ECE 3318
Applied Electricity and Magnetism

Spring 2023

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Notes 26
Electric Stored Energy

Stored Energy

Goal:

- Calculate the amount of stored electric energy in space, due to the presence of an electric field.

(Recall that in statics the electric field is produced by a charge density.)

Stored Energy (cont.)

Charge formula:

$$U_E = \frac{1}{2}\int_V \rho_v \Phi \, dV$$ (volume charge density)

$$U_E = \frac{1}{2}\int_S \rho_s \Phi \, dS$$ (surface charge density)

Note:
We don't have a line-charge version of the charge formulas, since a line charge would have an infinite stored energy (as would a point charge).

The charge formulas require $\Phi(\underline{\infty}) = 0$

Electric-field formula:

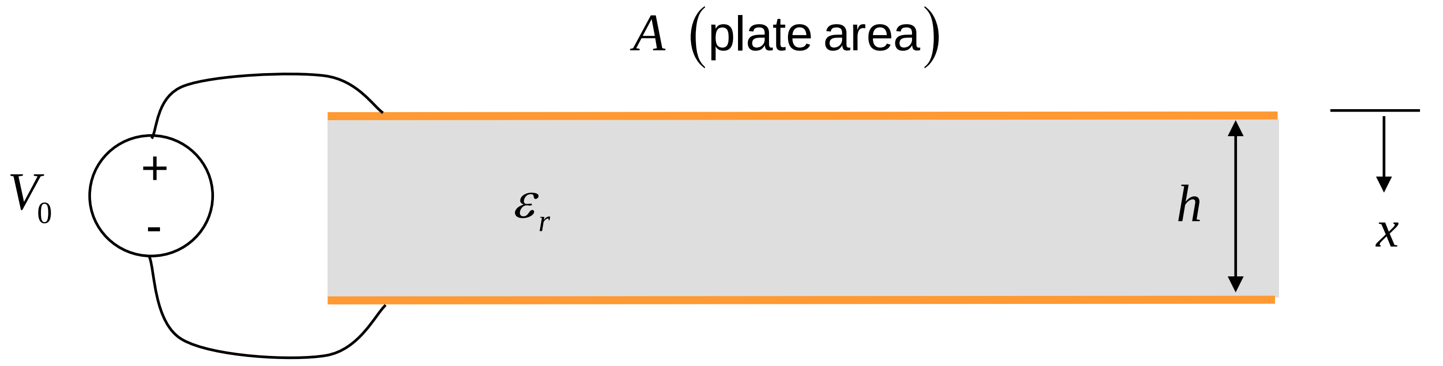
$$U_E = \frac{1}{2}\int_V \underline{D} \cdot \underline{E} \, dV$$

Note:
The charge and electric field formulas will give the same answer, provided that the integration regions include all places where there is a charge or a field.

Please see the textbook for a derivation of these formulas.

Example

Find the stored energy.



We assume an ideal parallel-plate capacitor (no fields outside).

$$\underline{E} = \underline{\hat{x}}\left(\frac{V_0}{h}\right)$$

$$\underline{D} = \varepsilon_0 \varepsilon_r \underline{\hat{x}}\left(\frac{V_0}{h}\right)$$

Method #1

Use the electric field formula:

$$U_E = \frac{1}{2}\int_V \underline{D} \cdot \underline{E}\, dV$$

so

$$U_E = \frac{1}{2}\int_V \varepsilon_0 \varepsilon_r \left(\frac{V_0}{h}\right)^2 dV$$

Example (cont.)

