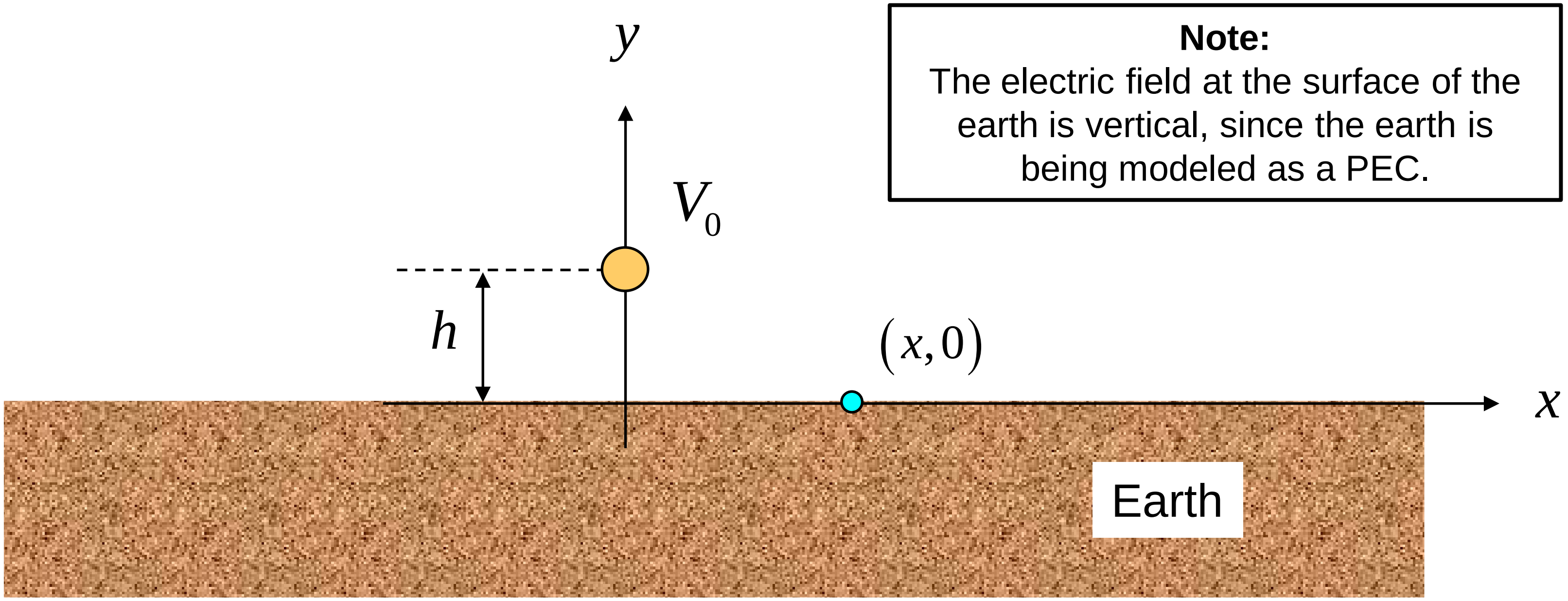


$$R = \sqrt{x^2 + h^2}$$

High-Voltage Power Line (cont.)

Final result:

$$\underline{E}(x,0) = -\underline{\hat{y}}\left(\frac{1}{h}\right)\frac{2V_0}{\ln\left(\frac{2h-a}{a}\right)}\left(\frac{1}{1+(x/h)^2}\right)$$

