ECE 3318
Applied Electricity and Magnetism

Spring 2023

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Dept. of ECE

Notes 27
DC Currents

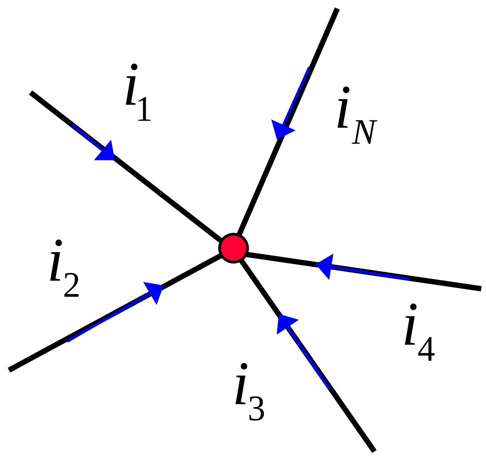
KCL Law

Wires meet at a “node”

$$\sum_{n=1}^{N} i_n = 0$$

or

$$i_{in}^{tot} = 0$$



Note:
The “node” can be a single point or a larger region.

Or we can say $i_{out}^{tot} = 0$

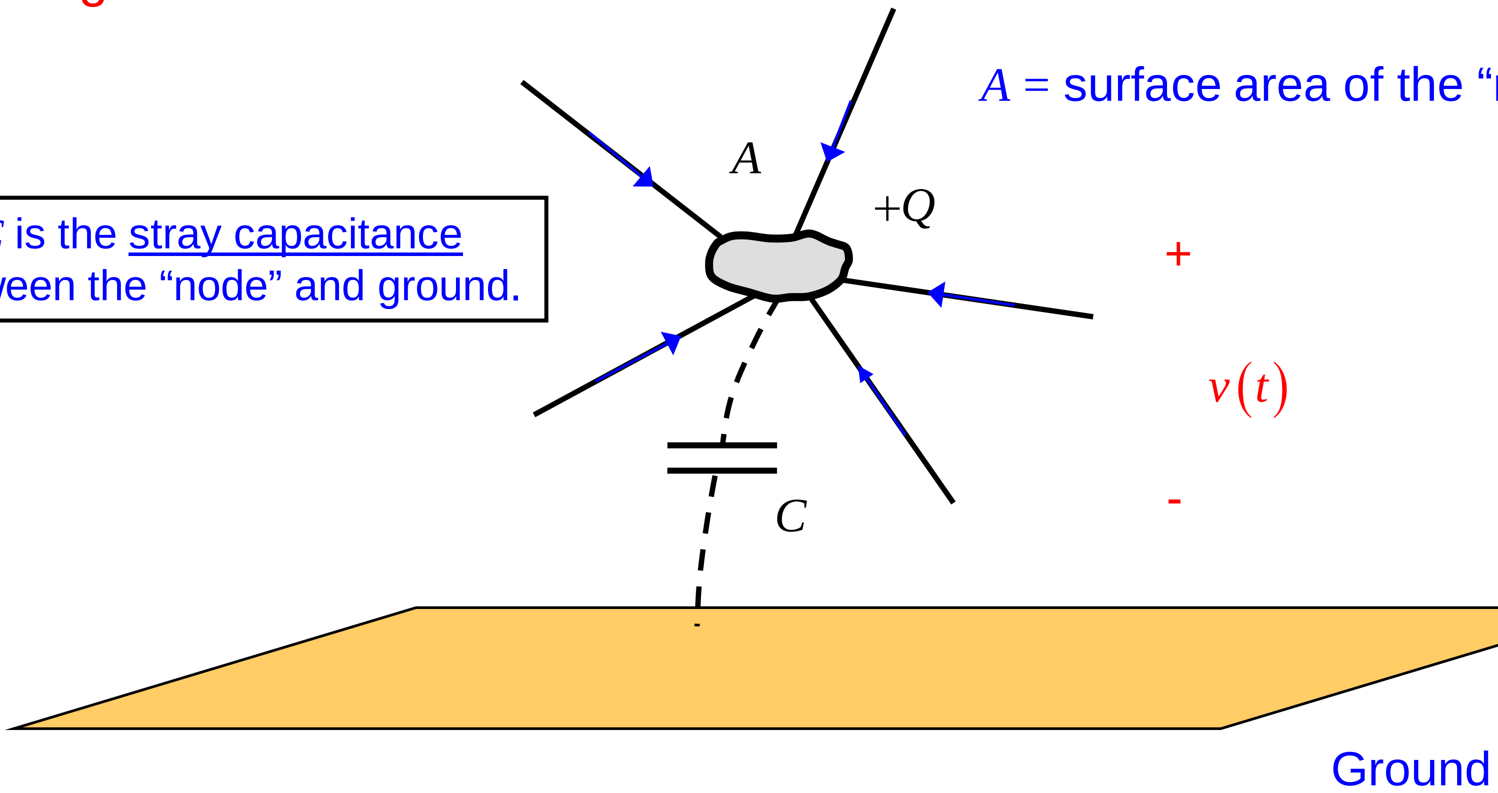
A proof of the KCL law is given next.

KCL Law (cont.)

A single node is considered.

C is the stray capacitance between the "node" and ground.

A = surface area of the "node."



The node forms the top (anode) plate of a *stray capacitor*.

$$i_{in}^{tot} = C\frac{dv}{dt}$$

KCL Law (cont.)

Two cases for which the KCL law is valid:

$$i_{in}^{tot} = C\frac{dv}{dt}$$

1) In "steady state" (no time change)

$$\frac{dv}{dt} = 0$$

$$\Rightarrow \quad i_{in}^{tot} = 0$$

2) As area of node $A \to 0$

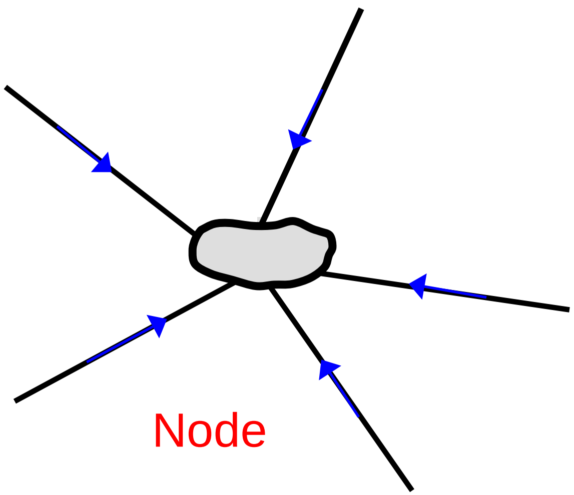
$$A \to 0 \quad \Rightarrow \quad C \to 0$$

$$\Rightarrow \quad i_{in}^{tot} = 0$$

KCL Law (cont.)

In general, the KCL law will be accurate if the size of the "node" is small compared with the wavelength λ_0.

Currents enter a node at some frequency f.



f	λ_0
60 Hz	5000 [km]
1 kHz	300 [km]
1 MHz	300 [m]
1 GHz	30 [cm]