Hence

$$U_E = \frac{1}{2}\varepsilon_0\varepsilon_r\left(\frac{V_0}{h}\right)^2(Ah) \quad [\mathrm{J}]$$

We can also write this as

$$U_E = \frac{1}{2}\left(\varepsilon_0\varepsilon_r\frac{A}{h}\right)V_0^2$$

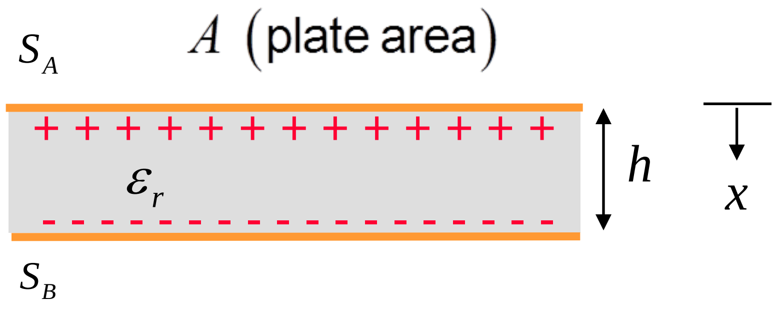
Recall that

$$C = \varepsilon_0\varepsilon_r\frac{A}{h}$$

Therefore, we have

$$U_E = \frac{1}{2}CV_0^2$$

Example (cont.)



Method #2

Use charge formula:

$$U_E = \frac{1}{2}\int_S \rho_s \Phi \, dS$$

(We use the surface-charge form of the formula.)

We then have:

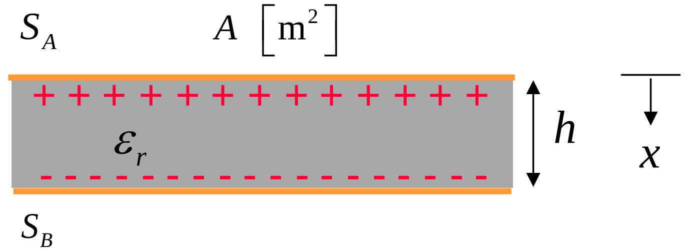
$$U_E = \frac{1}{2}\Phi_A \int_{S_A} \rho_s \, dS + \frac{1}{2}\Phi_B \int_{S_B} \rho_s \, dS$$

Example (cont.)

$$U_E = \frac{1}{2}\Phi_A Q_A + \frac{1}{2}\Phi_B Q_B$$

$$= \frac{1}{2}\Phi_A Q_A - \frac{1}{2}\Phi_B Q_A$$

$$= \frac{1}{2}Q\left(\Phi_A - \Phi_B\right)$$



so

$$U_E = \frac{1}{2}QV_0$$

Also, we can write this as

$$U_E = \frac{1}{2}QV_0$$

$$= \frac{1}{2}\left(CV_0\right)V_0$$

or

$$U_E = \frac{1}{2}CV_0^2$$

Note: This same derivation (method 2) actually holds for <u>any shape capacitor</u>.

Alternative Capacitance Formula

Using the result from the previous example, we can write

$$C = \frac{2U_E}{V^2}$$

where

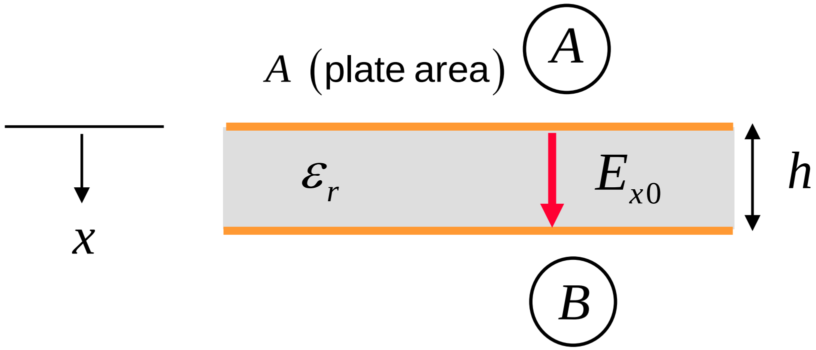
U_E = stored energy in capacitor

V = voltage on capacitor

This gives us an <u>alternative</u> method for calculating capacitance.

Alternative Capacitance Formula (cont.)

Find C (using the alternative formula).