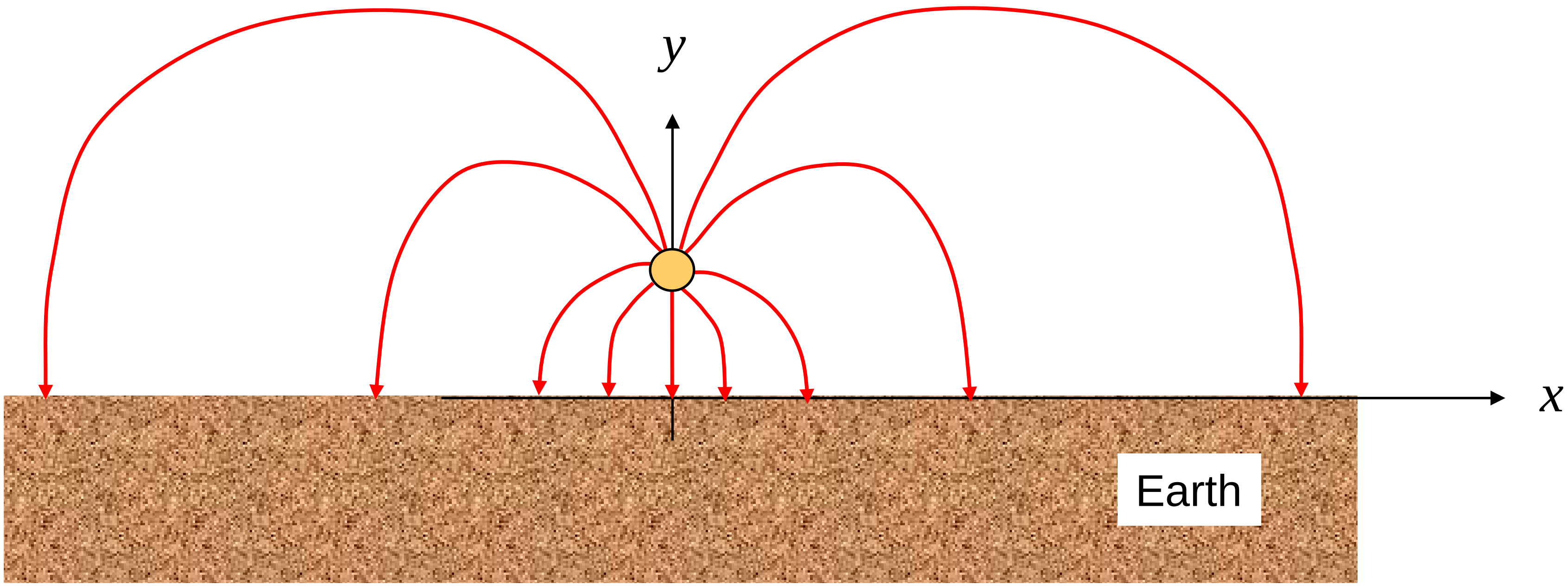


Note:
The electric field at the surface of the earth is vertical, since the earth is being modeled as a PEC.

High-Voltage Power Line (cont.)

Flux plot

$$\underline{E}(x,0) = -\underline{\hat{y}}\left(\frac{1}{h}\right)\frac{2V_0}{\ln\left(\frac{2h-a}{a}\right)}\left(\frac{1}{1+(x/h)^2}\right)$$



High-Voltage Power Line (cont.)

Example

(a) Find the electric field at the surface of the earth.

(b) Find the electric field at the bottom surface of the line.

Parameters:

$$V_0 = 230 \text{ [kV]}$$

$$h = 50 \text{ [m]}$$

$$a = 1 \text{ [cm]}$$

In MKS units:

$$V_0 = 2.30\times10^5 \text{ [V]}$$

$$h = 50 \text{ [m]}$$

$$a = 0.01 \text{ [m]}$$

High-Voltage Power Line (cont.)

Part (a)

At the surface of the Earth we have

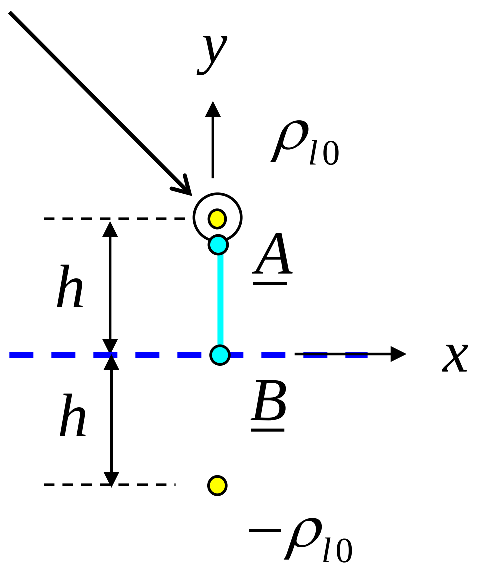
$$\underline{E}(x,0) = -\underline{\hat{y}}\left(\frac{1}{h}\right)\frac{2V_0}{\ln\left(\dfrac{2h-a}{a}\right)}\left(\frac{1}{1+(x/h)^2}\right)$$

This gives us :

$$\underline{E}(x,0) = -\underline{\hat{y}}(998.89)\left(\frac{1}{1+(x/h)^2}\right) \quad [\mathrm{V/m}]$$

High-Voltage Power Line (cont.)