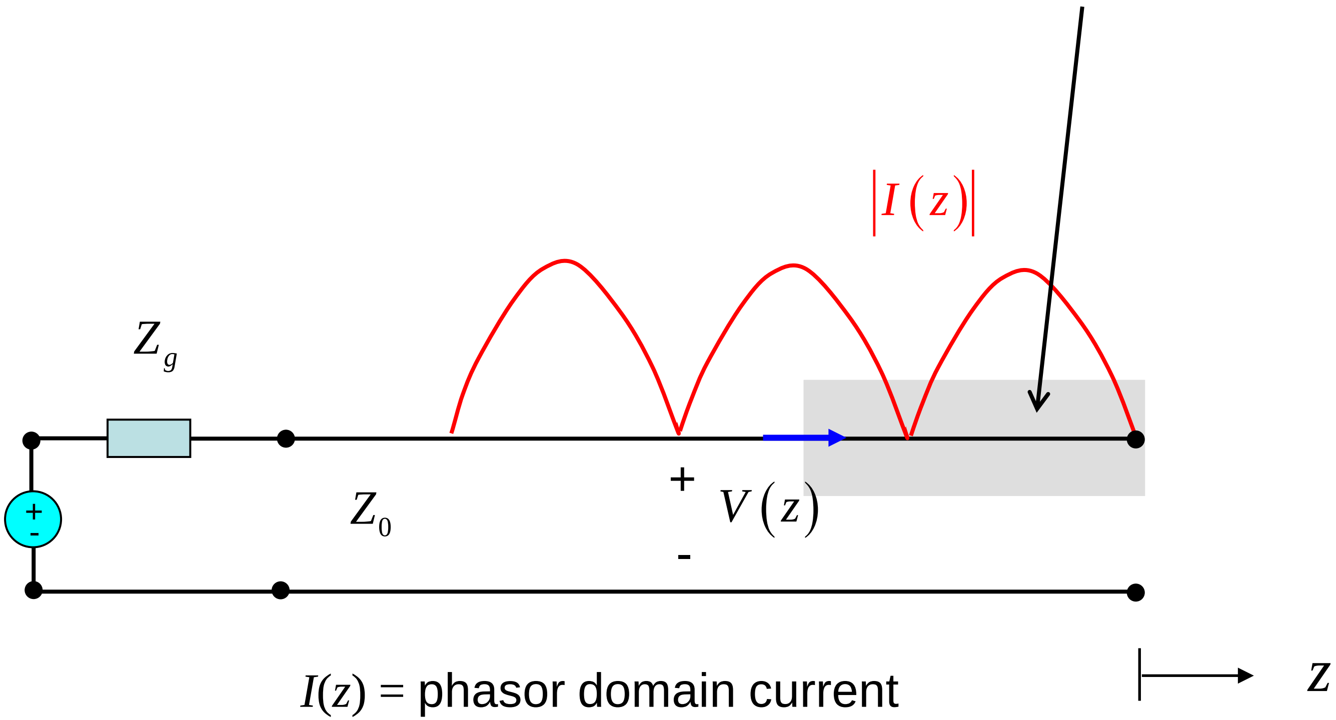
$$\lambda_0 = \frac{c}{f} = \frac{2.99792458 \times 10^8}{f}$$

KCL Law (cont.)

Example where the KCL is <u>not</u> valid

Open-circuited transmission line (ECE 3317)

Current enters this shaded region ("node") but does not leave.



$I(z)$ = phasor domain current

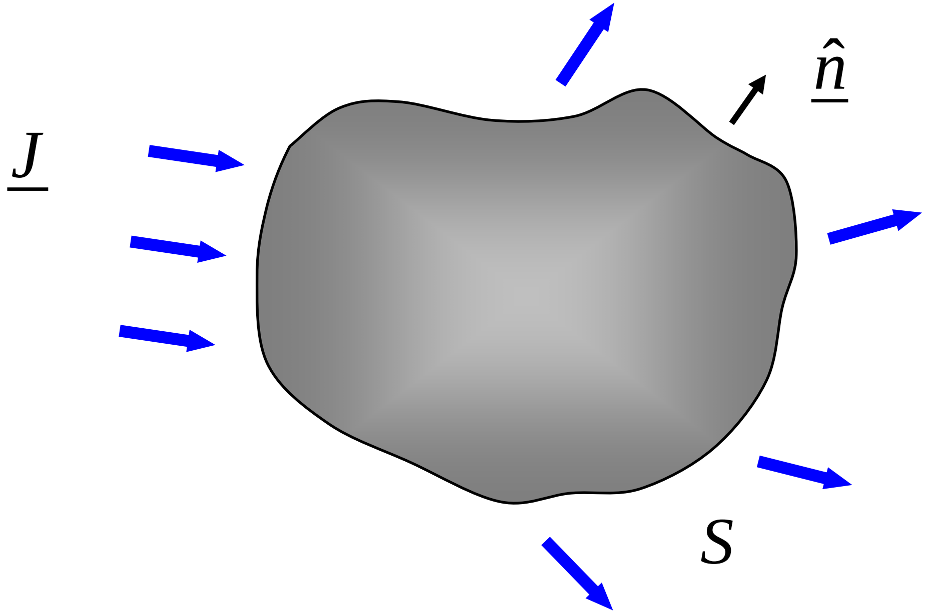
KCL Law (cont.)

General volume (3D) form of KCL equation:

$$i_{out} = \oint_S \underline{J} \cdot \underline{\hat{n}} \, dS = 0$$

(valid for D.C. currents)

The total current flowing out (or in) must be zero: whatever flows in must flow out.



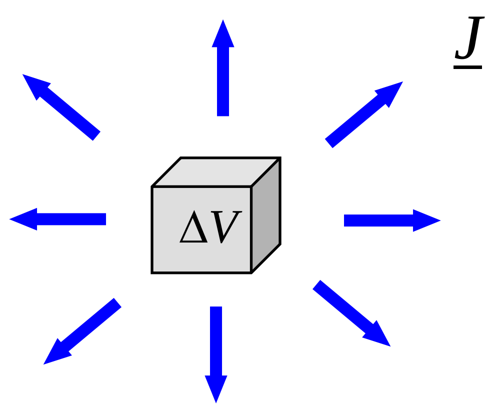
KCL Law (Differential Form)

To obtain the *differential form* of the KCL law for static (D.C.) currents, start with the definition of divergence:

$$\nabla \cdot \underline{J} \equiv \lim_{\Delta V \to 0} \frac{1}{\Delta V} \oint_{\Delta S} \underline{J} \cdot \underline{\hat{n}} \, dS$$

For the right-hand side:

$$\oint_{\Delta S} \underline{J} \cdot \underline{\hat{n}} \, dS = i_{out} = 0$$



Hence

$$\nabla \cdot \underline{J} = 0$$

(valid for D.C. currents)

Important Current Formulas

These two formulas hold in general (not only at DC):

Ohm's Law

$$\underline{J} = \sigma \underline{E}$$

(This is an experimental law that was introduced earlier in the semester.)

Charge-Current Formula

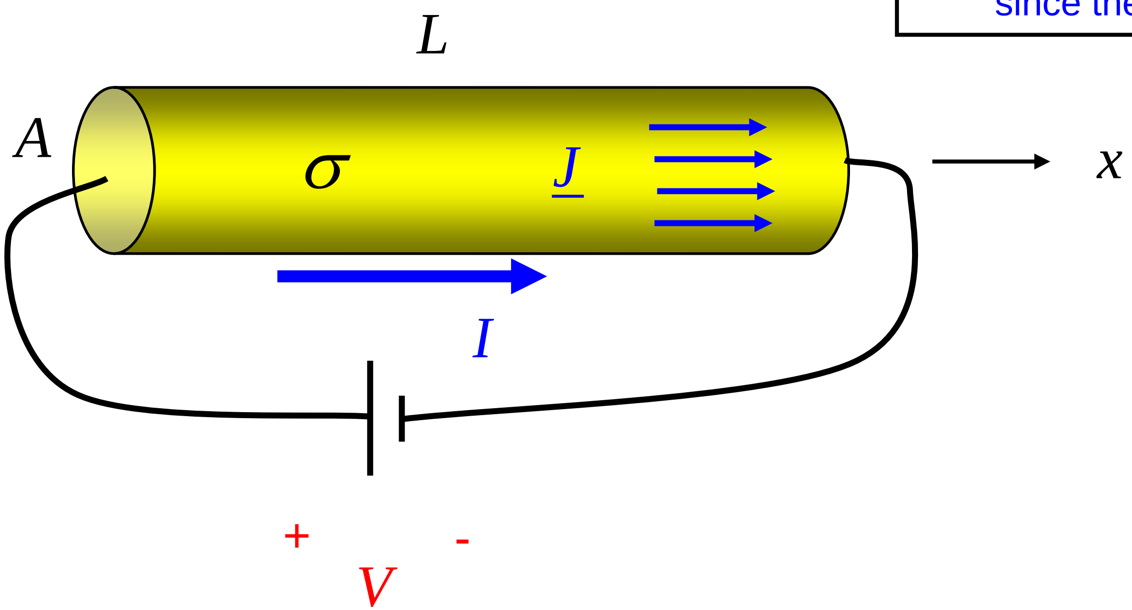
$$\underline{J} = \rho_v \underline{v}$$

(This was derived earlier in the semester.)

