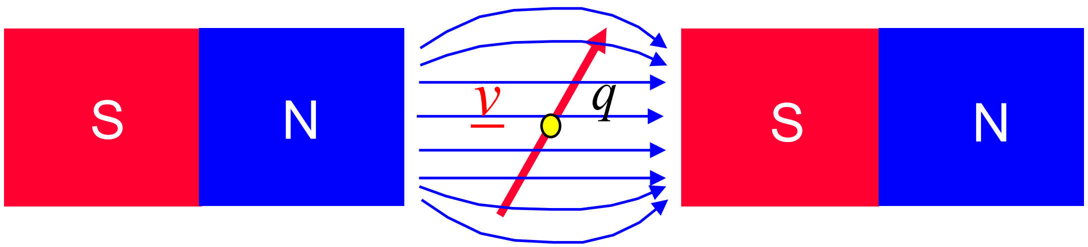
ECE 3318
Applied Electricity and Magnetism

Spring 2023

Prof. David R. Jackson
Dept. of ECE

Notes 28
Magnetic field and Ampere's Law

Magnetic Field



Note: Flux lines come out of north poles!

Lorentz force Law:

$$\underline{F} = q\,\underline{v} \times \underline{B}$$

This experimental law defines $\underline{B}$.

$\underline{B}$ is the magnetic flux density vector.

In general, (with both $\underline{E}$ and $\underline{B}$ present):

$$\underline{F} = q\left(\underline{E} + \underline{v} \times \underline{B}\right)$$

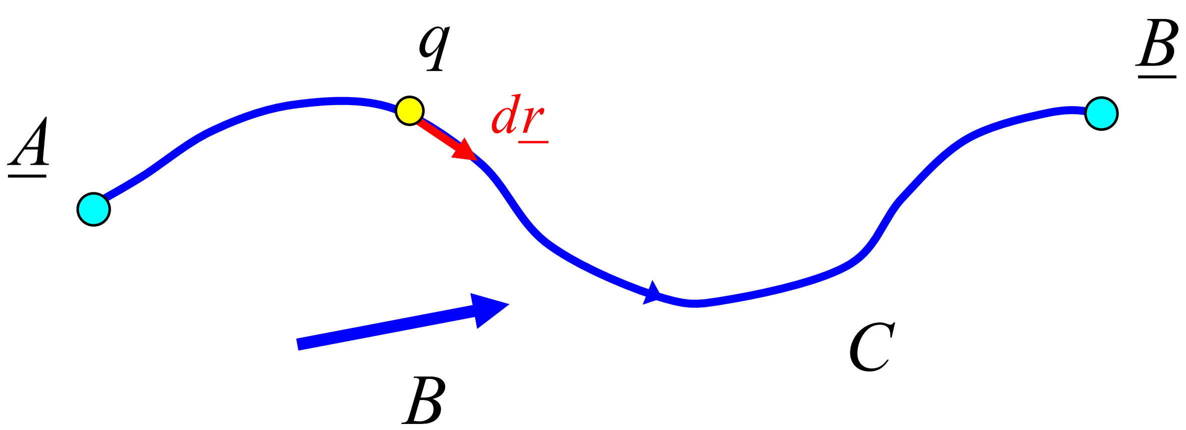
The units of $\underline{B}$ are Webers/m^2 or Tesla $[\mathrm{T}]$.

Magnetic Field (cont.)

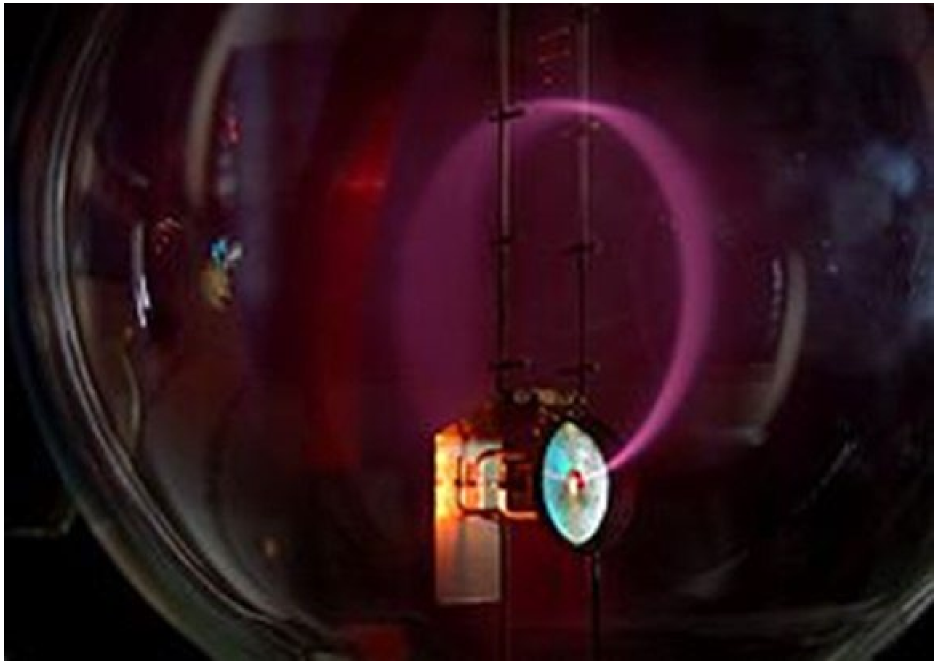
A magnetic field can never give energy to a particle that is moving on a path.

$$\Delta W = \underline{F} \cdot d\underline{r} = q\left(\underline{v} \times \underline{B}\right) \cdot d\underline{r} = q\left(\underline{v} \times \underline{B}\right) \cdot \frac{d\underline{r}}{dt} dt$$

$$\Rightarrow \quad \Delta W = q\underbrace{\left(\underline{v} \times \underline{B}\right) \cdot \underline{v}}_{\text{zero}}\, dt = 0$$



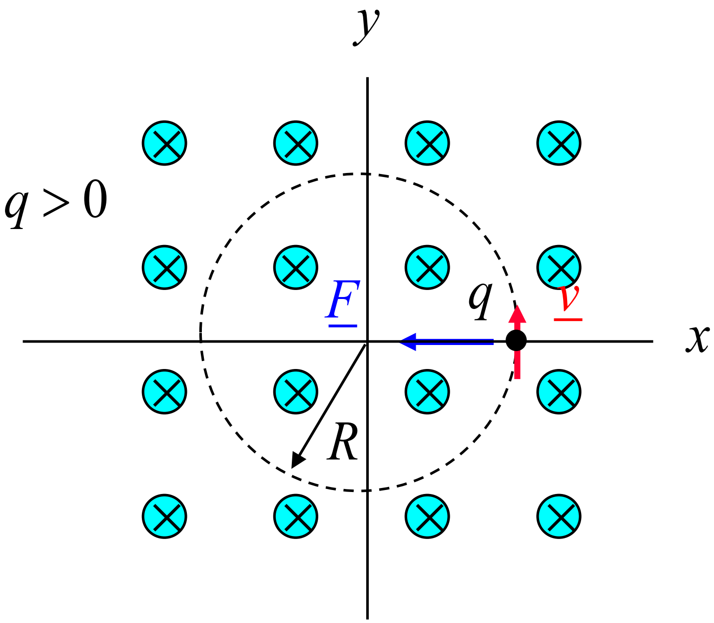
Magnetic Field (cont.)



Beam of electrons moving in a circle, due to the presence of a magnetic field. Purple light is emitted along the electron path, due to the electrons colliding with gas molecules in the bulb.

(From Wikipedia)

$$\underline{F} = q\left(\underline{v} \times \underline{B}\right)$$



$$\underline{B} = -\underline{\hat{z}} B_0$$

$$\underline{v} = \underline{\hat{\phi}}\, v_0$$

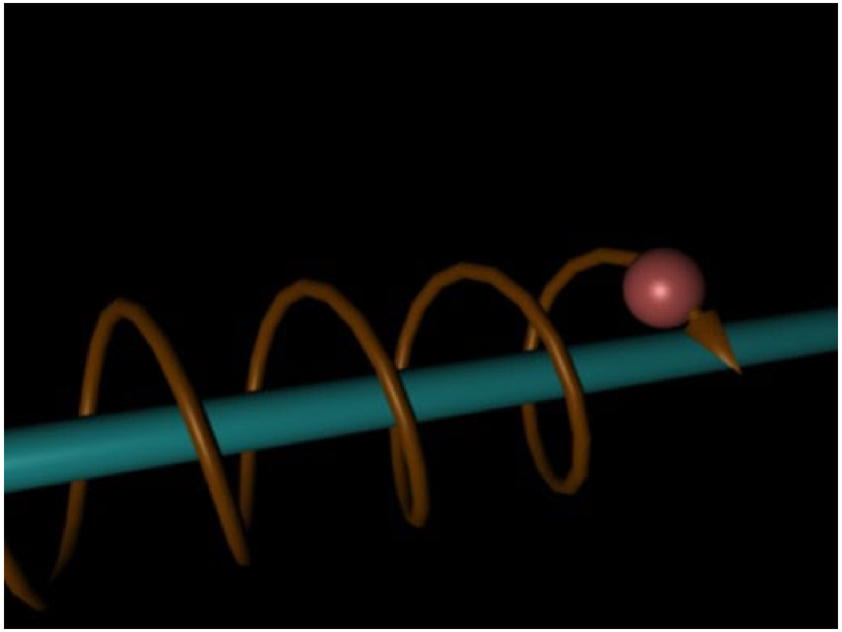
$$\underline{F} = -\underline{\hat{\rho}}\left(q v_0 B_0\right)$$

A stable orbit is a circular path:

$$\frac{m v_0^2}{R} = q v_0 B_0$$

Magnetic Field (cont.)