Part (b)



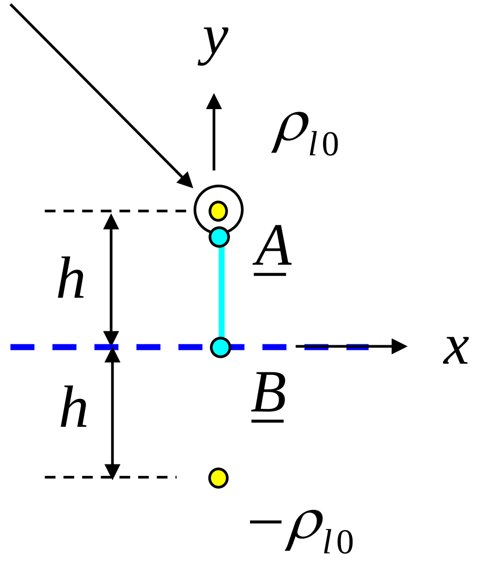
Along the vertical line we have (from slide 19):

$$\underline{E}(0,y) = -\underline{\hat{y}}\left(\frac{\rho_{l0}}{2\pi\varepsilon_0(h-y)}\right) + \underline{\hat{y}}\left(\frac{-\rho_{l0}}{2\pi\varepsilon_0(h+y)}\right)$$

Hence, at the bottom of the line ($y = h$-a) we have:

$$\underline{E}^{bot} = \underline{E}(0,h-a) = -\underline{\hat{y}}\left(\frac{\rho_{l0}}{2\pi\varepsilon_0 a}\right) + \underline{\hat{y}}\left(\frac{-\rho_{l0}}{2\pi\varepsilon_0(2h-a)}\right) \approx -\underline{\hat{y}}\left(\frac{\rho_{l0}}{2\pi\varepsilon_0 a}\right)$$

High-Voltage Power Line (cont.)



$$\underline{E}^{bot} \approx -\underline{\hat{y}}\left(\frac{\rho_{l0}}{2\pi\varepsilon_0 a}\right)$$

so that

$$\underline{E}^{bot} \approx -\underline{\hat{y}}\left(\frac{1}{a}\right)\frac{V_0}{\ln\left(\dfrac{2h-a}{a}\right)}$$

Recall:

$$\rho_{\ell 0} = \frac{2\pi\varepsilon_0 V_0}{\ln\left(\dfrac{2h-a}{a}\right)}$$

This gives us:

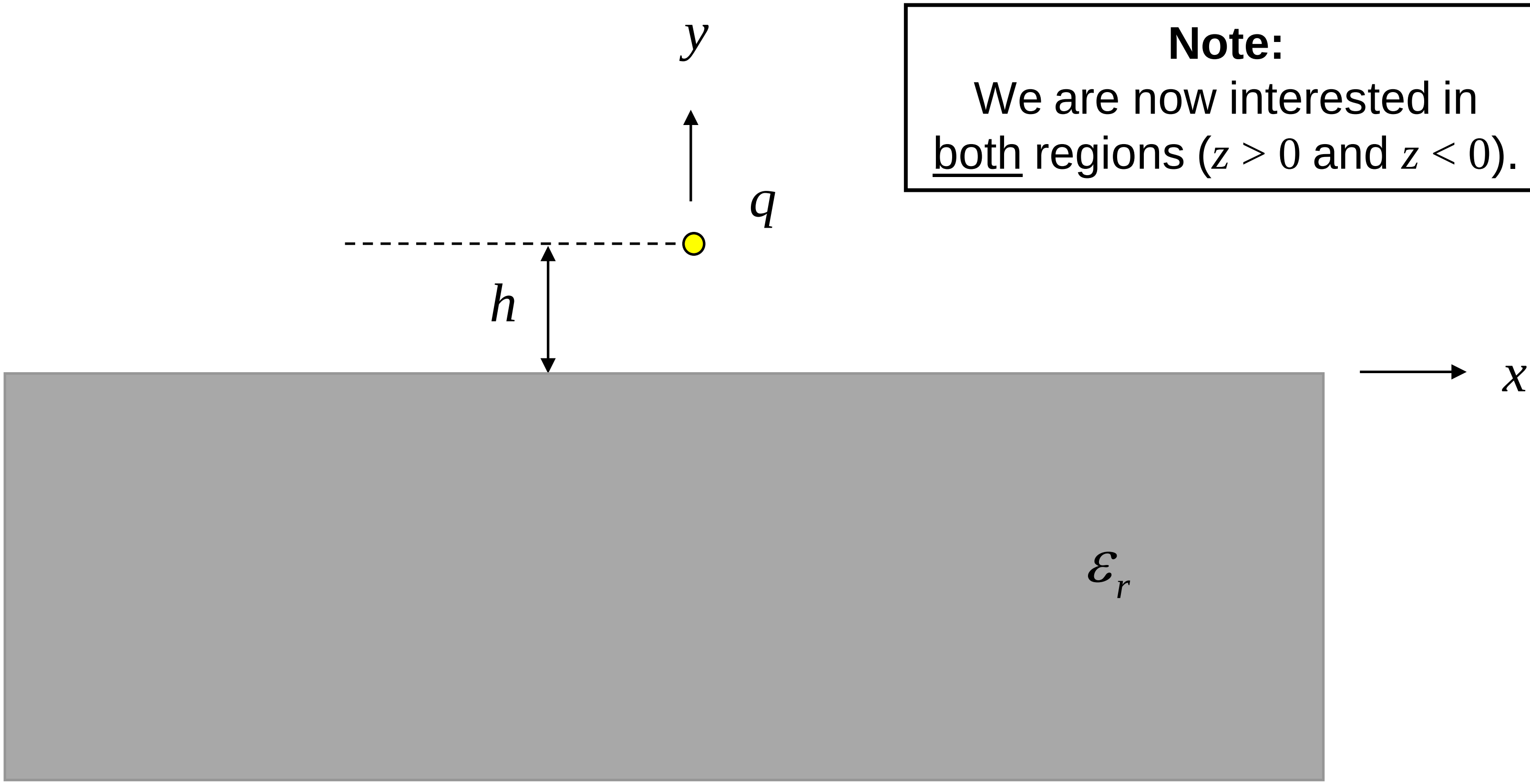
$$\underline{E}^{bot} = -\underline{\hat{y}}\left(2.50\times10^6\right)[\mathrm{V/m}]$$

Will corona discharge take place? Recall: $E_c = 3.0\times10^6$ [V/m]

Image Theory for Dielectric Region

Point charge over a semi-infinite dielectric region

(Please see the textbook for a derivation.)