Resistor Formula

A long narrow resistor: $\underline{E} \approx \underline{\hat{x}}\, E_{x0}$

Note:
The electric field is constant (does not change with *x*) since the current must be uniform (KCL law).



$$E_{x0} = \frac{V}{L}$$

$$J_x = \sigma E_{x0} = \sigma \left(\frac{V}{L} \right)$$

$$I = J_x A = \sigma \left(\frac{V}{L} \right) A$$

Solve for V from the last equation: $V = I \left(\dfrac{L}{\sigma A} \right)$

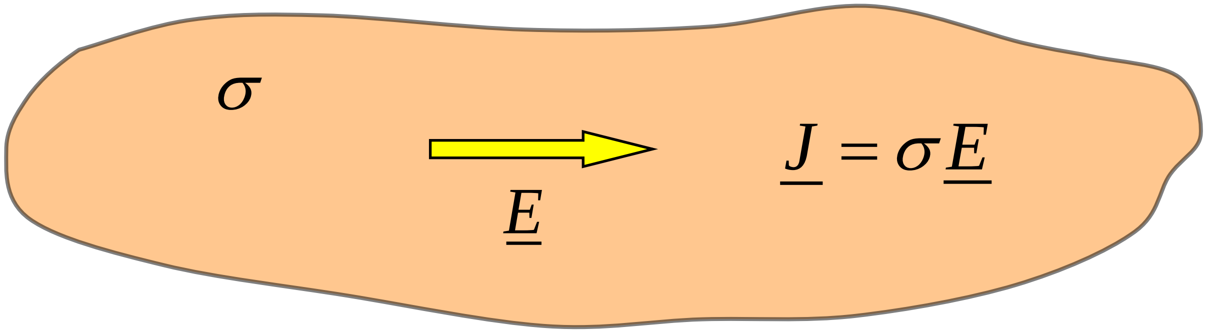
We also have that $R = \dfrac{V}{I}$

Hence we have

$$R = \left(\frac{L}{\sigma A} \right) \quad [\Omega]$$

Joule's Law

(Please see Appendix A for a derivation.)



Conducting body

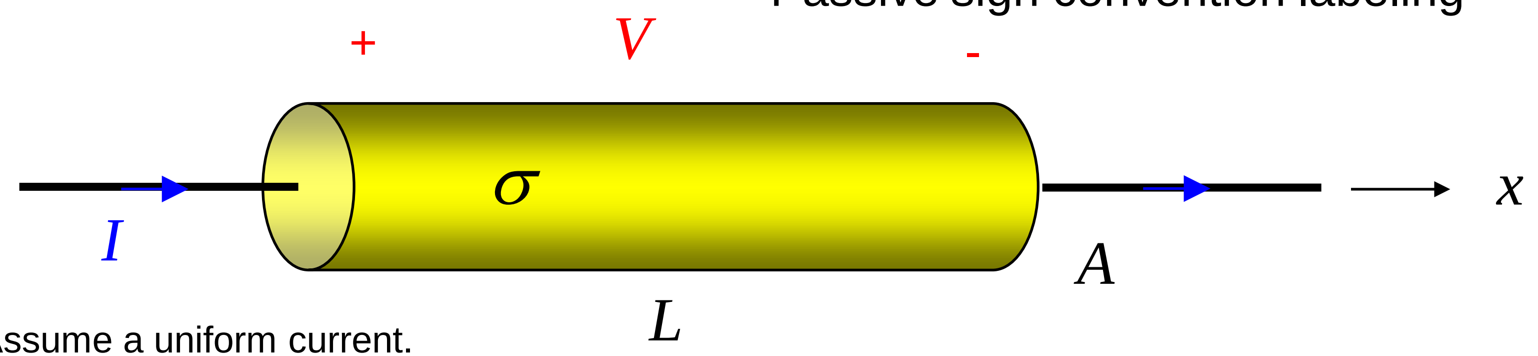
The power dissipated inside the body as heat is:

$$P_d = \int_V \underline{J} \cdot \underline{E} \, dV = \int_V \sigma \left| \underline{E} \right|^2 dV = \int_V \frac{\left| \underline{J} \right|^2}{\sigma} \, dV \quad [\mathrm{W}]$$

Power Dissipation by Resistor

Resistor

Passive sign convention labeling



Assume a uniform current.

$$\Delta V = AL$$

$$P_d = J_x E_x \Delta V = \left(\frac{I}{A}\right)\left(\frac{V}{L}\right)\Delta V = IV$$

Hence

$$P_d = VI$$

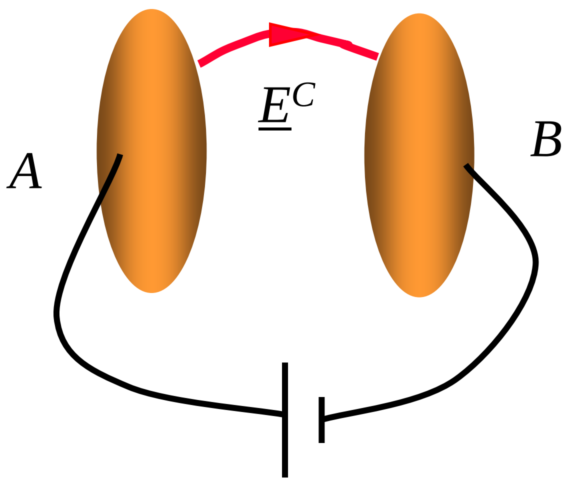
$$P_d = RI^2$$

Note: The passive sign convention applies to the VI formula.

RC Analogy

Insulating material

$$\varepsilon(\underline{r}) = F(\underline{r})$$



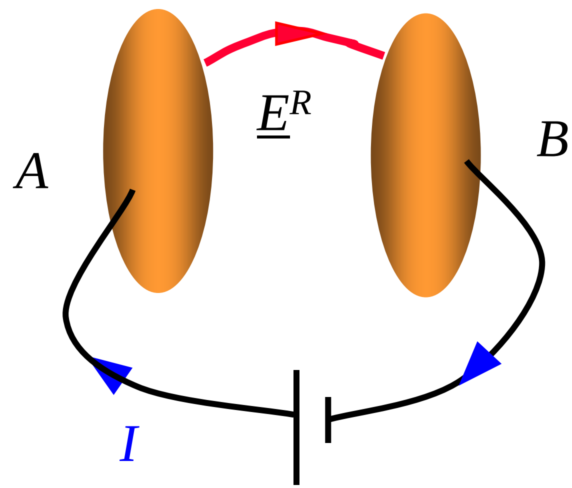
$+ \quad V_{AB} \quad -$

"C problem"

(same conductors)

Conducting material

$$\sigma(\underline{r}) = F(\underline{r})$$



$+ \quad V_{AB} \quad -$

"R problem"

Goal:

Assuming we know how to solve the C problem (i.e., find C), can we solve the R problem (find R)?

RC Analogy

(Please see Appendix B for a derivation.)

Recipe for calculating resistance:

1) Calculate the capacitance of the corresponding C problem.
2) Replace ε everywhere with σ to obtain G.
3) Take the reciprocal to obtain R.

In symbolic form:

$$\varepsilon \to \sigma$$
$$C \to G$$

RC Formula

(Please see Appendix B for a derivation.)

This is a special case: A homogeneous medium of conductivity σ surrounds the two conductors (there is only one value of σ).

