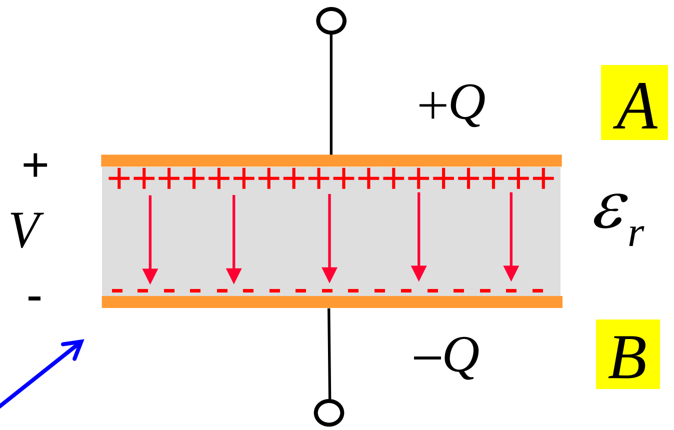
# ECE 3318
# Applied Electricity and Magnetism

**Spring 2023**

Prof. David R. Jackson
Dept. of ECE

## Notes 25
Capacitance

# Capacitance




Capacitor

$C$ [F]

**Note**:
The "$A$" conductor has the positive charge (connected to anode).

A capacitor is two metal objects with equal and opposite charge. (It does not always have to look like this!)

$$C \equiv \frac{Q}{V} \quad [\mathrm{F}]$$

$$\left(1\ [\mathrm{F}] = 1\,[\mathrm{C/V}]\right)$$

Notation:

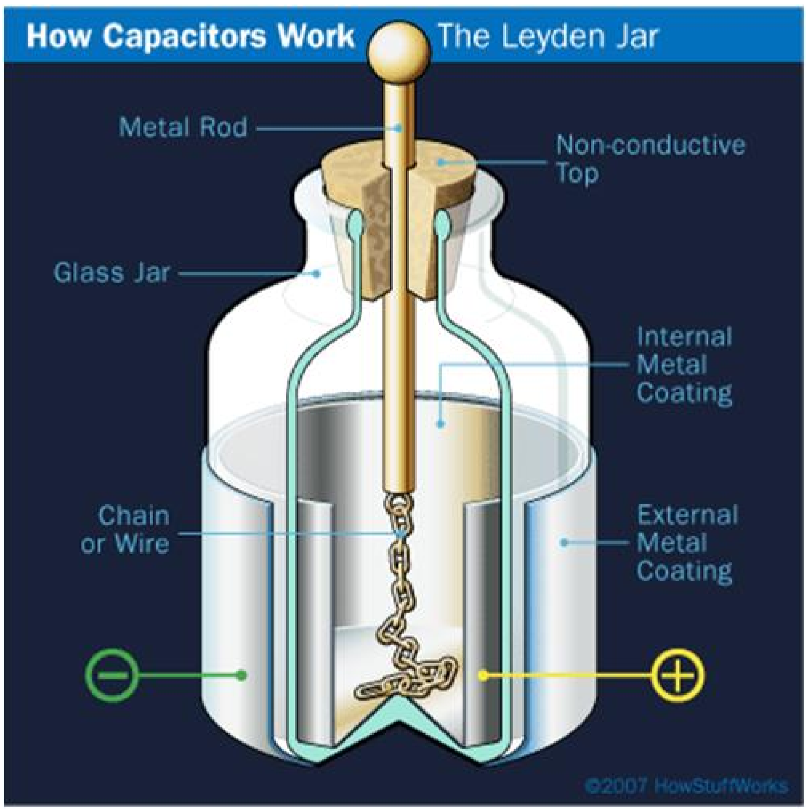
$$Q = Q_A$$
$$V = V_{AB}$$

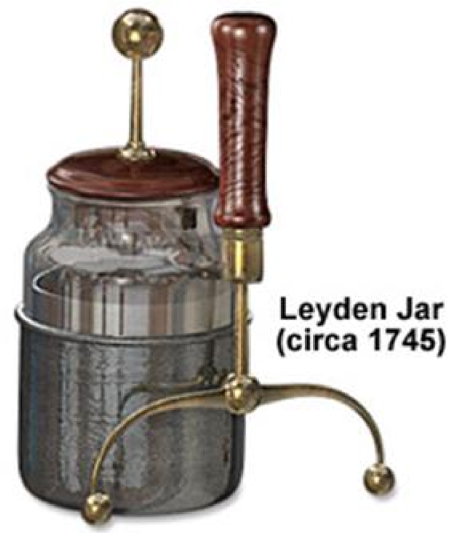
(both positive)