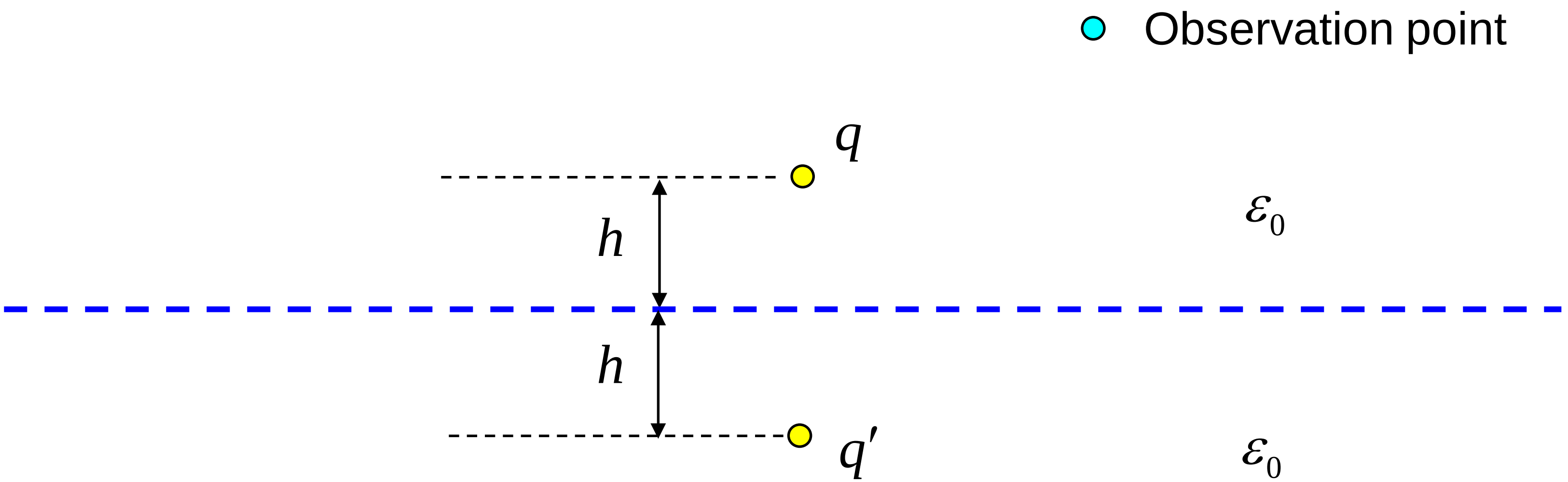


Note:
We are now interested in <u>both</u> regions ($z > 0$ and $z < 0$).

Note: A line charge could also be done the same way.

Image Theory for Dielectric Region (cont.)

Solution for air region ($z > 0$):