The most general stable path is a helix,
with the helix axis aligned with the magnetic field.



$\underline{B}$

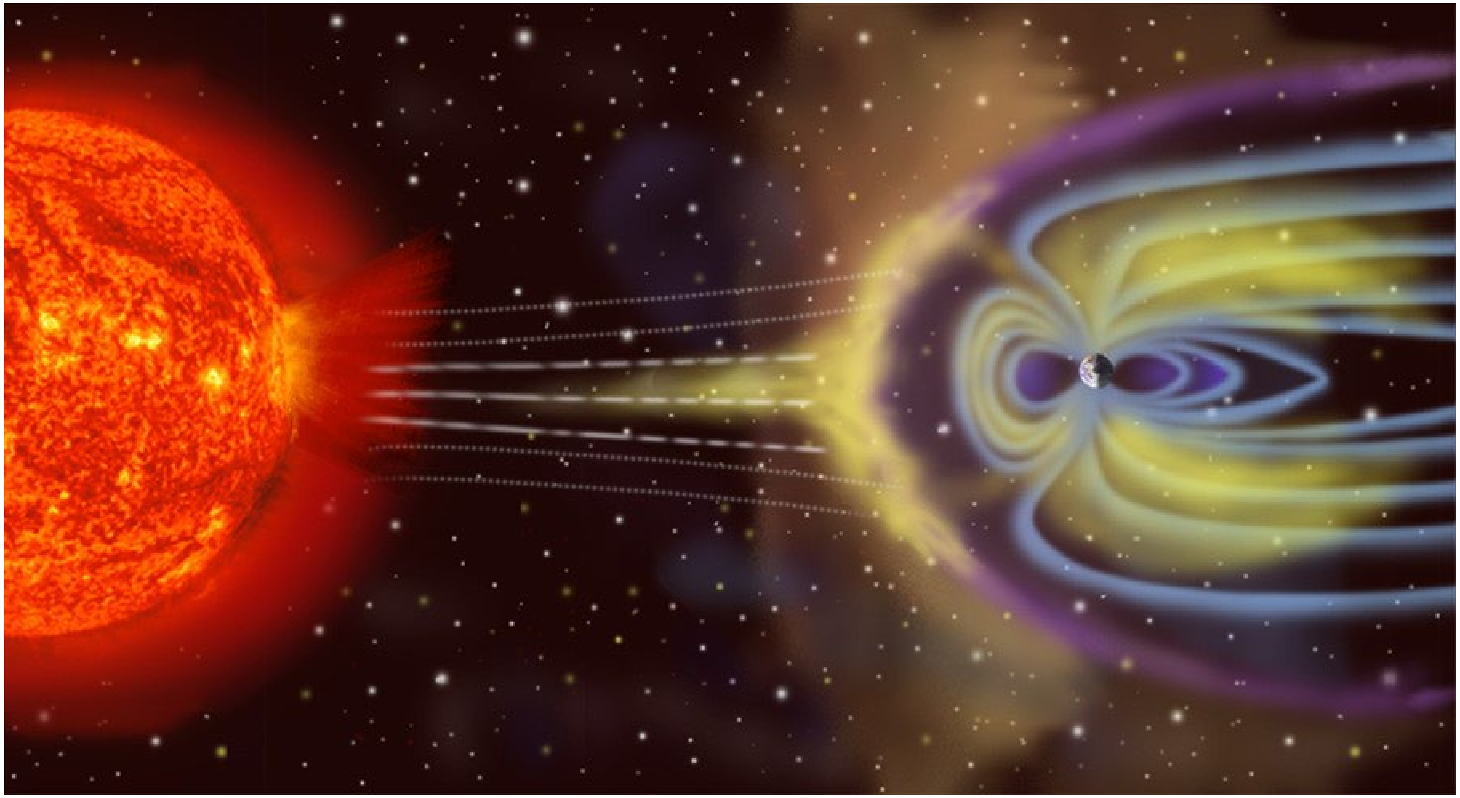
$$\underline{F} = q\,\underline{v} \times \underline{B}$$

There is no force in the axis direction (hence a constant velocity in this direction).

Magnetic field lines thus “guide” charged particles.

Magnetic Field (cont.)

The earth's magnetic field protects us from charged particles from the sun (called the solar wind).



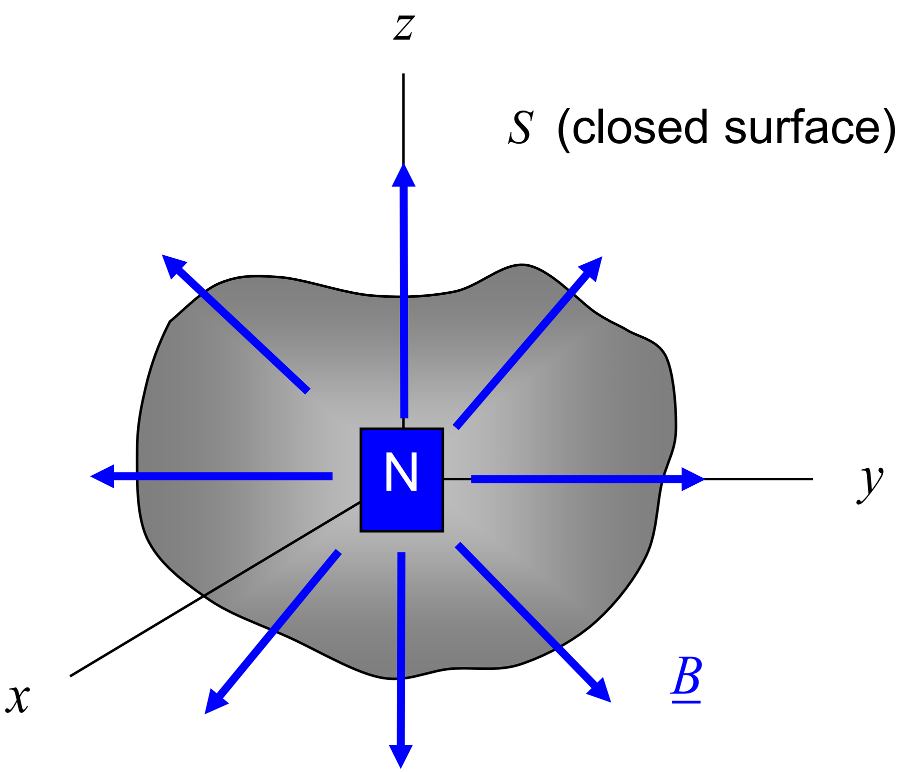
The particles spiral along the directions of the magnetic field, and are thus directed towards the poles.

Magnetic Field (cont.)

This also explains the auroras seen near the north pole (aurora borealis) and the south pole (aurora australis).

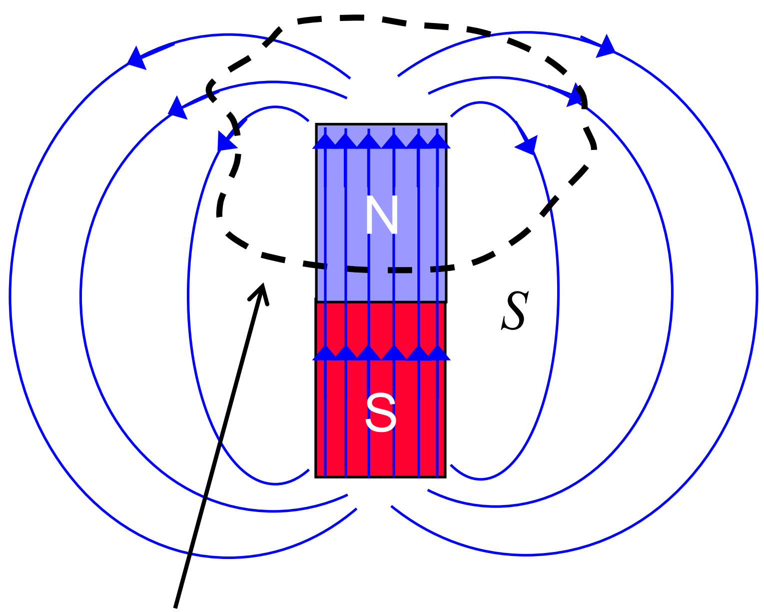
The particles from the sun that reach the earth are directed towards the poles.

Magnetic Gauss Law



Magnetic pole (not possible)!

$$\oint_S \underline{B} \cdot \underline{\hat{n}}\, dS = 0$$



No net magnetic flux out!

Note: Magnetic flux lines come out of a north pole and go into a south pole.