**Note:** The value of $C$ is always positive!

# Leyden Jar

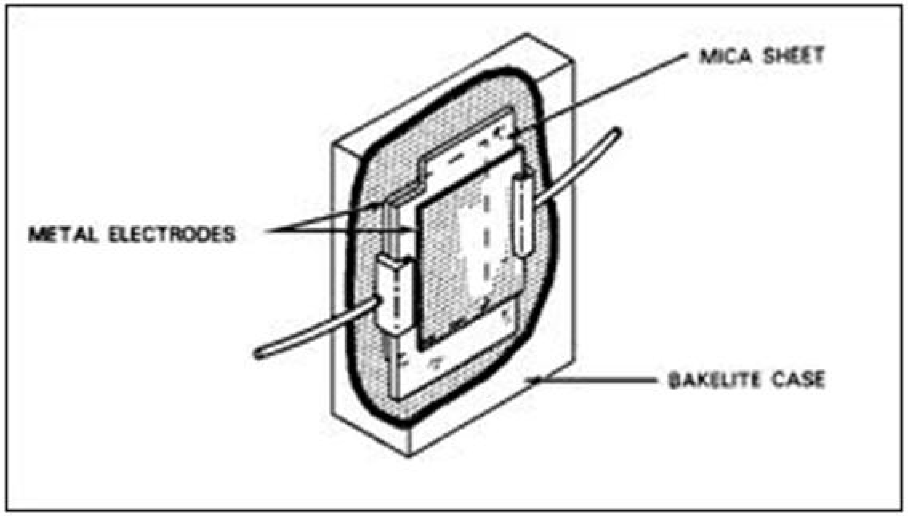
The Leyden Jar was one of the earliest capacitors. It was invented in 1745 by Pieter van Musschenbroek at the University of Leiden in the Netherlands (1746).




# Typical Capacitors

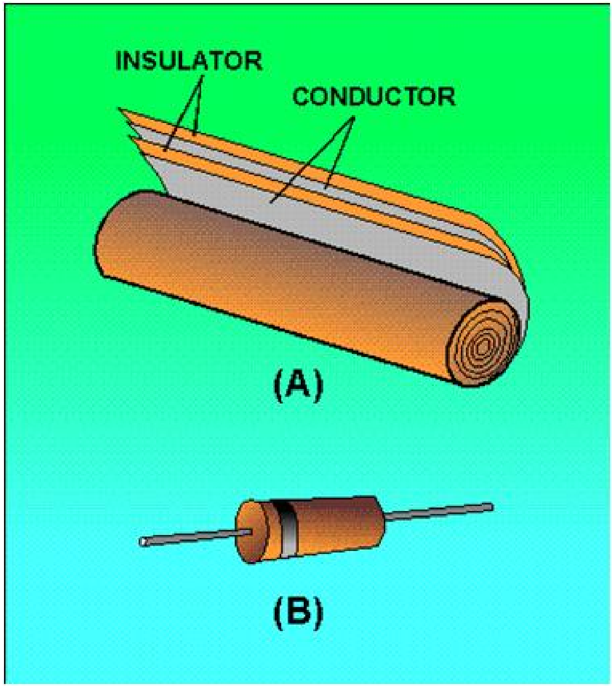

## Ceramic capacitors


The ceramic capacitor is often manufactured in the shape of a disk. After leads are attached to each side of the capacitor, the capacitor is completely covered with an insulating moisture-proof coating. Ceramic capacitors usually range in value from 1 picofarad to 0.01 microfarad and may be used with voltages as high as 30,000 volts.