σ = conductivity in resistor problem

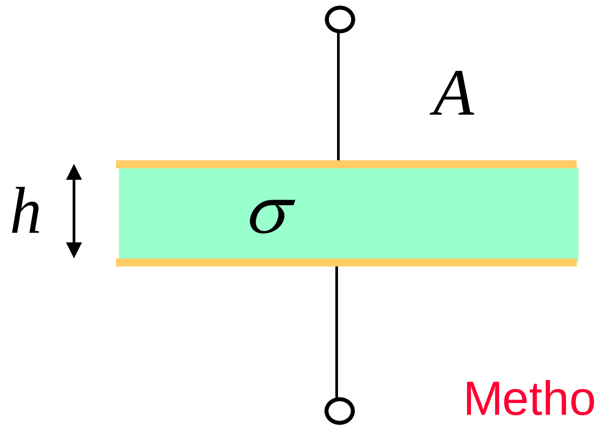
ε = permittivity in capacitor problem

The resistance R of the resistor problem is related to the capacitance C of the capacitor problem as follows:

$$RC = \left(\frac{\varepsilon}{\sigma} \right)$$

Example

Find R

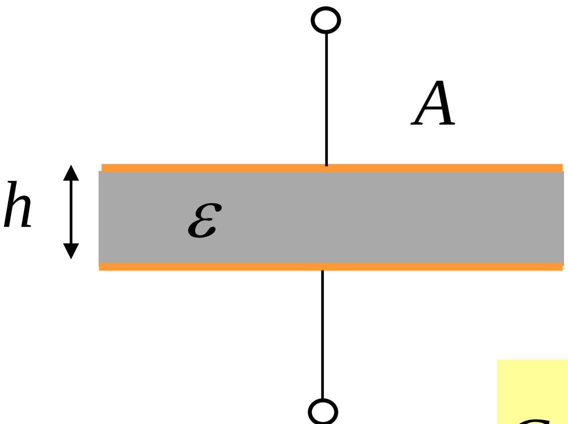


Method #1 (RC analogy or "recipe")

$$\varepsilon \rightarrow \sigma$$

$$C \rightarrow G$$

C problem:



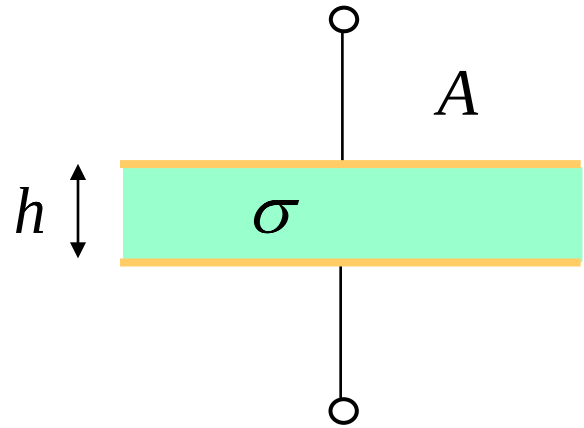
$$C = \frac{\varepsilon A}{h}$$

$$C = \frac{\varepsilon A}{h} \Rightarrow G = \frac{\sigma A}{h}$$

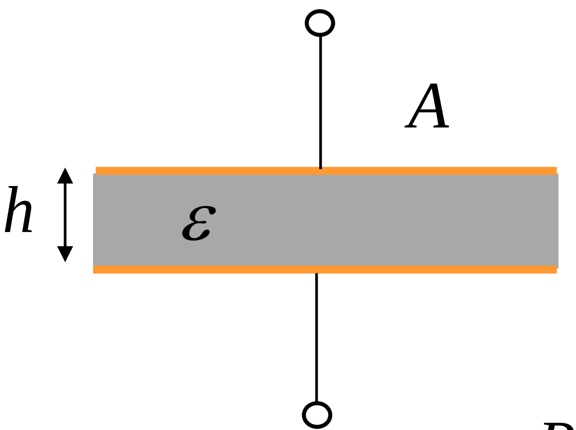
Hence $$R = \frac{h}{\sigma A}$$

Example (cont.)

Find R



C problem:



$$C = \frac{\varepsilon A}{h}$$

Method #2 (RC formula)

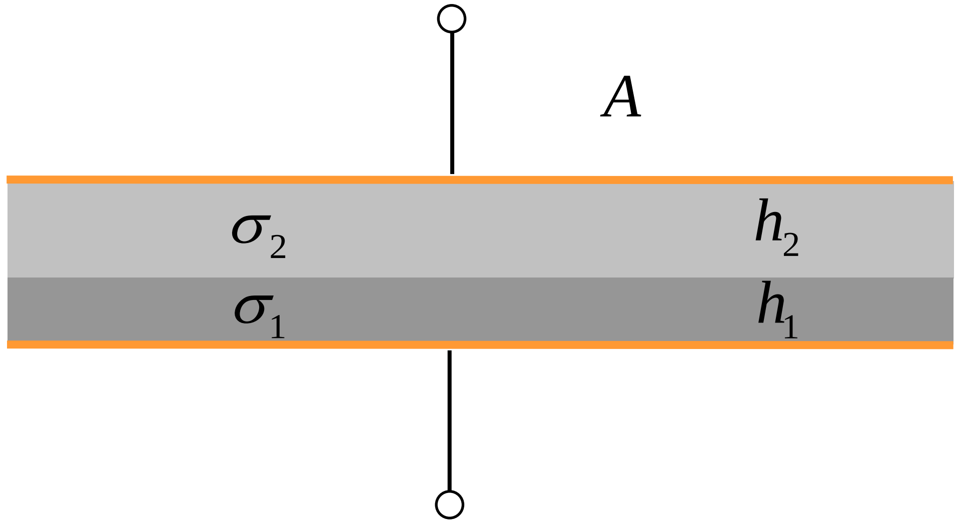
$$RC = \left(\frac{\varepsilon}{\sigma}\right)$$

$$R\left(\frac{\varepsilon A}{h}\right) = \left(\frac{\varepsilon}{\sigma}\right) \Rightarrow R = \frac{h}{\sigma A}$$

Note that the ε cancels out!

Example

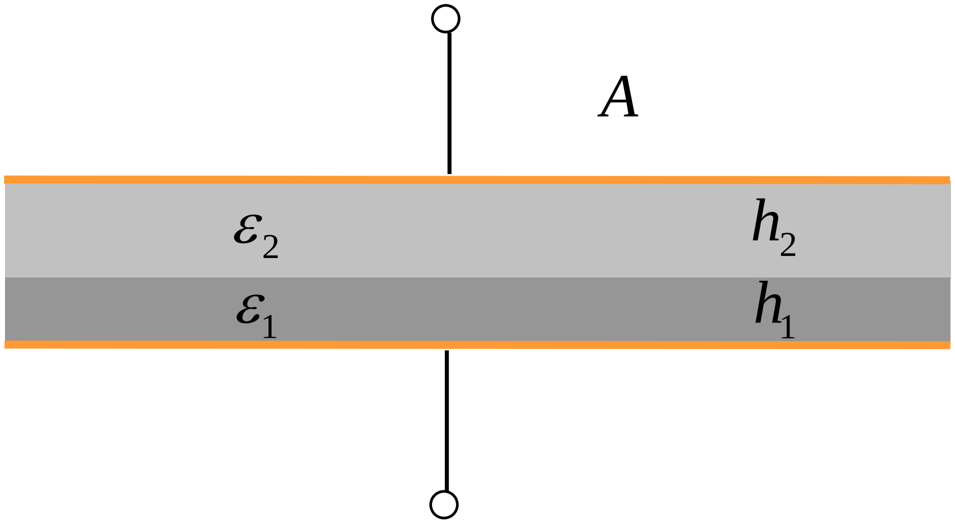
Find the resistance



Note:
We cannot use the RC formula, since there is more than one region (not a single conductivity).

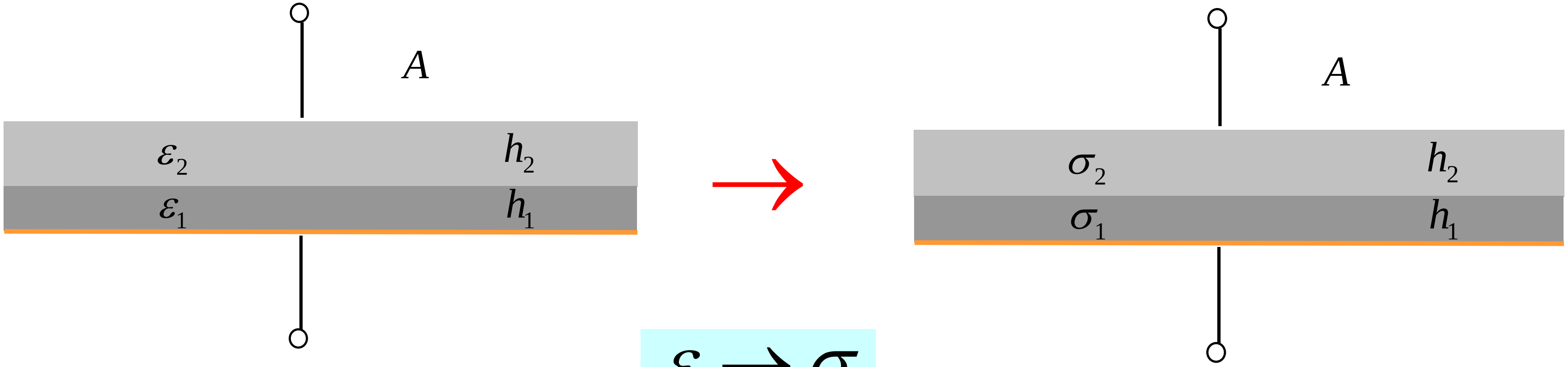
Example (cont.)