Magnetic Gauss Law: Differential Form

$$\oint_S \underline{B} \cdot \underline{\hat{n}}\, dS = 0$$

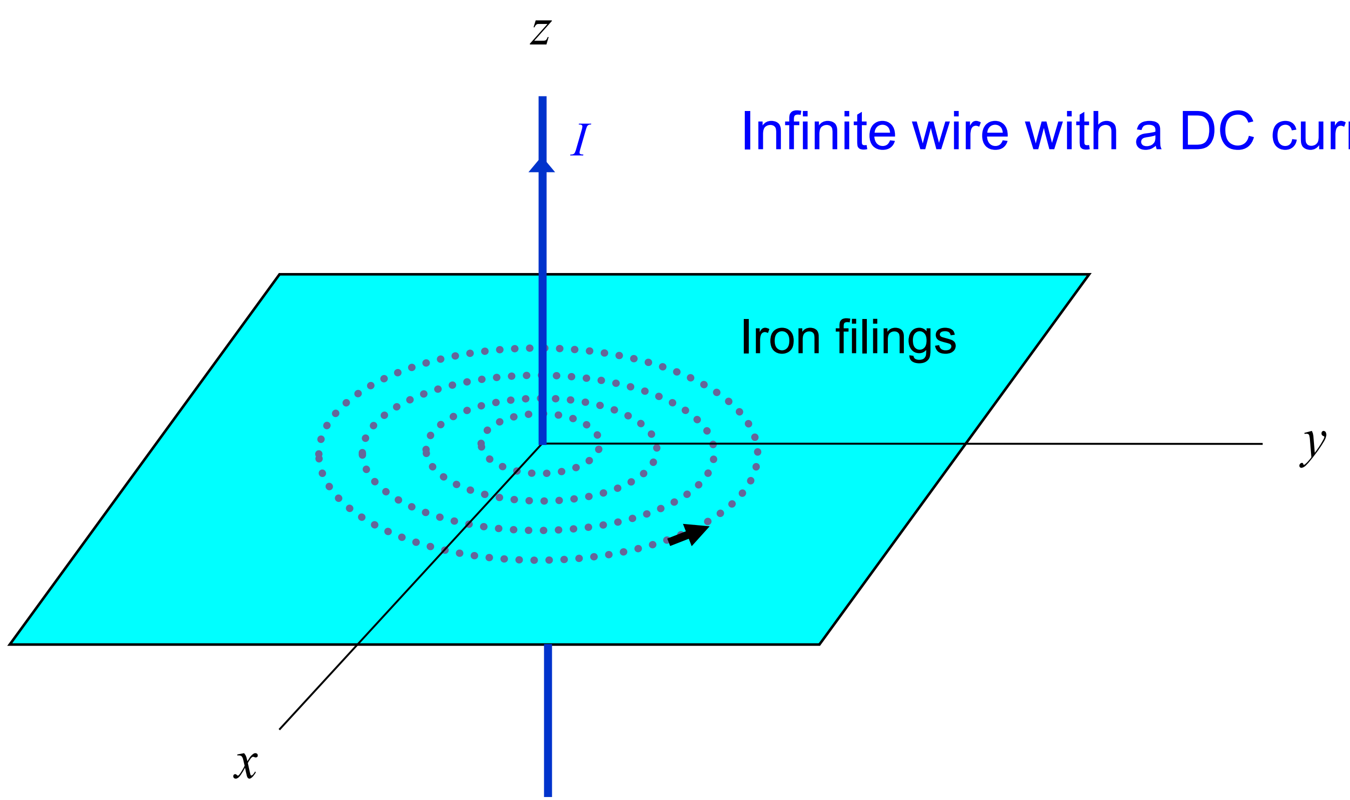
From the definition of divergence we then have

$$\nabla \cdot \underline{B} \equiv \lim_{\Delta V \to 0} \frac{1}{\Delta V} \oint_S \underline{B} \cdot \underline{\hat{n}}\, dS$$

Hence

$$\boxed{\nabla \cdot \underline{B} = 0}$$

Ampere's Law



Experimental law:

$$\underline{B} = \underline{\hat{\phi}}\left(\frac{I}{2\pi\rho}\right)\mu_0$$

$$\mu_0 = 4\pi \times 10^{-7} \quad [\mathrm{H/m}]$$

(This is an exact value: please see next slide.)

Ampere's Law (cont.)

Note: The definition of the Amp* is as follows:

1 [A] current produces: $B_\phi = 2\times 10^{-7}\quad [\mathrm{T}]$

$\text{at}\quad \rho = 1\quad [\mathrm{m}]$

* This is actually the definition before May 20, 2019. Please see Notes 3 for more discussion of this.

Hence

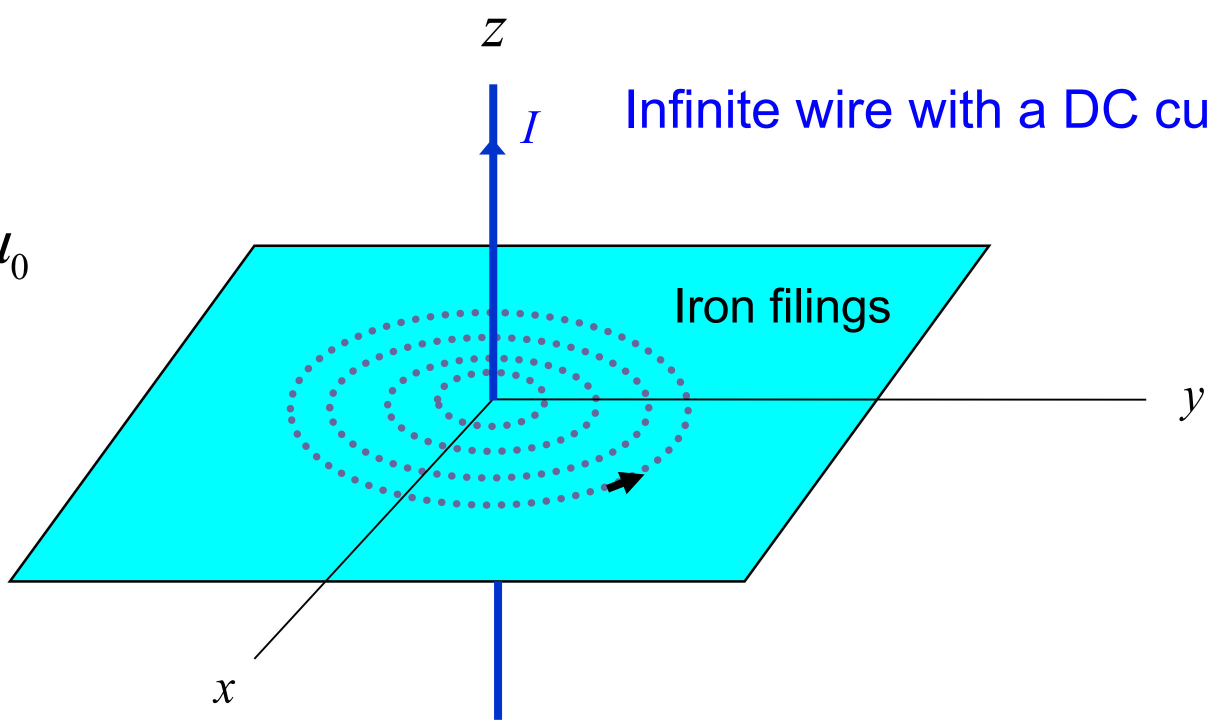
$$B_\phi = \left(\frac{I}{2\pi\rho}\right)\mu_0 \quad \Rightarrow \quad 2\times 10^{-7}\,[\mathrm{T}] = \frac{1[\mathrm{A}]}{2\pi\left(1[\mathrm{m}]\right)}\,\mu_0$$

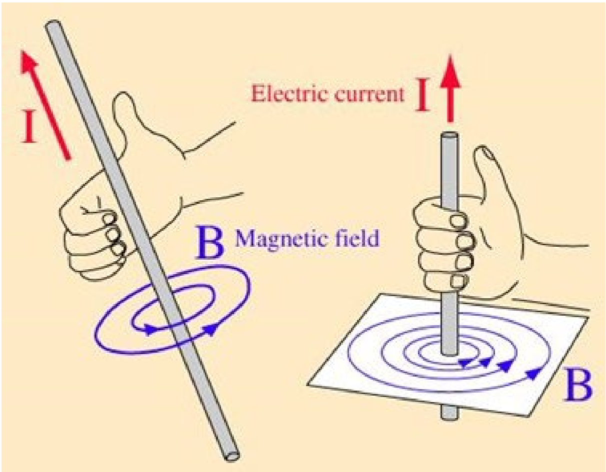
so

$$\mu_0 = 4\pi\times 10^{-7}\quad [\mathrm{H/m}]$$

Ampere's Law (cont.)

$$\underline{B} = \underline{\hat{\phi}}\left(\frac{I}{2\pi\rho}\right)\mu_0$$





Note:
There is a "right-hand rule" for predicting the <u>direction</u> of the magnetic field from a wire.

(The thumb is in the direction of the current, and the fingers are in the direction of the field.)

Ampere's Law (cont.)

Define:

$$\underline{H} \equiv \frac{1}{\mu_0}\underline{B}$$

(This is the definition of $\underline{H}$ in free space.)

Hence

$$\underline{B} = \mu_0 \underline{H}$$

$\underline{H}$ is called the "magnetic field".

The units of $\underline{H}$ are [A/m].

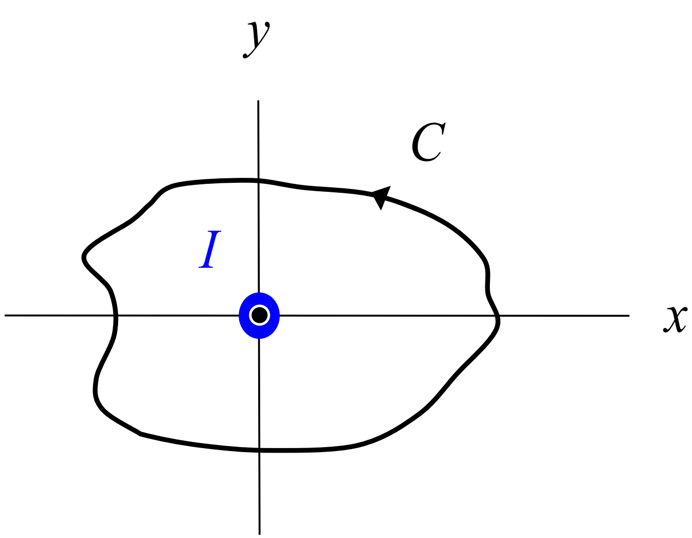
$$\underline{H} = \underline{\hat{\phi}}\left(\frac{I}{2\pi\rho}\right) \quad [\text{A/m}]$$

(for single infinite wire)

Ampere's Law (cont.)