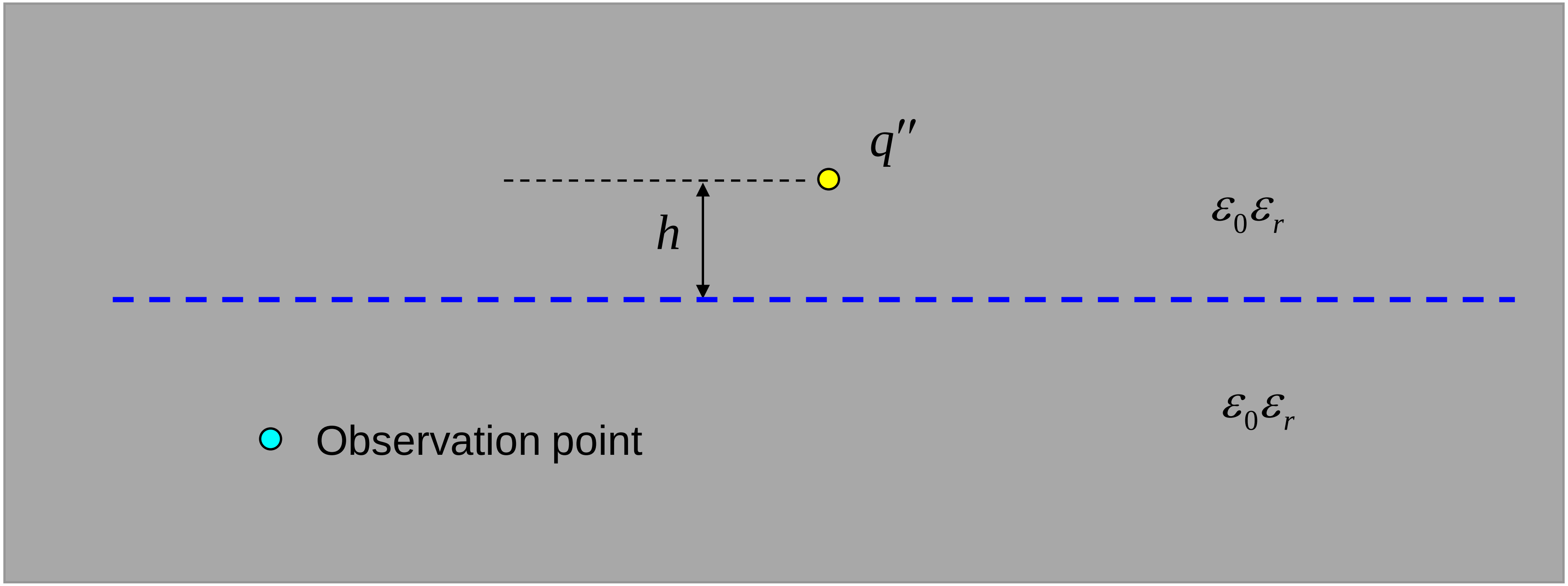


$$q' = -q\left(\frac{\varepsilon_r - 1}{\varepsilon_r + 1}\right)$$

Note:
A very large permittivity acts like a PEC ($q' = -q$).

Image Theory for Dielectric Region (cont.)

Solution for dielectric region ($z < 0$):

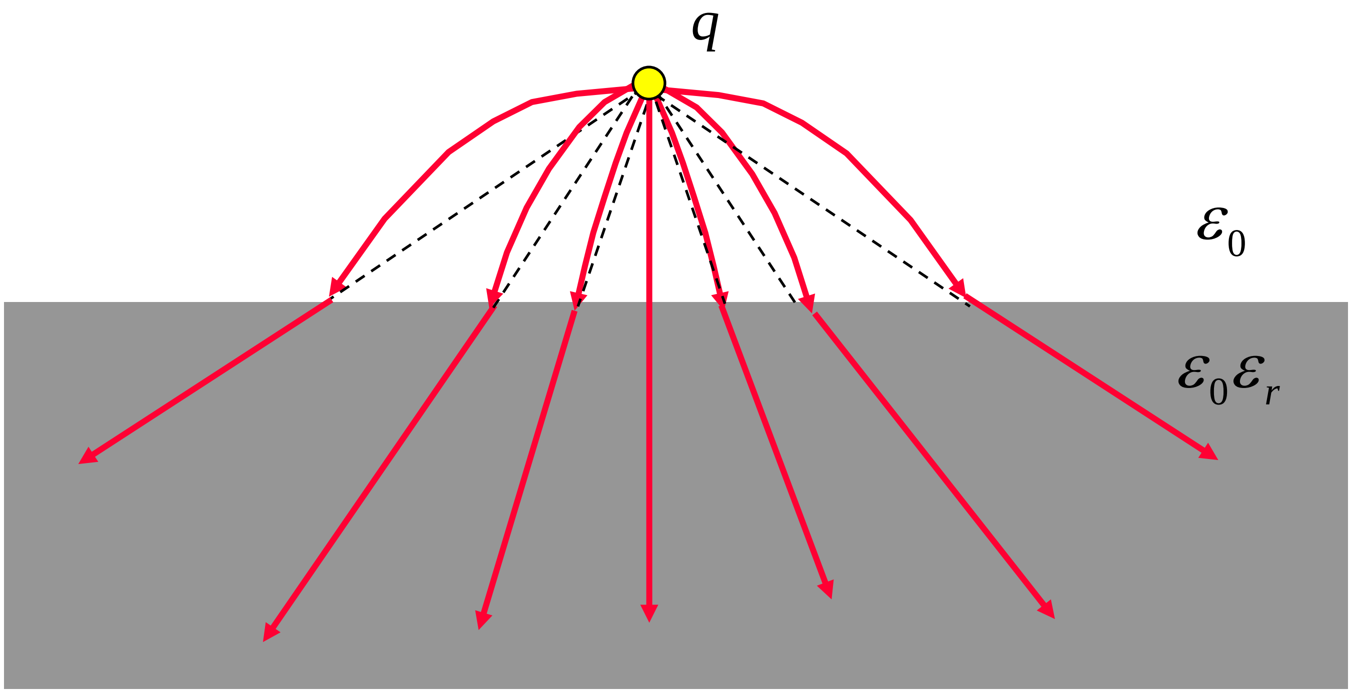


$$q'' = q\left(\frac{2\varepsilon_r}{\varepsilon_r + 1}\right)$$

Image Theory for Dielectric Region (cont.)

Sketch of electric field

Note: The flux lines are straight lines in the dielectric region.



The flux lines are bent away from the normal (as proven earlier from B.C.s).