# Typical Capacitors (cont.)

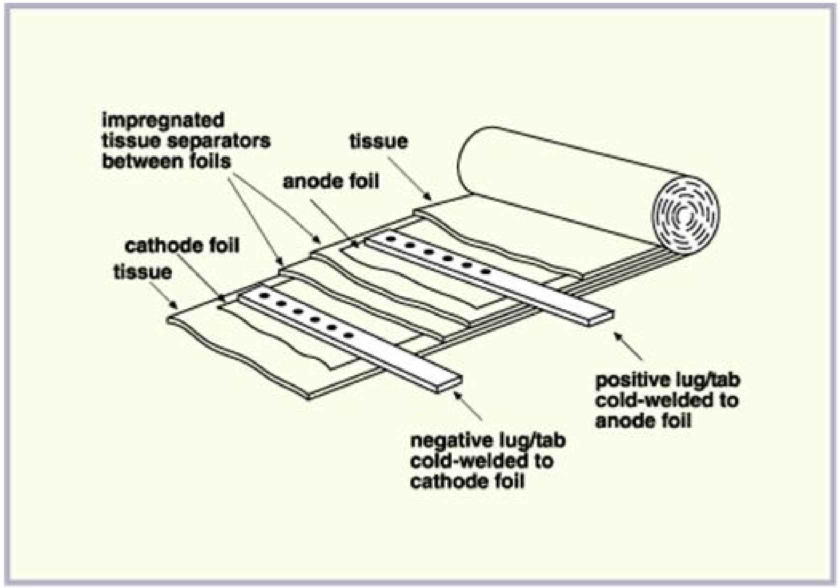
Paper capacitors


A paper capacitor is made of flat thin strips of metal foil conductors that are separated by waxed paper (the dielectric material). Paper capacitors usually range in value from about 300 picofarads to about 4 microfarads. The working voltage of a paper capacitor rarely exceeds 600 volts.

# Typical Capacitors (cont.)

## Electrolytic capacitors



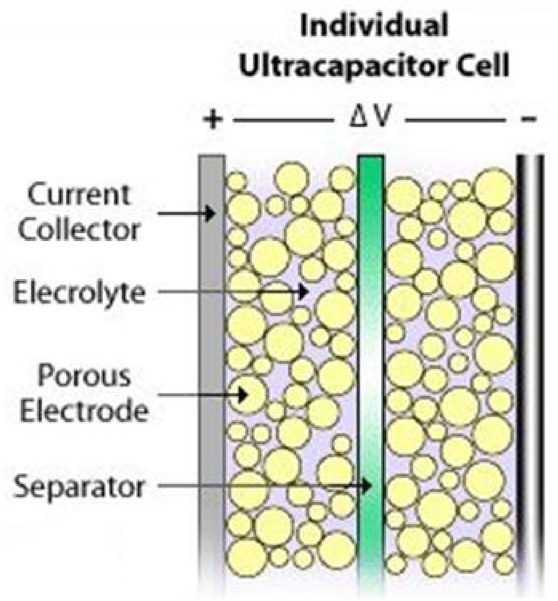


**Note:** One lead (anode) is often longer than the other to indicate polarization.

This type of capacitor uses an electrolyte (sometimes wet but often dry) and requires a <u>biasing voltage</u> (there is an anode and a cathode). A thin oxide layer forms on the anode due to an electrochemical process, creating the dielectric. Dry electrolytic capacitors vary in size from about 4 microfarads to several thousand microfarads and have a working voltage of approximately 500 volts.

# Typical Capacitors (cont.)

## Supercapacitors (Ultracapacitors)

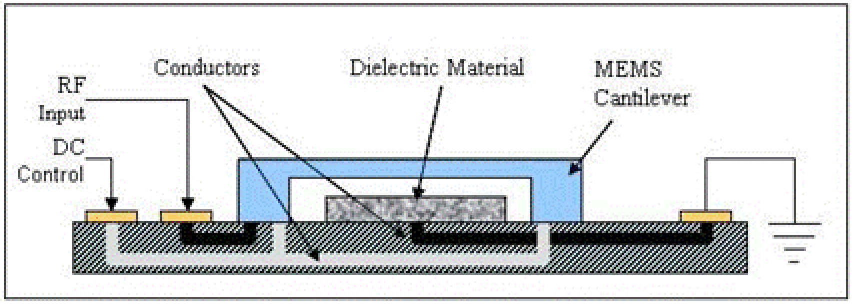



Maxwell Technologies "MC" and "BC" series supercapacitors (up to 3000 farad capacitance)

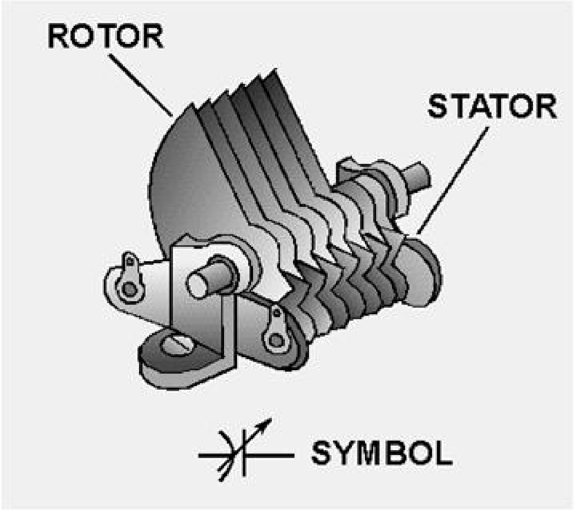
Compared to conventional electrolytic capacitors, the energy density is typically on the order of thousands of times greater.