Next, consider an infinite wire <u>inside</u> of an arbitrary closed path.

Wire inside path



$$\oint_C \underline{H} \cdot \underline{dr} = \oint_C \left(\underline{\hat{\phi}}\, H_\phi\right) \cdot \left(\underline{\hat{\phi}}\, \rho\, d\phi + \underline{\hat{\rho}}\, d\rho + \underline{\hat{z}}\, dz\right)$$

$$= \oint_C H_\phi\, \rho\, d\phi$$

$$= \int_0^{2\pi} \left(\frac{I}{2\pi \cancel{\rho}}\right) \cancel{\rho}\, d\phi$$

$$= \frac{I}{2\pi} \int_0^{2\pi} d\phi = \frac{I}{2\pi}(2\pi)$$

$$= I$$

A current (wire) is inside a closed path.

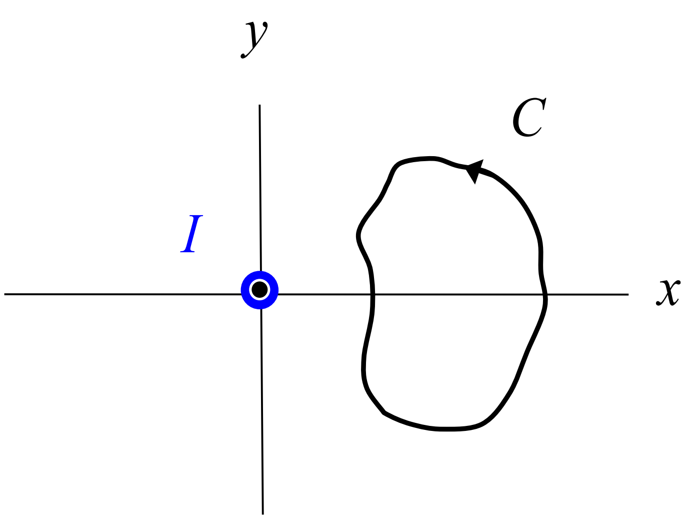
We integrate *counterclockwise*.

Observation:
The answer does not depend on the shape of the path!

Ampere's Law (cont.)

Next, consider an infinite wire <u>outside</u> of an arbitrary closed path.

Wire outside path



A current (wire) is outside a closed path.

$$\oint_C \underline{H} \cdot \underline{dr} = \int_0^0 \frac{I}{2\pi} d\phi$$

$$= 0$$

Note:
The angle ϕ smoothly goes from zero back to zero as we go around the path, starting on the x axis.

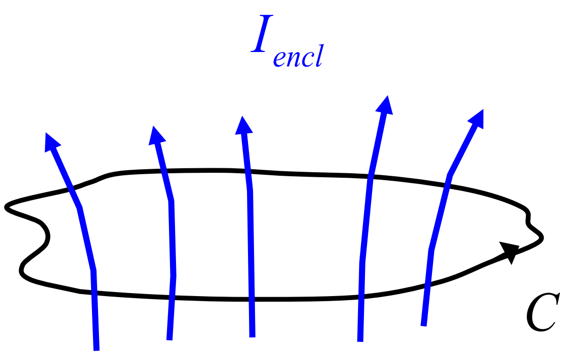
Observation:
The answer does not depend on the shape of the path!

Ampere's Law (cont.)

Hence

$$\oint_C \underline{H} \cdot \underline{dr} = I_{encl}$$

Ampere's Law (DC currents)



Although the law was derived for an infinite vertical wire of current, the assumption is made that it holds for any current.

This is now an experimental law.

Note:
The same DC current I_{encl} goes through any surface that is attached to C
(This follows from the KCL law.)

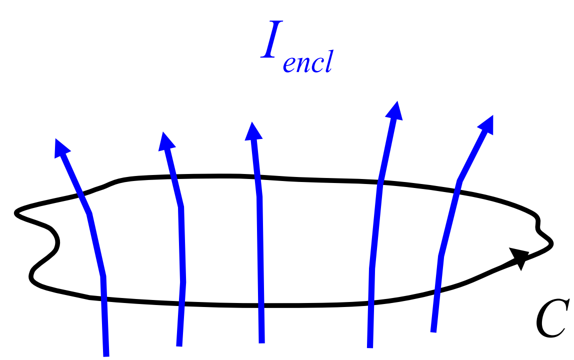
Ampere's Law (cont.)

Right-hand rule for Ampere's law:

The fingers of the right hand are in the direction of the path C, and the thumb gives the reference direction for the current that is enclosed by the path.

(If the contour C goes counterclockwise, then the reference direction for current is pointing up.)

$$\oint_C \underline{H} \cdot \underline{dr} = I_{encl}$$



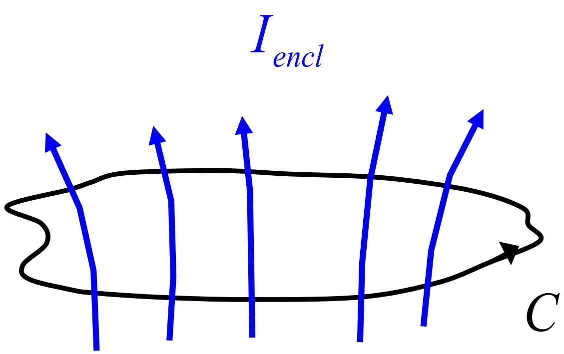
"Right-Hand Rule"

Ampere's Law Summary

Summary of Ampere's law:

$$\oint_C \underline{H} \cdot \underline{dr} = I_{encl}$$

The direction of the current enclosed is chosen by the right-hand rule.



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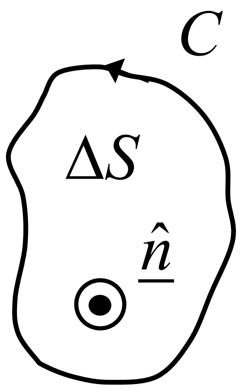
Amperes' Law: Differential Form



ΔS is a small planar surface.

$\underline{\hat{n}}$ is constant

$$\oint_C \underline{H} \cdot \underline{dr} = I_{encl}$$

$$\int_{\Delta S} \left(\nabla \times \underline{H}\right) \cdot \underline{\hat{n}}\, dS = I_{encl}$$ (from Stokes's theorem)

$$\int_{\Delta S} \left(\nabla \times \underline{H}\right) \cdot \underline{\hat{n}}\, dS = \int_{\Delta S} \underline{J} \cdot \underline{\hat{n}}\, dS$$

$$\left(\nabla \times \underline{H}\right) \cdot \underline{\hat{n}}\ \Delta S \approx \underline{J} \cdot \underline{\hat{n}}\ \Delta S$$

Let $\Delta S \to 0$

$$\Rightarrow \left(\nabla \times \underline{H}\right) \cdot \underline{\hat{n}} = \underline{J} \cdot \underline{\hat{n}}$$

Since the unit normal is arbitrary (it could be any of the three unit vectors), we have

$$\boxed{\nabla \times \underline{H} = \underline{J}}$$

Maxwell's Equations (Statics)

$$\nabla \cdot \underline{D} = \rho_v$$

Electric Gauss law

$$\nabla \cdot \underline{B} = 0$$

Magnetic Gauss law

$$\nabla \times \underline{E} = \underline{0}$$

Faraday's law

$$\nabla \times \underline{H} = \underline{J}$$

Ampere's law

Maxwell's Equations (Dynamics)

$$\nabla \cdot \underline{D} = \rho_v$$

Electric Gauss law

$$\nabla \cdot \underline{B} = 0$$

Magnetic Gauss law

$$\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}$$

Faraday's law

$$\nabla \times \underline{H} = \underline{J} + \frac{\partial \underline{D}}{\partial t}$$

Ampere's law

This term is called "displacement current".

Ampere's Law: Finding $\underline{H}$

$$\oint_C \underline{H} \cdot \underline{dr} = I_{encl}$$

Rules:

1) The "Amperian path" C must be a closed path.

2) The sign of I_{encl} is from the right-hand rule.

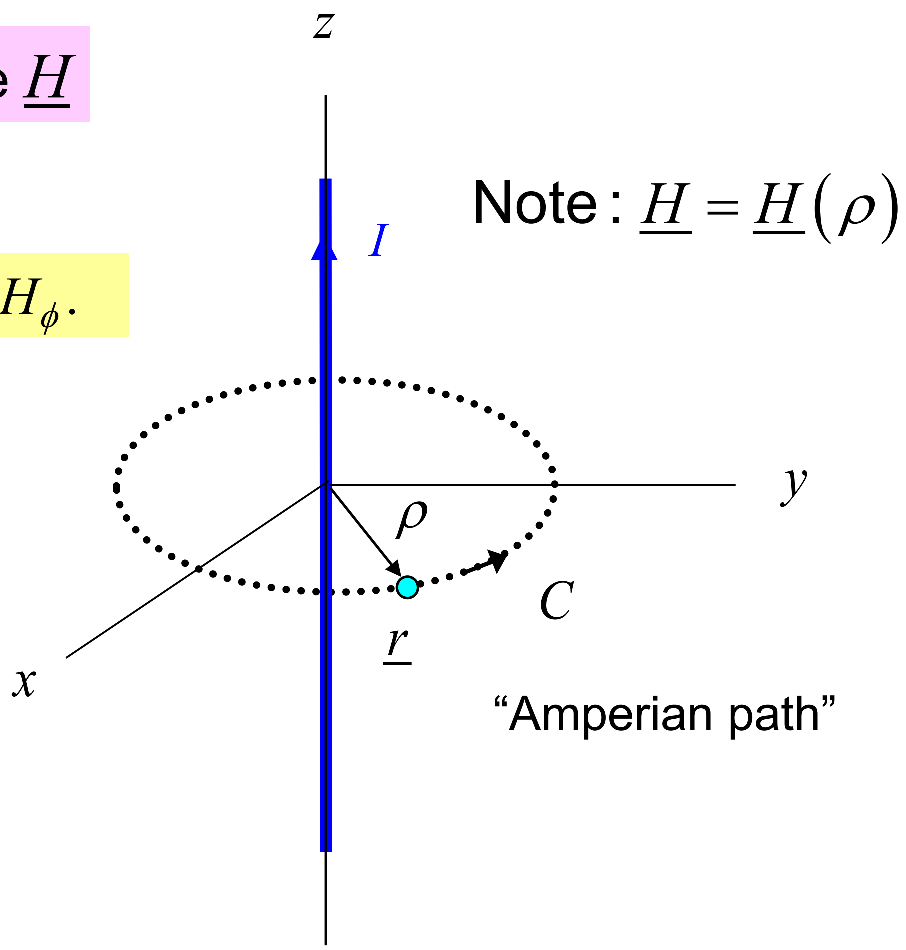
3) Pick C in the direction of $\underline{H}$ (to the extent possible).