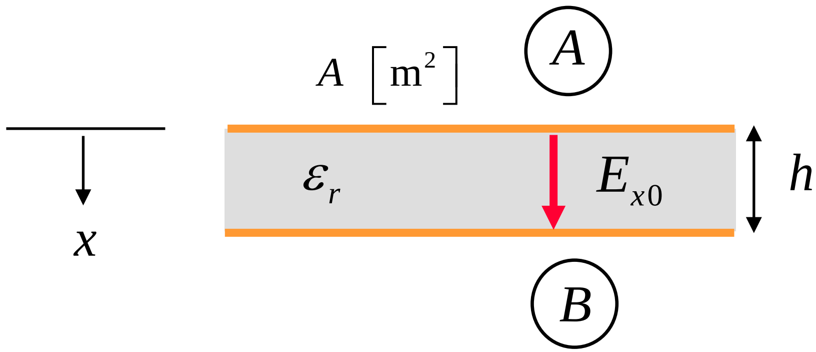
Ideal parallel plate capacitor

Assume: $\underline{E} = \underline{\hat{x}}\, E_{x0}$

$$V = V_{AB} = \int_{\underline{A}}^{\underline{B}} \underline{E} \cdot \underline{dr} = \int_0^h E_x \, dx = \int_0^h E_{x0} \, dx$$

$$= E_{x0} \, h$$

Alternative Capacitance Formula (cont.)



$$U_E = \frac{1}{2}\int_V \underline{D}\cdot\underline{E}\,dV = \frac{1}{2}\int_V \varepsilon\left(\underline{E}\cdot\underline{E}\right)dV = \frac{1}{2}\int_V \varepsilon\left|\underline{E}\right|^2 dV = \frac{1}{2}\varepsilon E_{x0}^2\left(Ah\right)$$

Therefore

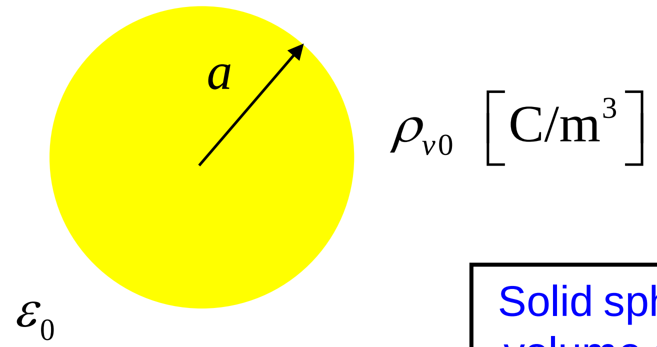
$$C = \frac{2U_E}{V^2} = \frac{2\left(\frac{1}{2}\varepsilon E_{x0}^2\left(Ah\right)\right)}{\left(E_{x0}\,h\right)^2}$$

so

$$C = \frac{\varepsilon A}{h}$$

Example

Find the stored energy.



$$U_E = \frac{1}{2}\int_V \underline{D}\cdot\underline{E}\,dV$$

Solid sphere of uniform volume charge density (not a capacitor!)

$$\underline{E} = \underline{E}(r) = \underline{\hat{r}}\,E_r(r)$$

Gauss's Law:

$r < a$

$$\underline{E} = \underline{\hat{r}}\left(\frac{\rho_{v0}\left(\frac{4}{3}\pi r^3\right)}{4\pi\varepsilon_0 r^2}\right)$$

$r > a$

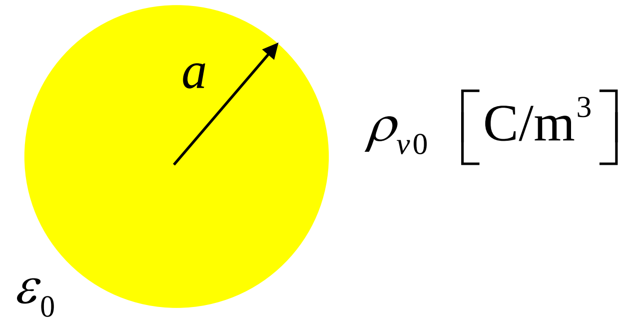
$$\underline{E} = \underline{\hat{r}}\left(\frac{\rho_{v0}\frac{4}{3}\pi a^3}{4\pi\varepsilon_0 r^2}\right)$$