# Typical Capacitors (cont.)

## MEMS capacitor

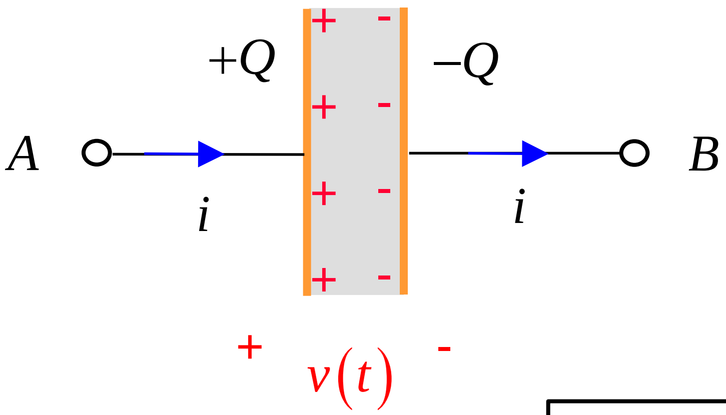



## Variable capacitor




## Capacitors for substations

# Current - Voltage Equation




$$i = \frac{dQ}{dt}$$

$$= \frac{d}{dt}(Cv)$$

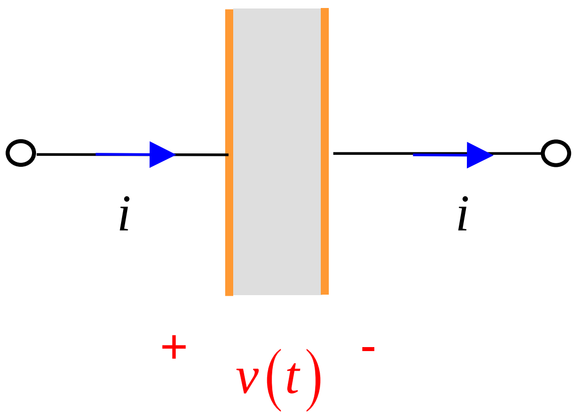
$$= C\frac{dv}{dt}$$

The reference directions for voltage and current correspond to “passive sign convention” in circuit theory.

**Note:**
$Q$ is the charge that flows from left to right, into plate $A$.

# Current - Voltage Equation (cont.)

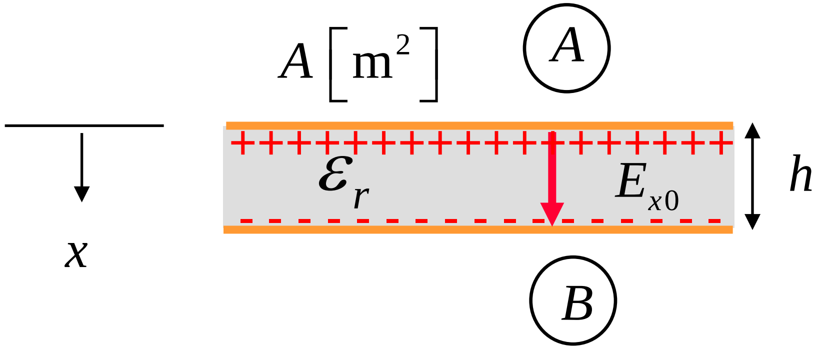
Hence we have $$i(t) = C\frac{dv}{dt}$$




"Passive sign convention"

# Example

Find $C$




$$C \equiv \frac{Q}{V}$$

Ideal parallel plate capacitor

Method #1 (start with $\underline{E}$)