Note:
Ampere's law is only useful when the problem is very symmetric (there is only one unknown component of magnetic field).

Example

Calculate $\underline{H}$

1) First solve for H_ϕ.



An infinite line current along the z axis

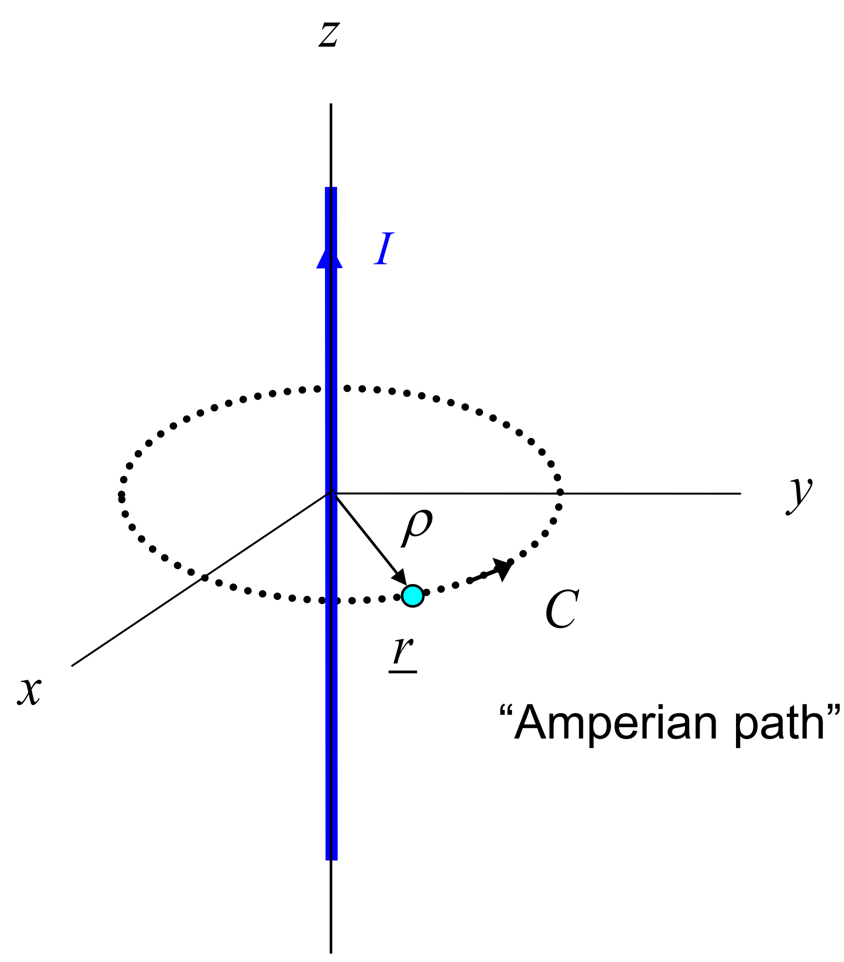
Example (cont.)

$$\oint_C \underline{H} \cdot \underline{dr} = I_{encl}$$

$$\oint_C \left(\underline{H}\right) \cdot \left(\underline{\hat{\phi}}\, \rho \, d\phi\right) = I_{encl} = +I$$

$$\int_0^{2\pi} H_\phi \, \rho \, d\phi = I$$

$$H_\phi \, \rho \left(2\pi\right) = I$$



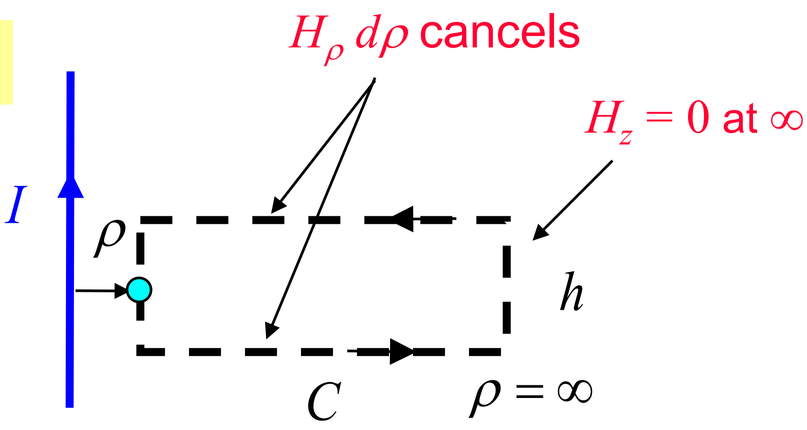
$$\boxed{H_\phi = \frac{I}{2\pi\rho}}$$

Note:
Whenever we integrate counterclockwise on a circular path, the LHS of Ampere's law will always be the same:

$$\text{LHS} = H_\phi \left(2\pi\rho\right)$$

Example (cont.)

2) $H_z = 0$



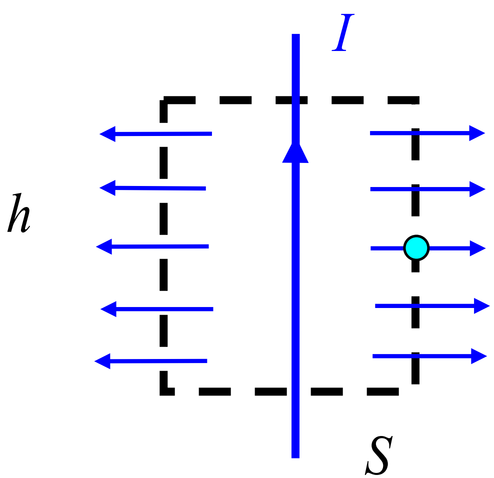
$$\oint_C \underline{H} \cdot \underline{dr} = I_{encl} = 0$$

$$\Downarrow$$

$$H_z\big|_\rho (-h) + \cancel{H_z\big|_\infty} (h) = 0$$

$$\Rightarrow \; H_z\big|_\rho = 0$$

3) $H_\rho = 0$

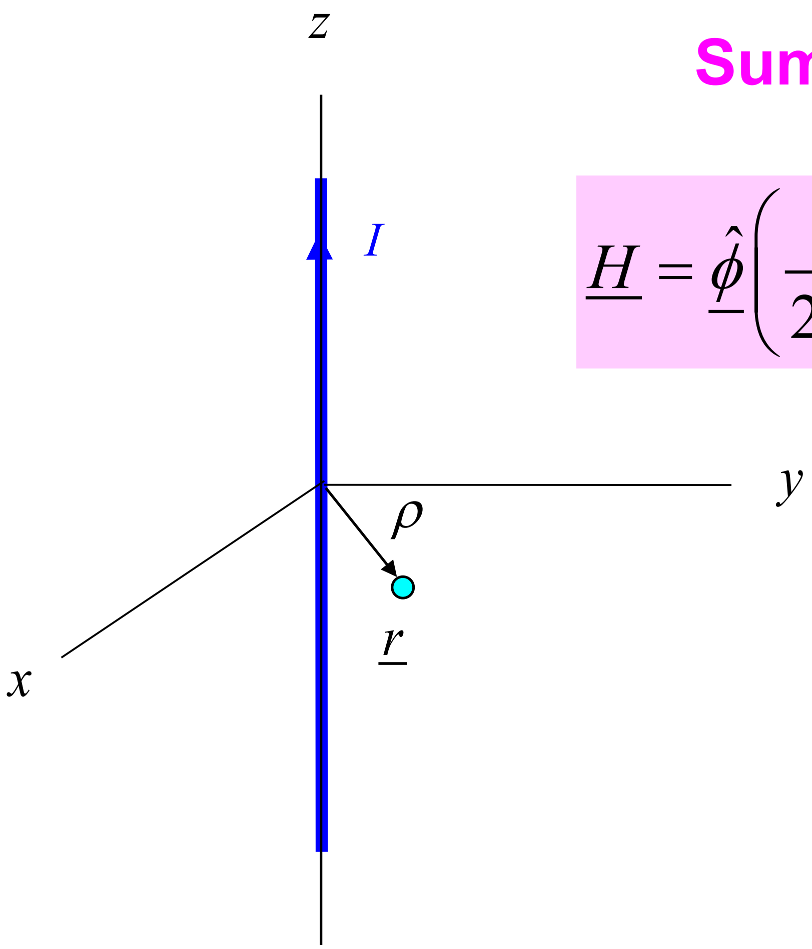


Magnetic Gauss law:

$$\oint_S \underline{B} \cdot \underline{\hat{n}} \, dS = 0$$

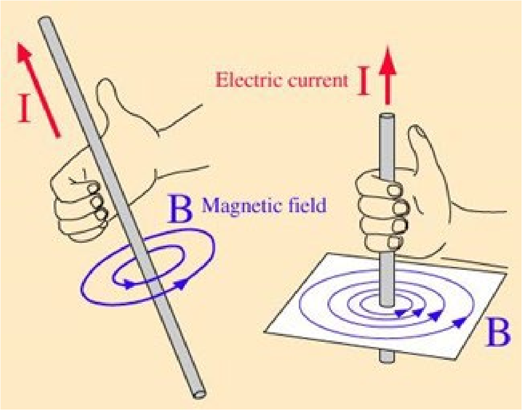
$$B_\rho \left(2\pi\rho h\right) = 0$$

Example (cont.)