C problem:



$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} \qquad C_1 = \frac{\varepsilon_1 A}{h_1} \qquad C_2 = \frac{\varepsilon_2 A}{h_2}$$

Example (cont.)



$$\varepsilon \to \sigma$$

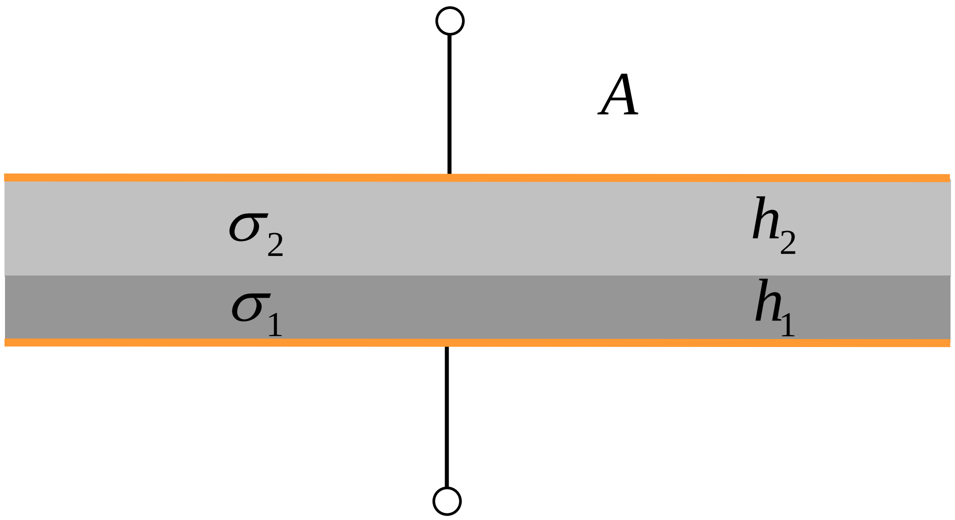
$$C \to G$$

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} \qquad C_1 = \frac{\varepsilon_1 A}{h_1} \quad C_2 = \frac{\varepsilon_2 A}{h_2}$$

$$\downarrow \qquad\qquad \downarrow$$

$$\frac{1}{G} = \frac{1}{G_1} + \frac{1}{G_2} \qquad G_1 = \frac{\sigma_1 A}{h_1} \quad G_2 = \frac{\sigma_2 A}{h_2}$$

Example (cont.)



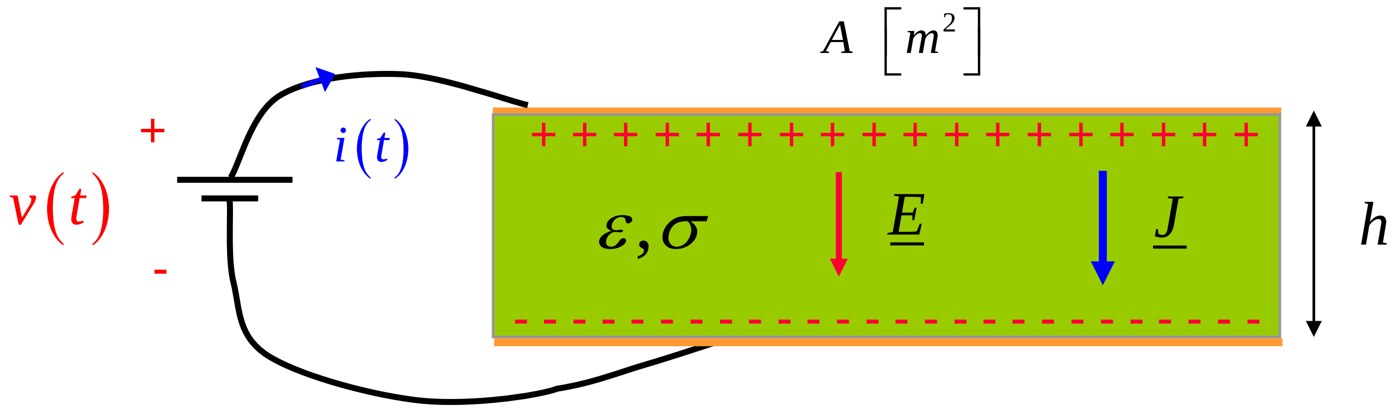
Hence, we have

$$R = R_1 + R_2$$

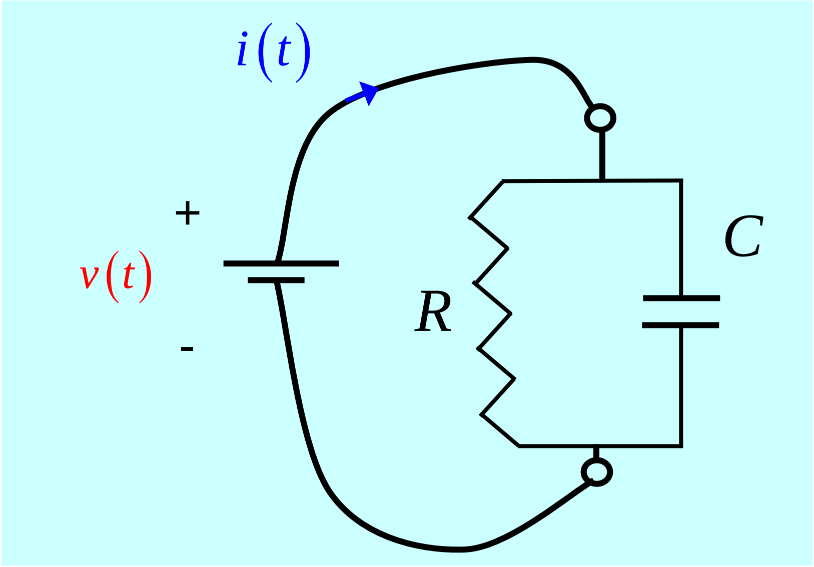
$$R_1 = \frac{h_1}{\sigma_1 A}$$

$$R_2 = \frac{h_2}{\sigma_2 A}$$

Lossy Capacitor



This is modeled by a parallel equivalent circuit
(The proof is in Appendix C.)



$$C = \frac{\varepsilon A}{h}$$

$$R = \frac{h}{\sigma A}$$

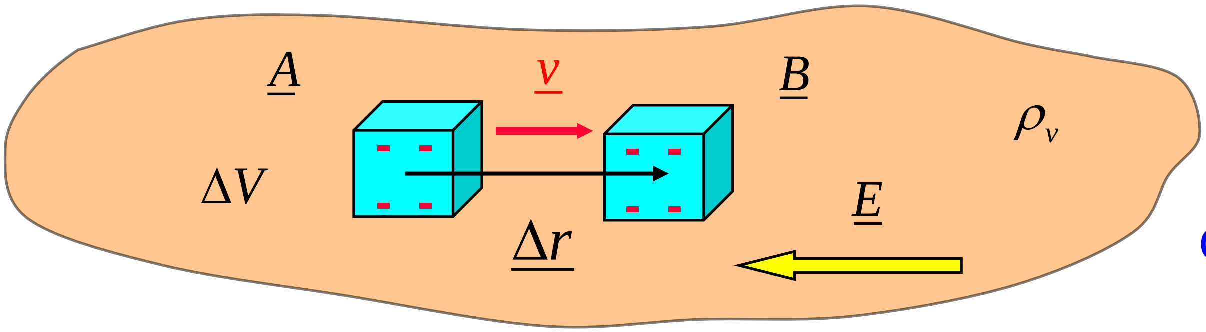
Note:
The time constant is

$$\tau = RC$$

Appendix A

Derivation of Joule's Law

Joule's Law



ρ_v = charge density from electrons in conduction band

Conducting body

$$\underline{J} = \sigma \underline{E}$$

ΔW = work (energy) given to a small volume of charge as it moves inside the conductor from point $\underline{A}$ to point $\underline{B}$.