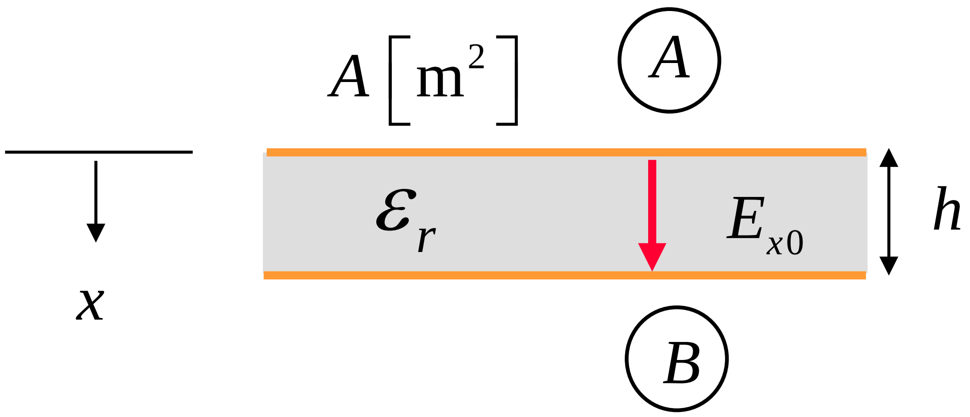
Note that the top plate has charge on the lower surface, and the bottom plate has charge on the top surface.

Assume: $\underline{E} = \underline{\hat{x}}\, E_{x0}$

$$V = V_{AB} = \int_{\underline{A}}^{\underline{B}} \underline{E} \cdot \underline{dr} = \int_0^h E_x\, dx = \int_0^h E_{x0}\, dx$$

$$= E_{x0}\, h$$

# Example (cont.)




$$
\begin{aligned}
Q = Q^A = \rho_s^A A &= A\,\underline{D}\cdot\underline{\hat{n}}_A \\
&= \varepsilon_0\varepsilon_r A\,\underline{\hat{n}}_A\cdot\underline{E} \\
&= \varepsilon_0\varepsilon_r A\,\underline{\hat{x}}\cdot\underline{E} \\
&= \varepsilon_0\varepsilon_r A E_x \\
&= \varepsilon_0\varepsilon_r A E_{x0}
\end{aligned}
$$

**Remember:**
The unit normal vector in the boundary condition formula always points outward from the conductor. On the bottom surface of the top plate, this means pointing down.

Hence

$$C = \frac{Q}{V} = \frac{\varepsilon_0\varepsilon_r A E_{x0}}{E_{x0} h}$$

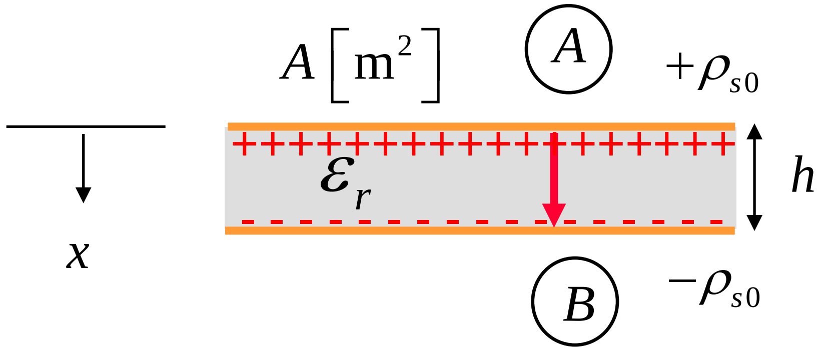
$$C = \varepsilon_0\varepsilon_r\left(\frac{A}{h}\right) \quad [\mathrm{F}]$$

**Note:**
$C$ is only a function of the geometry and the permittivity!

# Example (cont.)

Method #2 (start with $\rho_s$)

$$C \equiv \frac{Q}{V}$$




$$\underline{D} \cdot \underline{\hat{n}}_A = \rho_s^A = \rho_{s0}$$

$$\left(\underline{\hat{x}} D_x\right) \cdot \underline{\hat{x}} = \rho_{s0}$$

$$D_x = \rho_{s0}$$

$$\varepsilon_0 \varepsilon_r E_x = \rho_{s0}$$

$$\rho_s^A = \rho_{s0}$$

$$Q = Q^A = \rho_{s0} A$$

Hence

$$E_x = E_{x0} = \frac{\rho_{s0}}{\varepsilon_0 \varepsilon_r}$$

# Example (cont.)

$$V = V_{AB} = \int_{\underline{A}}^{\underline{B}} \underline{E} \cdot \underline{dr} = \int_0^h E_x dx = \int_0^h E_{x0}\, dx = h\, E_{x0}$$