Summary

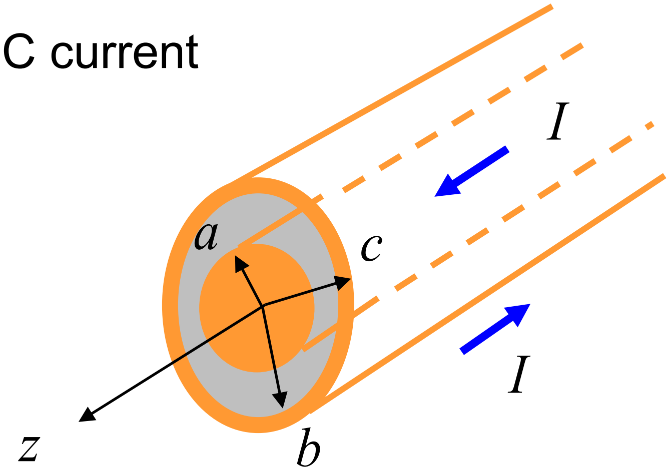
$$\underline{H} = \underline{\hat{\phi}}\left(\frac{I}{2\pi\rho}\right) \quad [\mathrm{A/m}]$$

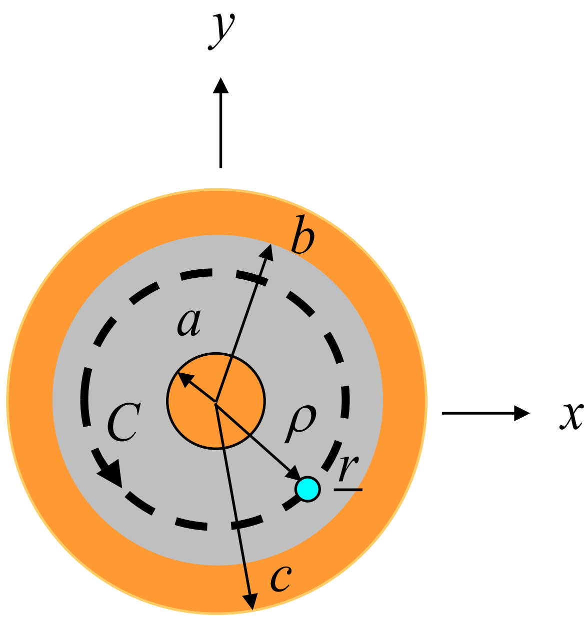


Example

Coaxial cable

DC current





This inner wire is solid.
The outer shield (jacket) of the coax has a thickness of $t = c - b$.

$$\rho < a \qquad J_z = J_z^A = \frac{I}{\pi a^2} \quad \left[\mathrm{A/m^2}\right]$$

Note: The permittivity of the material inside the coax does not matter here.

$$b < \rho < c \qquad J_z = J_z^B = \frac{-I}{\pi c^2 - \pi b^2} \quad \left[\mathrm{A/m^2}\right]$$

Note:
At DC, the current density inside the conductor is uniform, since the electric field is uniform (due to the fact that the voltage drop in the z direction is path independent).

Example (cont.)

$$\underline{H} = \underline{\hat{\phi}}\, H_\phi$$

The other components are zero, as in the wire example.

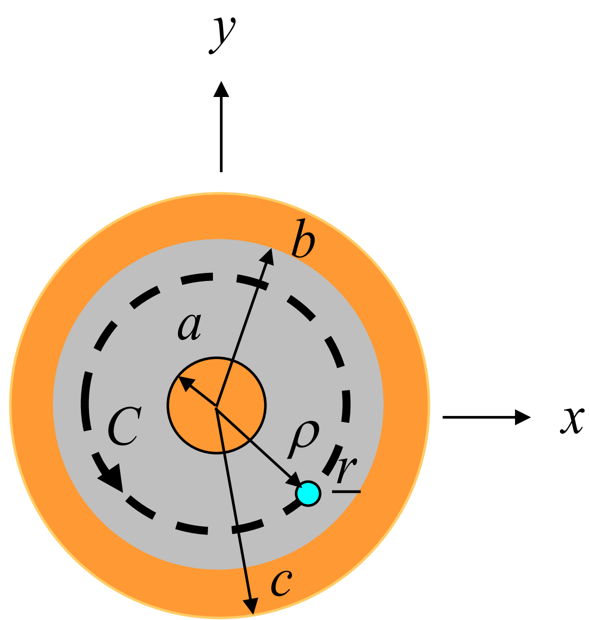
$$\oint_C \underline{H} \cdot \underline{dr} = I_{encl}$$

$$\oint_C \underline{H} \cdot \left(\underline{\hat{\phi}}\, \rho\, d\phi \right) = I_{encl}$$

$$\rho \int_0^{2\pi} H_\phi \, d\phi = I_{encl}$$

$$2\pi\rho H_\phi = I_{encl}$$

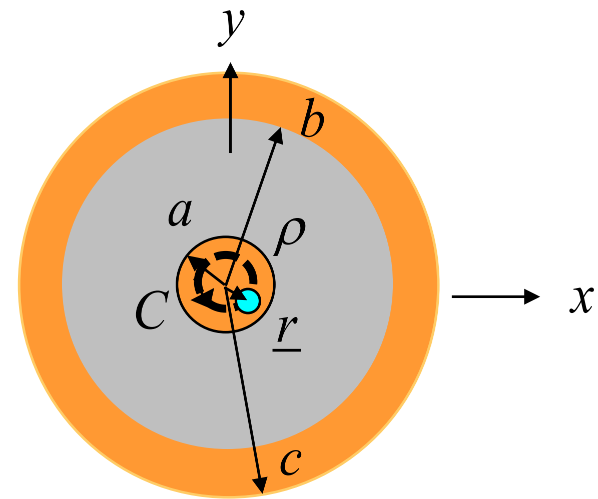
$$H_\phi = \frac{I_{encl}}{2\pi\rho}$$



This formula holds for <u>any</u> radius, as long as we get I_{encl} correct.

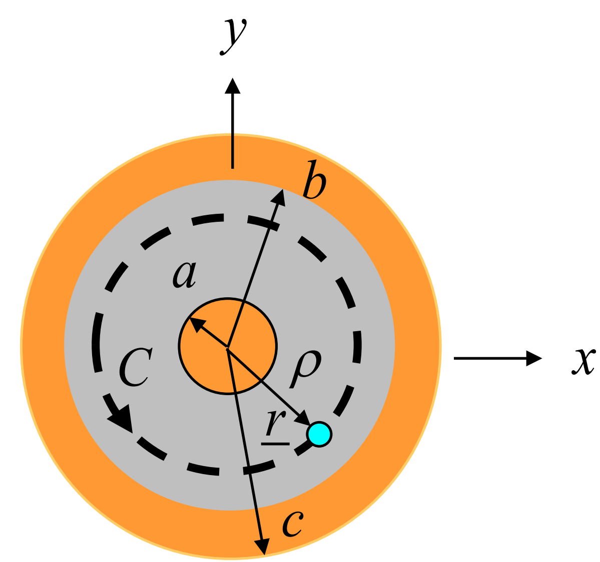
Example (cont.)

$\rho < a$



$$I_{encl} = J_z^A\left(\pi\rho^2\right) = \left(\frac{I}{\pi a^2}\right)\pi\rho^2$$

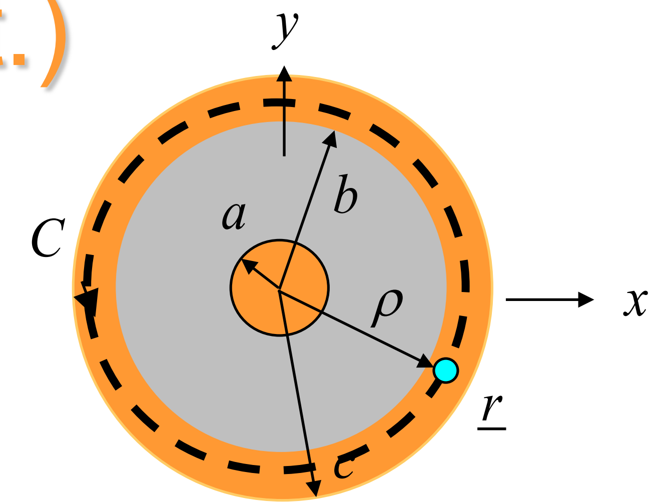
$a < \rho < b$



$$I_{encl} = +I$$

Example (cont.)