This goes to heat!

(There is no acceleration of charges in steady state, as this would cause current to change along the conductor, violating the KCL law.)

$$\Delta W = \Delta Q \; V_{AB}$$

$$\approx \left(\rho_v \, \Delta V\right) \int_{\underline{A}}^{\underline{B}} \underline{E} \cdot \underline{dr}$$

$$\approx \left(\rho_v \, \Delta V\right) \underline{E} \cdot \underline{\Delta r}$$

$$= \Delta V \left(\rho_v \, \underline{\Delta r}\right) \cdot \underline{E} = \Delta V \left(\rho_v \frac{\underline{\Delta r}}{\Delta t}\right) \cdot \underline{E} \, \Delta t$$

Joule's Law (cont.)

$$\Delta W = \Delta V\left(\rho_v \frac{\Delta \underline{r}}{\Delta t}\right)\cdot \underline{E}\,\Delta t = \Delta V\left(\rho_v \underline{v}\right)\cdot \underline{E}\,\Delta t$$

$$\text{power / volume} = \left(\frac{\Delta W}{\Delta t}\right)\left(\frac{1}{\Delta V}\right) = \left(\rho_v \underline{v}\right)\cdot \underline{E}$$

$$= \underline{J}\cdot\underline{E}$$

The total power dissipated is then

$$P_d = \int_V \underline{J}\cdot\underline{E}\,dV \;[\mathrm{W}]$$

We can also write

$$P_d = \int_V \sigma\left|\underline{E}\right|^2 dV = \int_V \frac{\left|\underline{J}\right|^2}{\sigma}\,dV \;\;[\mathrm{W}]$$

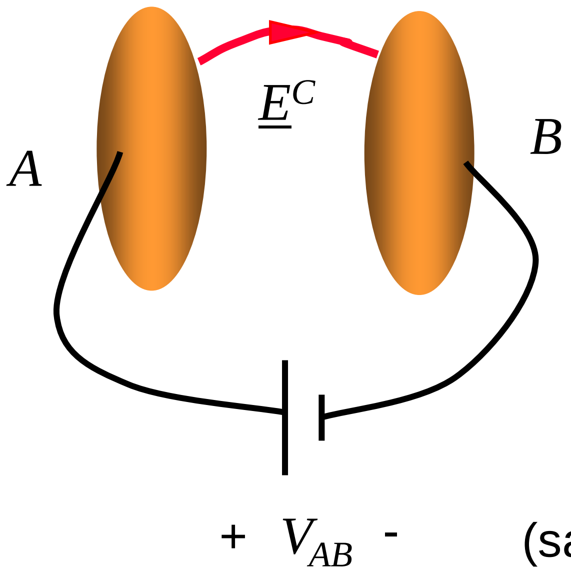
Appendix B

Derivation of RC Analogy and RC Formula

RC Analogy

Insulating material

$$\varepsilon(\underline{r}) = F(\underline{r})$$

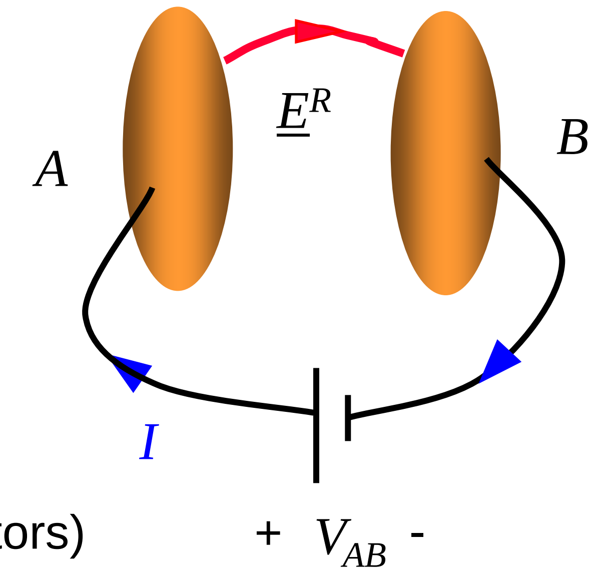


"C problem"

(same conductors)

Conducting material

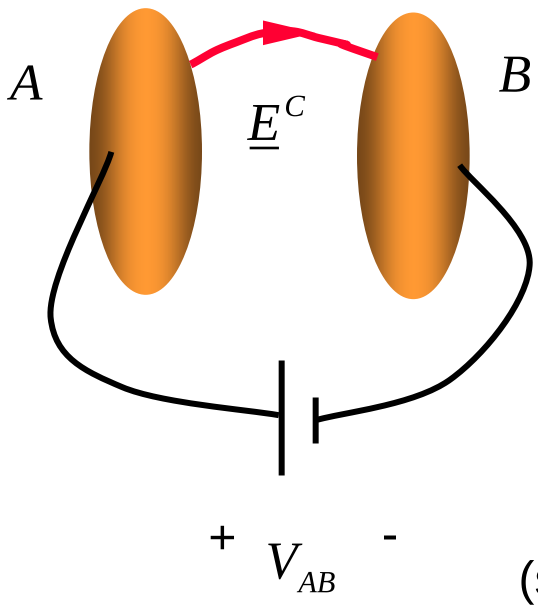
$$\sigma(\underline{r}) = F(\underline{r})$$



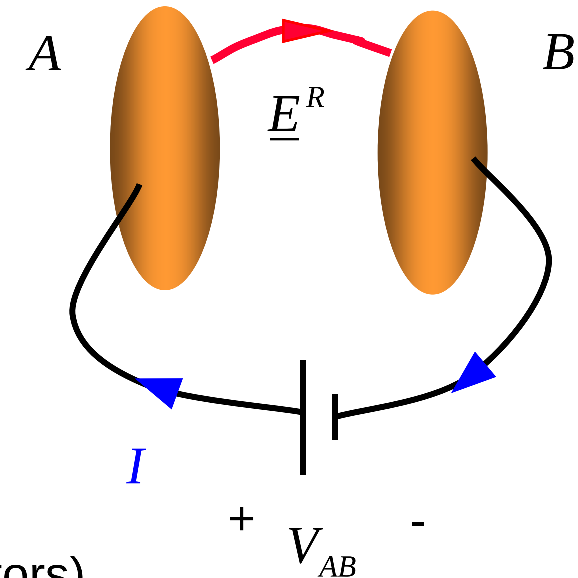
"R problem"

RC Analogy (cont.)

$\varepsilon(\underline{r}) = F(\underline{r})$



$\sigma(\underline{r}) = F(\underline{r})$



(same conductors)

"*C* problem"

"*R* problem"

Theorem: $\underline{E}^C = \underline{E}^R$ (same field in both problems)

RC Analogy (cont.)

Proof of "theorem"

"*C* problem"

$$\nabla \cdot \underline{D} = 0$$

$$\Rightarrow \quad \nabla \cdot \left(\varepsilon \underline{E} \right) = 0$$

"*R* problem"

$$\nabla \cdot \underline{J} = 0$$

$$\Rightarrow \quad \nabla \cdot \left(\sigma \underline{E} \right) = 0$$

- Same differential equation since $\varepsilon(\underline{r}) = \sigma(\underline{r})$
- Same B. C. since the same voltage is applied

Hence, $E^C = E^R$

(The two electric fields must be the same function from the uniqueness of the solution to the differential equation.)