Hence

$$V = h\, E_{x0} = h\left(\frac{\rho_{s0}}{\varepsilon_0 \varepsilon_r}\right)$$

Therefore,

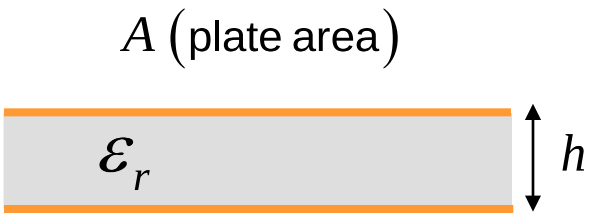
$$C = \frac{\rho_{s0} A}{h\left(\dfrac{\rho_{s0}}{\varepsilon_0 \varepsilon_r}\right)}$$

or

$$C = \varepsilon_0 \varepsilon_r \left(\frac{A}{h}\right) \quad [\mathrm{F}]$$

# Example

$\varepsilon_r = 6.0$ (mica)
$A = 1$ [cm$^2$]
$h = 0.01$ [mm]



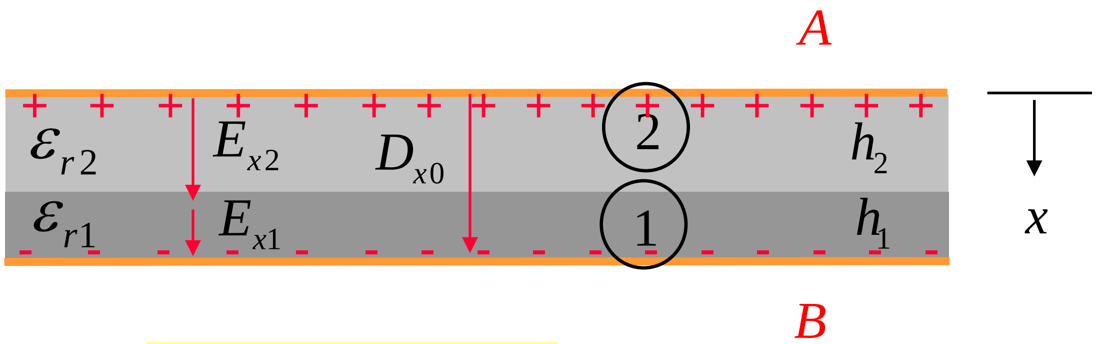

$$C = \varepsilon_0 \varepsilon_r \left( \frac{A}{h} \right)$$

$$= \left(8.854 \times 10^{-12}\right)(6.0)\left( \frac{1 \times 10^{-4}}{0.01 \times 10^{-3}} \right) \quad [\mathrm{F}]$$

$$C = 530.12\ [\mathrm{pF}] = 0.53012\ [\mathrm{nF}]$$

# Example

Find $C$




Starting assumption: $D_{x1} = D_{x2} = D_{x0}$ ($D_x$ is the same in both regions, from boundary conditions.)

**Goal:**
Put $Q$ and $V$ in terms of $D_{x0}$.

B. C. on plate $A$:

$$\underline{D} \cdot \underline{\hat{n}}_A = \rho_s$$

$$\underline{D}_2 \cdot \underline{\hat{x}} = \rho_s^A$$

$$D_{x2} = D_{x0} = \rho_s^A$$

$$Q = \rho_s^A A$$

$$C \equiv \frac{Q}{V}$$

$$\Rightarrow \quad Q = D_{x0} A$$

# Example (cont.)

$$V = E_{x1}h_1 + E_{x2}h_2$$

$$= \frac{D_{x1}}{\varepsilon_1}h_1 + \frac{D_{x2}}{\varepsilon_2}h_2$$

$$= \frac{D_{x0}}{\varepsilon_1}h_1 + \frac{D_{x0}}{\varepsilon_2}h_2$$

$$= D_{x0}\left[\frac{h_1}{\varepsilon_1} + \frac{h_2}{\varepsilon_2}\right]$$

$$V = D_{x0}\left[\frac{h_1}{\varepsilon_1} + \frac{h_2}{\varepsilon_2}\right]$$

# Example (cont.)

Hence:

$$C = \frac{Q}{V} = \frac{D_{x0}\ A}{\left(\frac{h_1}{\varepsilon_1} + \frac{h_2}{\varepsilon_2}\right) D_{x0}}$$