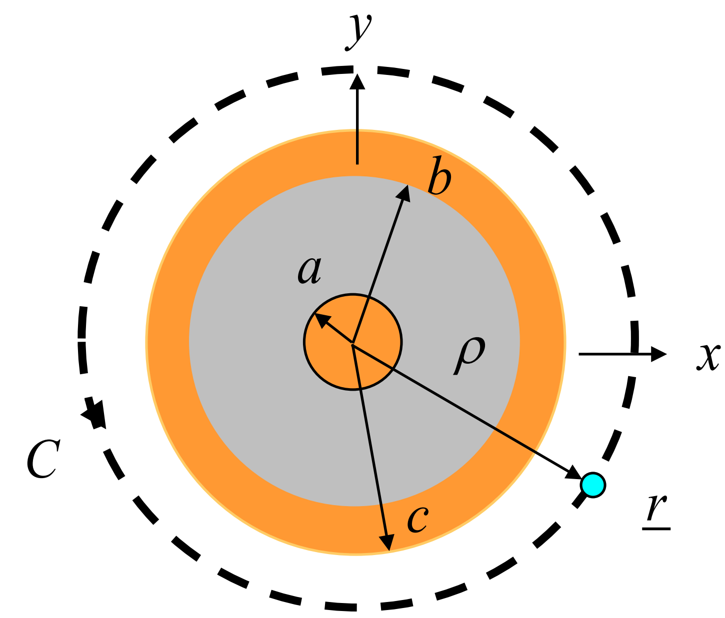
$b < \rho < c$



$$I_{encl} = +I + J_z^B\left(\pi\rho^2 - \pi b^2\right)$$

$$= +I + \frac{-I}{\pi c^2 - \pi b^2}\left(\pi\rho^2 - \pi b^2\right)$$

$\rho > c$



$$I_{encl} = +I + (-I) = 0$$

Example (cont.)

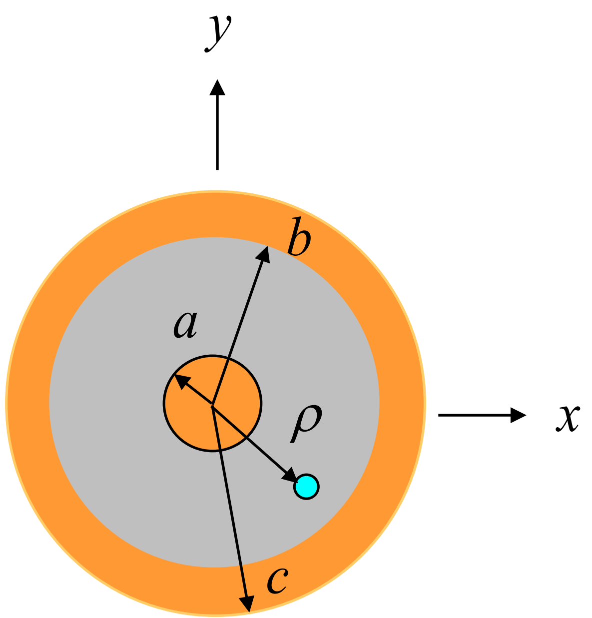
Summary

$\rho < a$ $$\underline{H} = \underline{\hat{\phi}}\frac{I}{2\pi\rho}\left(\frac{\rho^2}{a^2}\right) \quad [\text{A/m}]$$

$a < \rho < b$ $$\underline{H} = \underline{\hat{\phi}}\frac{I}{2\pi\rho} \quad [\text{A/m}]$$

$b < \rho < c$ $$\underline{H} = \underline{\hat{\phi}}\frac{I}{2\pi\rho}\left(1 - \frac{\pi\rho^2 - \pi b^2}{\pi c^2 - \pi b^2}\right) \quad [\text{A/m}]$$

$\rho > c$ $$\underline{H} = \underline{0} \quad [\text{A/m}]$$



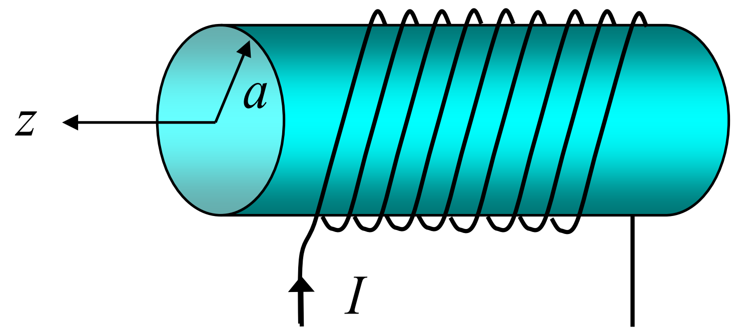
Note:
There is no magnetic field outside of the coax (a perfect "shielding property").

Example

Solenoid

Calculate $\underline{H}$

$n = \#$ turns/meter



$\mu = \mu_0 \mu_r$

Ideal solenoid:

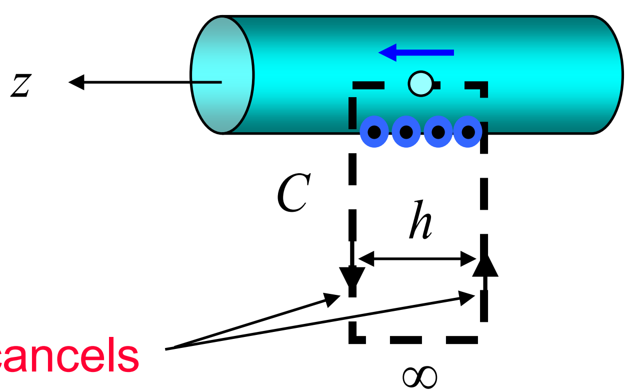
$n \rightarrow \infty$

Length $\rightarrow \infty$

Note:
We will say more about relative permeability later.

First, find H_z

$\rho < a$



$H_\rho \, d\rho$ cancels

H_z is zero at ∞

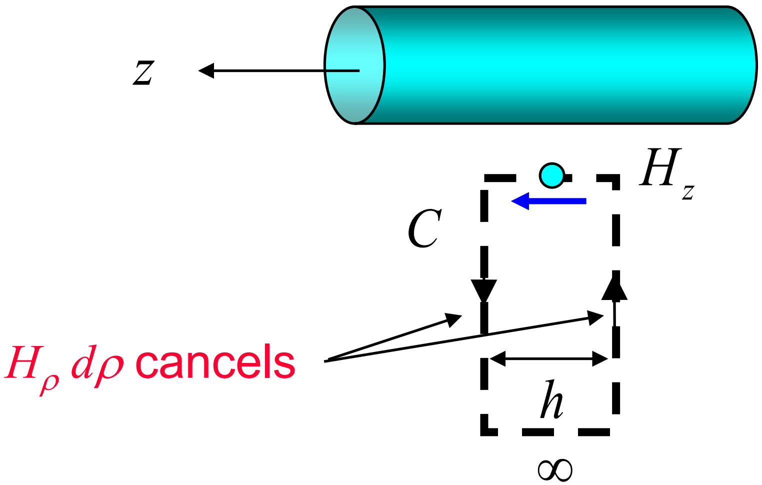
$$\oint_C \underline{H} \cdot \underline{dr} = I_{encl}$$

$$\Rightarrow H_z h = I_{encl} = +I\left(nh\right)$$

$$H_z = nI$$

Example (cont.)

$\rho > a$



$H_z = 0$ at ∞

$$H_z h = I_{encl}$$

$$I_{encl} = 0$$

$$\Rightarrow \quad H_z h = 0$$

so

$$H_z = 0$$

Hence

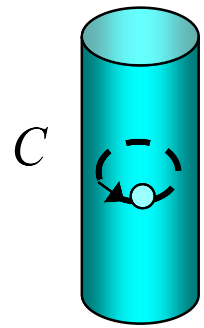
$$\begin{aligned} H_z &= (nI) \quad [\mathrm{A/m}], \quad \rho < a \\ &= 0, \quad \rho > a \end{aligned}$$

Example (cont.)

The other components of the magnetic field are zero:

1) $H_\phi = 0$ since

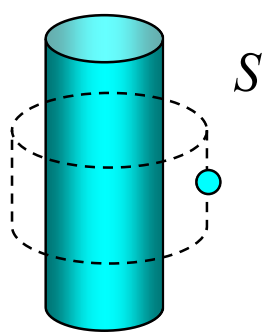
$$I_{encl} = 0$$



$$H_\phi 2\pi\rho = I_{encl}$$

2) $H_\rho = 0$ from

$$\oint_S \underline{B} \cdot \underline{\hat{n}}\, dS = 0$$

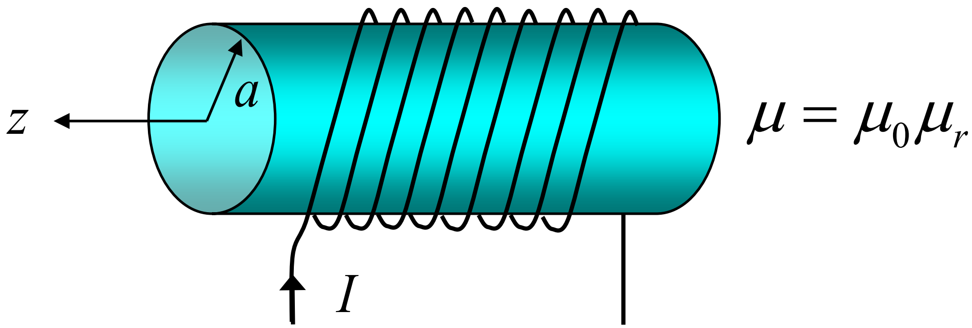


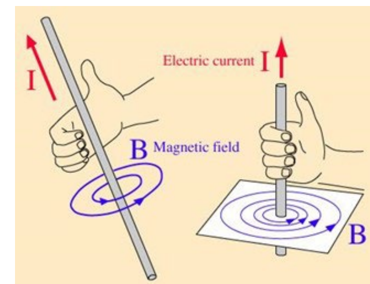
$$B_\rho 2\pi\rho h = 0$$

Example (cont.)