RC Analogy (cont.)

$$C = \frac{Q}{V} = \frac{Q_A}{V_{AB}}$$

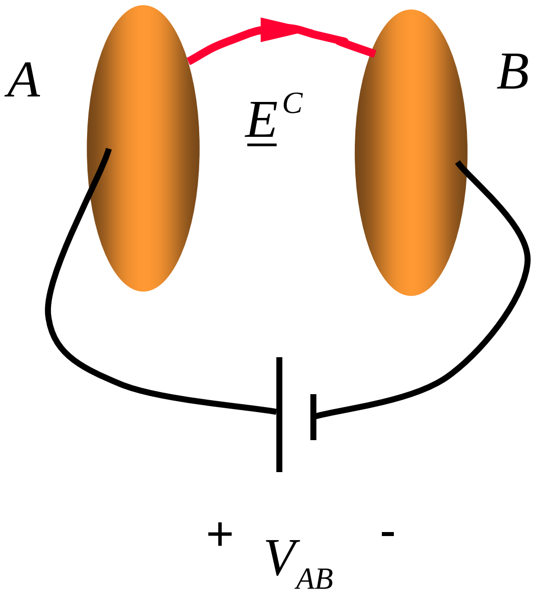
$$= \frac{\oint\limits_{S_A} \rho_s \, dS}{\int\limits_{\underline{A}}^{\underline{B}} \underline{E} \cdot \underline{dr}}$$

Use $\rho_s = \underline{D} \cdot \underline{\hat{n}} = \varepsilon \underline{E} \cdot \underline{\hat{n}}$

Hence

$$C = \frac{\oint\limits_{S_A} \varepsilon \underline{E} \cdot \underline{\hat{n}} \, dS}{\int\limits_{\underline{A}}^{\underline{B}} \underline{E} \cdot \underline{dr}}$$

$\varepsilon(\underline{r}) = F(\underline{r})$



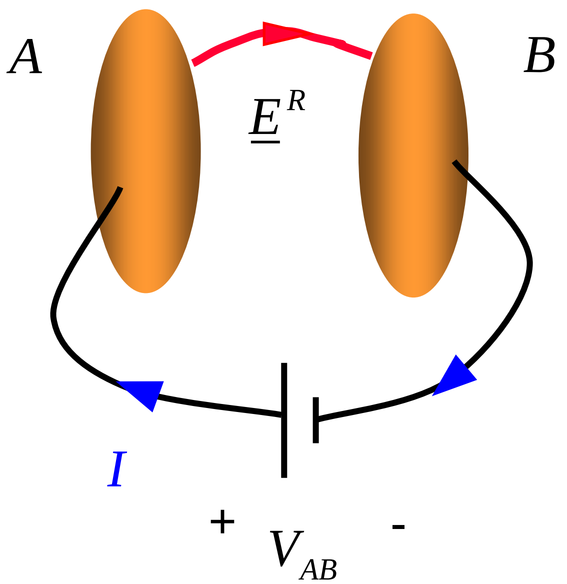
"C Problem"

RC Analogy (cont.)

$$R = \frac{V}{I} = \frac{V_{AB}}{I_A}$$

$$= \frac{\int_A^B \underline{E} \cdot \underline{dr}}{\oint_{S_A} \underline{J} \cdot \underline{\hat{n}}\, dS}$$

$$\sigma(\underline{r}) = F(\underline{r})$$



Use $\underline{J} = \sigma \underline{E}$

Hence

$$R = \frac{\int_A^B \underline{E} \cdot \underline{dr}}{\oint_{S_A} \sigma\, \underline{E} \cdot \underline{\hat{n}}\ dS}$$

"R Problem"

RC Analogy (cont.)

Compare:

$$C = \frac{\oint\limits_{S_A} \varepsilon \underline{E} \cdot \underline{\hat{n}} \; dS}{\int\limits_{\underline{A}}^{\underline{B}} \underline{E} \cdot \underline{dr}} \qquad G = \frac{1}{R} = \frac{\oint\limits_{S_A} \sigma \underline{E} \cdot \underline{\hat{n}} \; dS}{\int\limits_{\underline{A}}^{\underline{B}} \underline{E} \cdot \underline{dr}}$$

Recall that $\sigma\left(\underline{r}\right) = \varepsilon\left(\underline{r}\right) = F\left(\underline{r}\right)$

Hence $C = G$

RC Formula

This is a special case: A homogeneous medium of conductivity σ surrounds the two conductors (there is only one value of σ).

$$C = \varepsilon \frac{\oint_{S_A} \underline{E} \cdot \underline{\hat{n}}\, dS}{\int_{\underline{A}}^{\underline{B}} \underline{E} \cdot \underline{dr}} \qquad G = \sigma \frac{\oint_{S_A} \underline{E} \cdot \underline{\hat{n}}\, dS}{\int_{\underline{A}}^{\underline{B}} \underline{E} \cdot \underline{dr}}$$

Hence,

$$G = C\left(\frac{\sigma}{\varepsilon}\right)$$

or

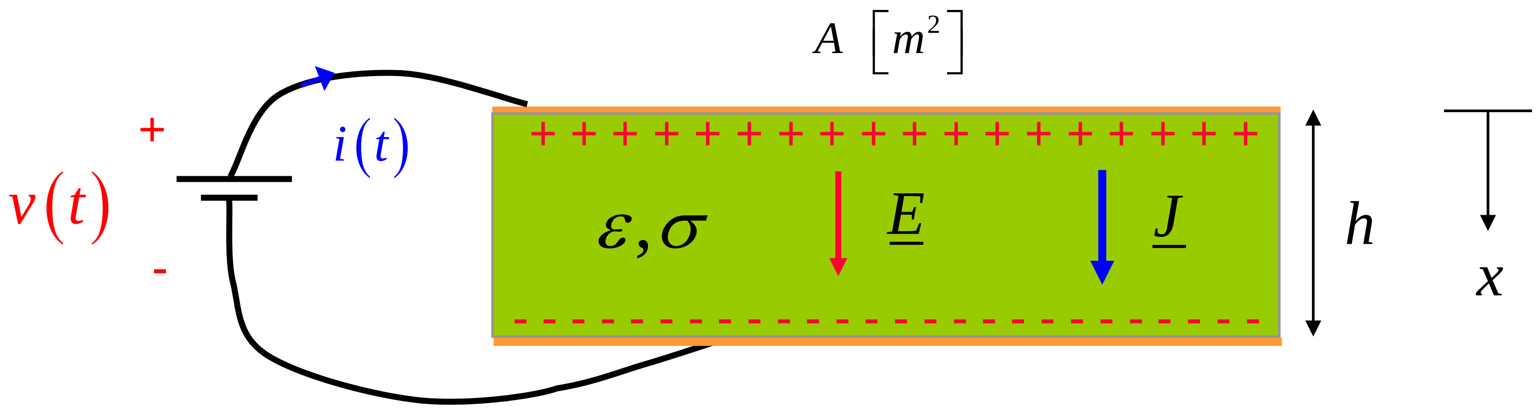
$$RC = \left(\frac{\varepsilon}{\sigma}\right)$$