$$= \frac{A}{\frac{h_1}{\varepsilon_1} + \frac{h_2}{\varepsilon_2}} = \frac{1}{\frac{h_1}{\varepsilon_1 A} + \frac{h_2}{\varepsilon_2 A}}$$

$$= \frac{1}{\frac{1}{\left(\frac{\varepsilon_1 A}{h_1}\right)} + \frac{1}{\left(\frac{\varepsilon_2 A}{h_2}\right)}}$$

or

$$C = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2}}$$

where

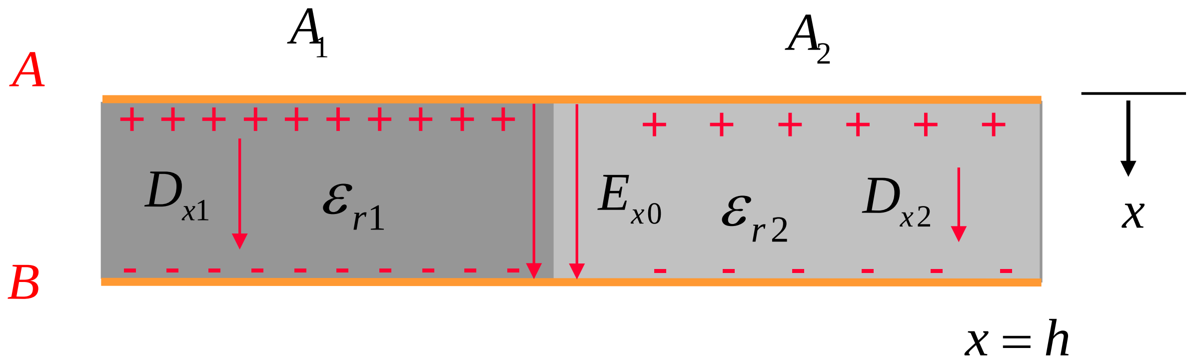
$$C_1 = \frac{\varepsilon_1 A}{h_1} \qquad C_2 = \frac{\varepsilon_2 A}{h_2}$$

Hence,

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}$$

# Example

Find $C$




Starting assumption: $E_{x1} = E_{x2} = E_{x0}$ ($E_x$ is the same in both regions, from boundary conditions.)

$$V = E_{x0}\, h$$

**Goal:**
Put $Q$ and $V$ in terms of $E_{x0}$.

$$\rho_{s1}^{A} = D_{x1} = \varepsilon_1 E_{x1} = \varepsilon_1 E_{x0}$$

$$\rho_{s2}^{A} = D_{x2} = \varepsilon_2 E_{x2} = \varepsilon_2 E_{x0}$$

$$C \equiv \frac{Q}{V}$$

# Example (cont.)

$$Q = \rho_{s1} A_1 + \rho_{s2} A_2$$

$$= \varepsilon_1 E_{x0} A_1 + \varepsilon_2 E_{x0} A_2$$

$$C = \frac{Q}{V} = \frac{\varepsilon_1 A_1 E_{x0} + \varepsilon_2 A_2 E_{x0}}{E_{x0} h}$$

or

$$C = \frac{\varepsilon_1 A_1}{h} + \frac{\varepsilon_2 A_2}{h}$$

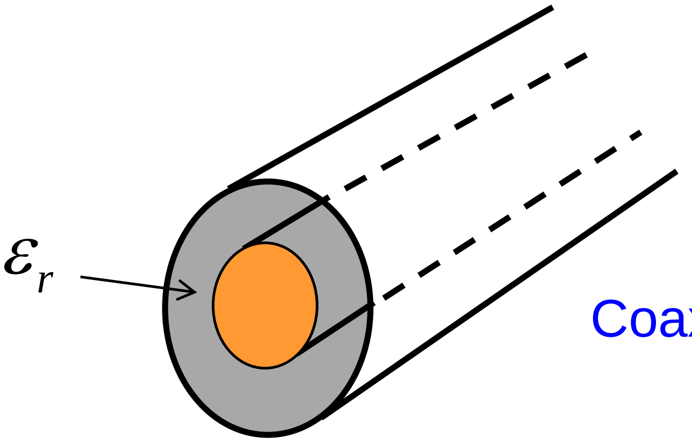
$$C = C_1 + C_2$$

where

$$C_1 = \frac{\varepsilon_1 A_1}{h}$$

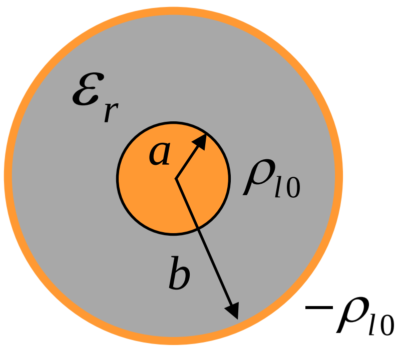
$$C_2 = \frac{\varepsilon_2 A_2}{h}$$

# Example




Find $C_l$

(capacitance / length)