Example (cont.)

$$U_E = \frac{\varepsilon_0}{2}\int_V \left|\underline{E}\right|^2 dV = \frac{\varepsilon_0}{2}\int_V E_r^2(r)\,dV$$

$$= \frac{\varepsilon_0}{2}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} E_r^2(r)\; r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= \frac{\varepsilon_0}{2}(2\pi)(2)\int_0^{\infty} E_r^2(r)\; r^2 dr$$



Hence we have

$$U_E = 2\pi\varepsilon_0\int_0^a \left(\frac{\rho_{v0}\frac{4}{3}\pi r^3}{4\pi\varepsilon_0 r^2}\right)^2 r^2 dr + 2\pi\varepsilon_0\int_a^{\infty}\left(\frac{\rho_{v0}\frac{4}{3}\pi a^3}{4\pi\varepsilon_0 r^2}\right)^2 r^2 dr$$

Example (cont.)

Result after simplifying:

$$U_E = \frac{\rho_{v0}^2}{\varepsilon_0}\left(\frac{4\pi}{15}\right)a^5$$

Denote $Q = \rho_{v0}\left(\frac{4}{3}\pi a^3\right)$

so that $\rho_{v0} = Q\left(\frac{3}{4\pi a^3}\right)$

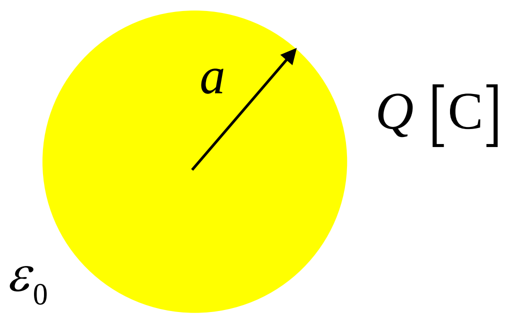
We then have:

$$U_E = \frac{1}{\varepsilon_0}\left(Q\left(\frac{3}{4\pi a^3}\right)\right)^2\left(\frac{4\pi}{15}\right)a^5$$

Example (cont.)

Final result:

$$U_E = \frac{1}{a}\left(\frac{Q^2}{\varepsilon_0}\right)\left(\frac{3}{20\pi}\right) \quad [\mathrm{J}]$$



Note:
This is the energy that it takes to assemble (force together) the charge inside the cloud from infinity.

Note:

$$U_E \to \infty$$

as $a \to 0$

It takes an infinite amount of energy to make an ideal point charge!

(The same conclusion holds for an ideal line charge.)

Example

Find the "radius" of an electron.

$$U_E = \frac{1}{a}\left(\frac{Q^2}{\varepsilon_0}\right)\left(\frac{3}{20\pi}\right) \quad \Rightarrow \quad a = \frac{Q^2}{U_E}\left(\frac{3}{20\pi\varepsilon_0}\right)$$

$$Q = q_e = -1.6022 \times 10^{-19} \;\; [\mathrm{C}]$$

Assume that all of the stored energy is in the form of electrostatic energy: $U_E = mc^2$

We have:

$$\left.\begin{array}{l} m = 9.1094 \times 10^{-31} \;\; [\mathrm{kg}] \\ c \equiv 2.99792458 \times 10^{8} \;\; [\mathrm{m/s}] \end{array}\right\} \Rightarrow \quad U_E = mc^2 = 8.1871 \times 10^{-14} \;\; [\mathrm{J}]$$

Hence $a = 1.69 \times 10^{-15} \;\; [\mathrm{m}]$ This is one form of the "classical electron radius".