Appendix C

Equivalent Circuit for a Lossy Capacitor

Lossy Capacitor

Derivation of equivalent circuit



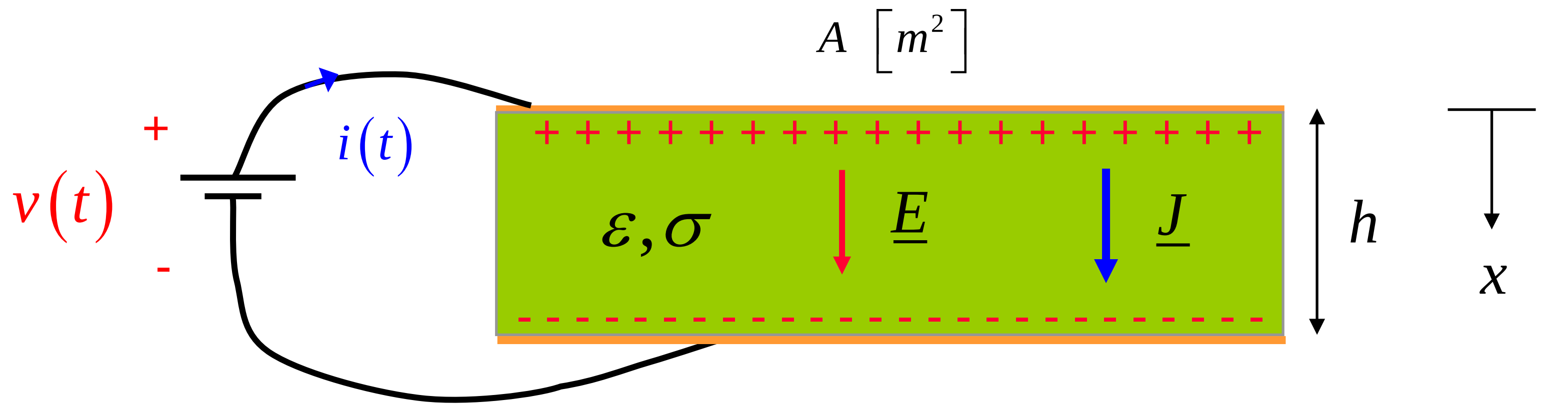
Total (net) current entering top (A) plate:

$$i_{in}^{A}(t) = i(t) - J_x A = \frac{dQ}{dt}$$

Therefore,

$$i(t) = J_x A + \frac{dQ}{dt}$$

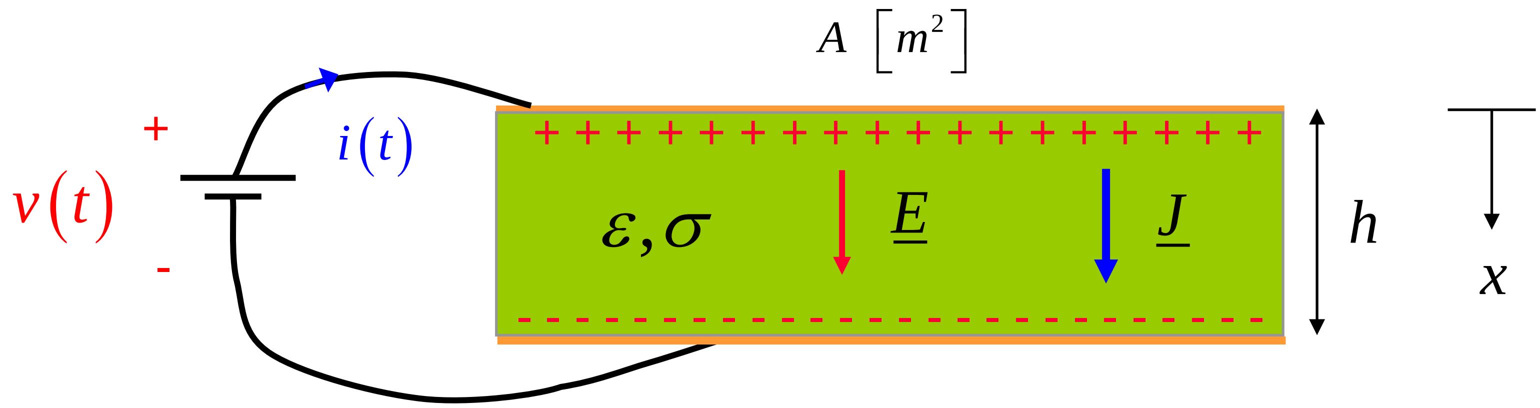
Lossy Capacitor (cont.)

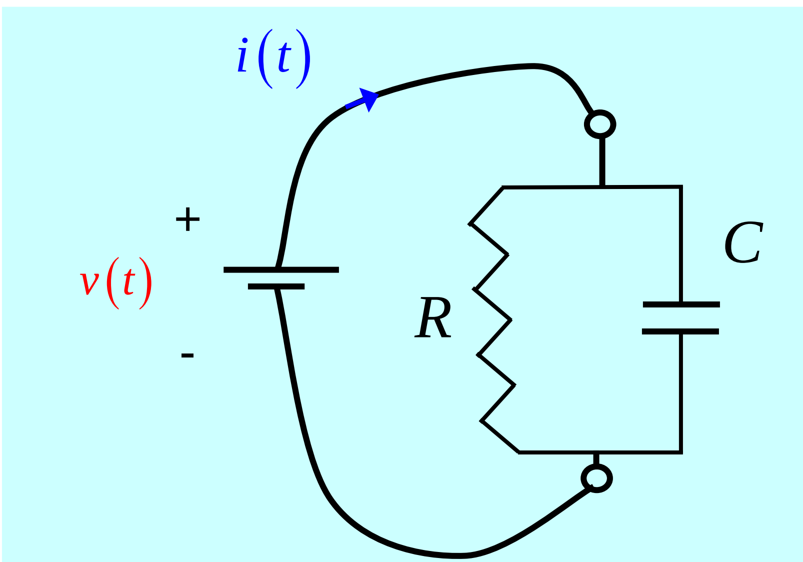


$$i(t) = J_x A + \frac{dQ}{dt}$$
$$= \sigma E_x A + A \frac{d\rho_s^A}{dt}$$
$$= \sigma \left(\frac{v}{h} \right) A + A \frac{dD_x}{dt}$$
$$= \frac{v}{\left(\frac{h}{\sigma A} \right)} + A\varepsilon \frac{dE_x}{dt}$$

$$i(t) = \frac{v}{\left(\frac{h}{\sigma A} \right)} + A\varepsilon \frac{dE_x}{dt}$$
$$= \frac{v}{R} + \left(\frac{A\varepsilon}{h} \right) \frac{dv}{dt}$$
$$= \frac{v}{R} + C \frac{dv}{dt}$$

Lossy Capacitor (cont.)

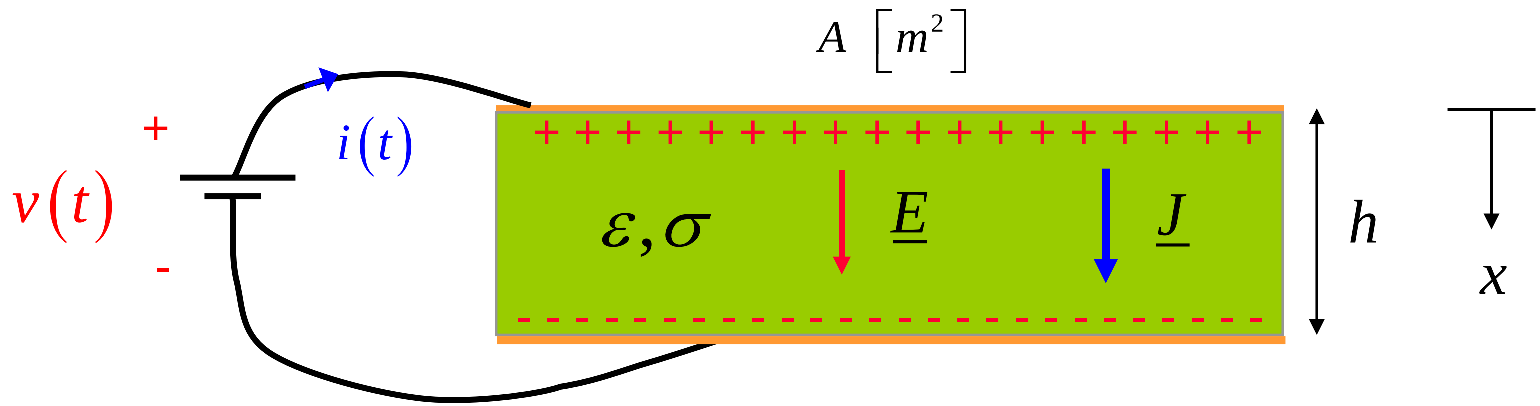




$$i(t) = \frac{v(t)}{R} + C\frac{dv(t)}{dt}$$

This is the KCL equation for a resistor in parallel with a capacitor.

Lossy Capacitor (cont.)



Note on displacement current:

We can also write the current as

$$i(t) = J_x A + \frac{dD_x}{dt} A$$

Conduction current

Displacement current

Ampere's law:

$$\nabla \times \underline{H} = \underline{J} + \frac{\partial \underline{D}}{\partial t}$$

Conduction current (density)

Displacement current (density)