$$\underline{E} = \underline{\hat{\rho}}\left(\frac{\rho_{l0}}{2\pi\varepsilon\rho}\right) = \underline{\hat{\rho}}\left(\frac{\rho_{l0}}{2\pi\varepsilon_0\varepsilon_r\rho}\right)$$

$$V = \int_{\underline{A}}^{\underline{B}} \underline{E}\cdot\underline{dr} = \int_a^b E_\rho \, d\rho$$

$$C_l \equiv \frac{\rho_{l0}}{V}$$

**Note :**

$\rho_{s0}^{a} = \rho_{l0} / (2\pi a)$

$\rho_{s0}^{b} = -\rho_{l0} / (2\pi b)$

# Example (cont.)

Hence we have

$$V = \frac{\rho_{\ell 0}}{2\pi\varepsilon_0\varepsilon_r}\int_a^b \frac{1}{\rho}d\rho$$

$$= \frac{\rho_{\ell 0}}{2\pi\varepsilon_0\varepsilon_r}\ln\left(\frac{b}{a}\right)$$

Therefore

$$C_l = \frac{\rho_{\ell 0}}{V} = \frac{\rho_{\ell 0}}{\left(\frac{\rho_{\ell 0}}{2\pi\varepsilon_0\varepsilon_r}\right)\ln\left(\frac{b}{a}\right)}$$

We then have

$$C_l = \frac{2\pi\varepsilon_0\varepsilon_r}{\ln\left(\frac{b}{a}\right)} \quad [\text{F/m}]$$

# Example (cont.)

Formula from ECE 3317:

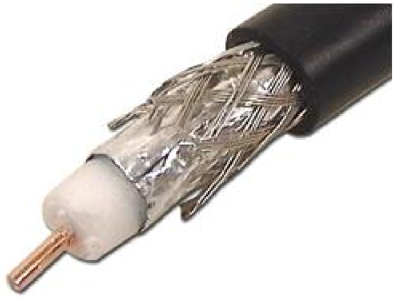
$$Z_0 = \sqrt{\frac{L_l}{C_l}}$$

(characteristic impedance)

The characteristic impedance is one of the most important numbers that characterizes a transmission line.

Coax for cable TV: $Z_0 = 75\ [\Omega]$
Twin lead for TV: $Z_0 = 300\ [\Omega]$

Coaxial cable



Twin lead

# Example (cont.)

Formula from ECE 3317:

$$v_p = \text{phase velocity} = \frac{1}{\sqrt{L_l C_l}} = \frac{1}{\sqrt{\mu\varepsilon}}$$

Assume

$$\mu = \mu_0$$

Hence

$$L_l = \frac{\mu_0 \varepsilon}{C_l}$$

$$\varepsilon = \varepsilon_0 \varepsilon_r$$

Using our expression for $C_l$ for the coax gives:

$$L_l = \frac{\mu_0}{2\pi} \ln\left(\frac{b}{a}\right) \quad [\text{H/m}]$$

# Example (cont.)

Hence, the characteristic impedance of the coaxial cable transmission line is:

$$Z_0 = \frac{\eta_0}{2\pi\sqrt{\varepsilon_r}} \ln\left(\frac{b}{a}\right) \quad [\Omega]$$

$$Z_0 = \sqrt{\frac{L_l}{C_l}}$$

$$C_l = \frac{2\pi\varepsilon_0\varepsilon_r}{\ln\left(\frac{b}{a}\right)} \quad [\text{F/m}]$$

$$L_l = \frac{\mu_0}{2\pi} \ln\left(\frac{b}{a}\right) \quad [\text{H/m}]$$

where

$$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 376.7603 \;\; [\Omega]$$ (intrinsic impedance of free space)

$$\varepsilon_0 \doteq 8.854187 \times 10^{-12} \;\; [\text{F/m}]$$

$$\mu_0 = 4\pi \times 10^{-7} \;\; [\text{H/m}]$$