Summary

$n = \#$ turns/meter





$$\underline{H} = \underline{\hat{z}}\left(nI\right) \quad \left[\text{A/m}\right], \quad \rho < a$$
$$= \underline{0}, \quad \rho > a$$

Note:
A right-hand rule can be used to determine the direction of the magnetic field (the same rule as for a straight wire).

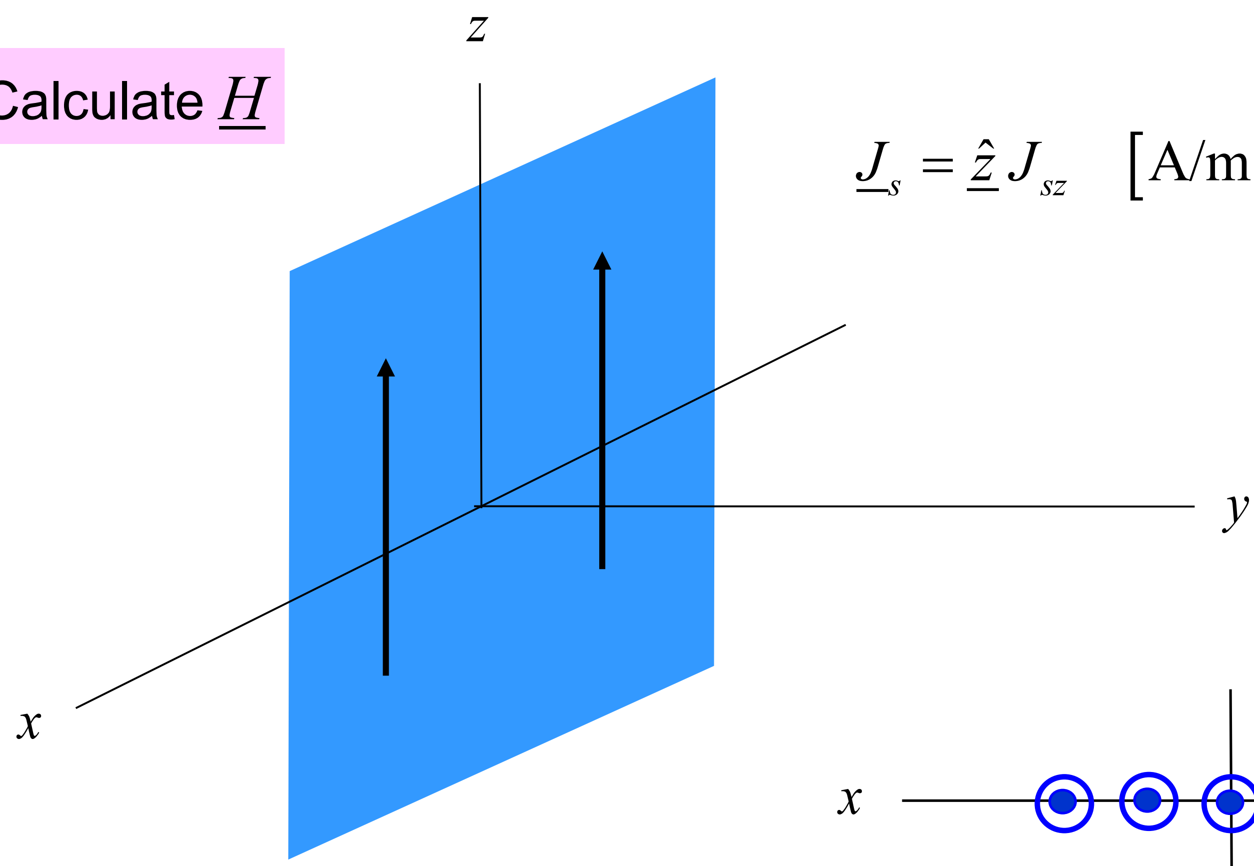
$$\underline{B} = \mu_0 \mu_r \underline{H}, \quad \rho < a$$

Note:
A larger relative permeability will give a larger magnetic flux density. This results is a stronger magnet, or a larger inductance.

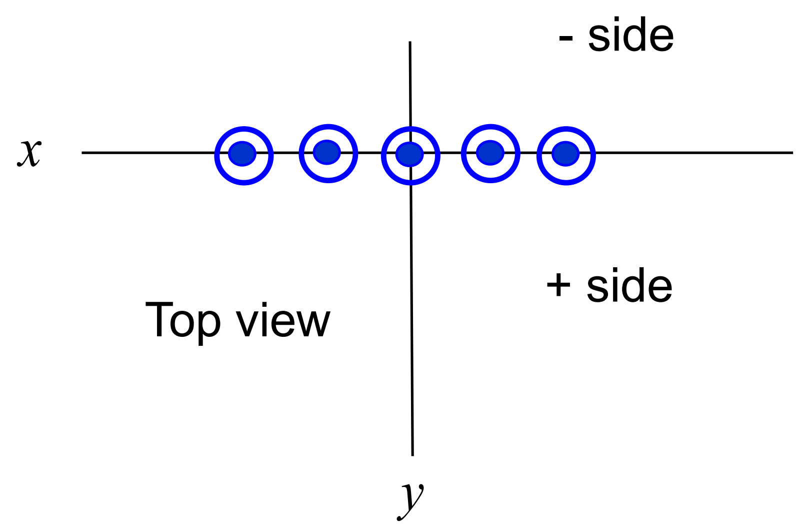
Example

Calculate $\underline{H}$

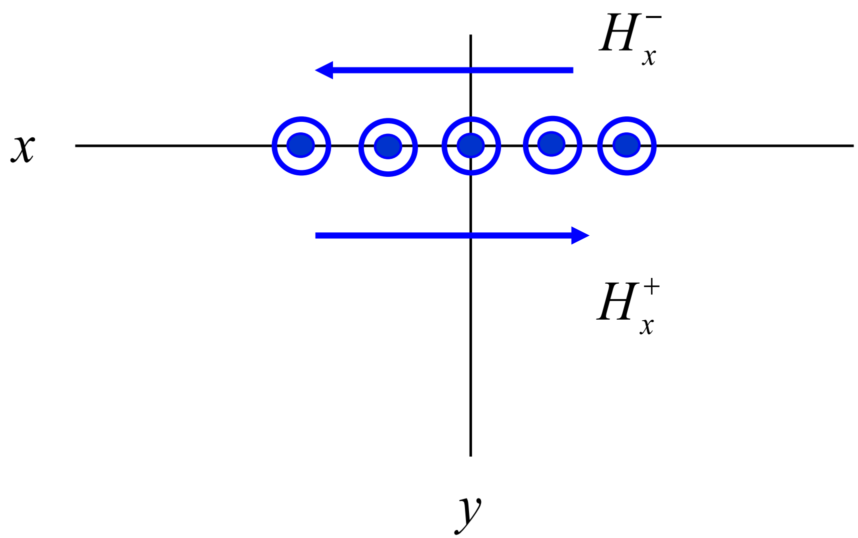
$$\underline{J}_s = \underline{\hat{z}}\, J_{sz} \quad [\mathrm{A/m}]$$



Infinite sheet of current



Example (cont.)

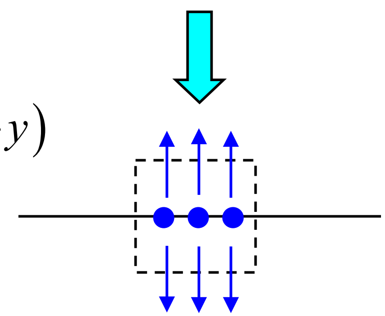


$$\underline{H} = \underline{\hat{x}}\, H_x(y)$$

$$H_z = 0$$ (superposition with line currents)

$$H_y = 0$$ (magnetic Gauss Law)

$$H_y(-y) = -H_y(+y)$$

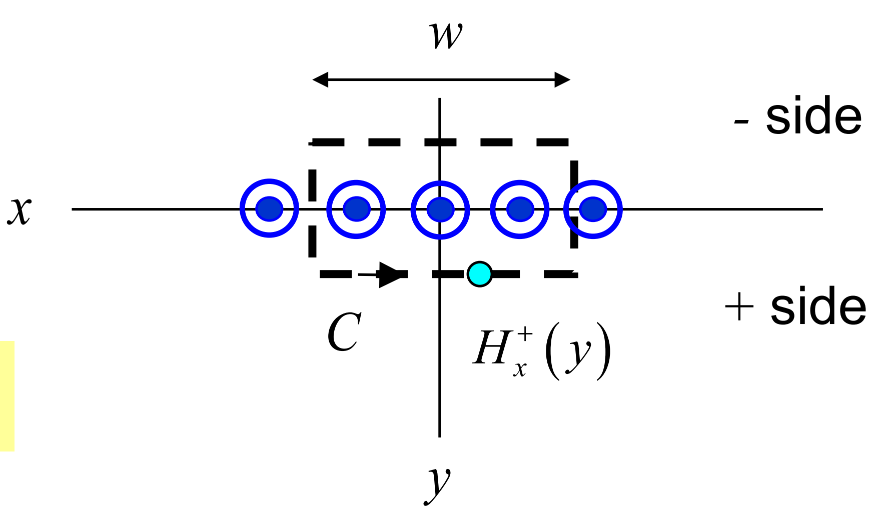


$$2H_y(+y)\,\Delta x \Delta z = 0$$

Also, by symmetry:

$$H_x(-y) = -H_x(+y)$$

Example (cont.)



$$\underline{H} = \hat{\underline{x}}\, H_x(y)$$

$$\oint_C \underline{H} \cdot \underline{dr} = I_{encl}$$

Note:
There is no contribution from the left and right edges (the edges are perpendicular to the field).

$$\int_{front} \underline{H} \cdot \underline{dr} = \int_{w/2}^{-w/2} H_x\, dx = -H_x^+ w$$

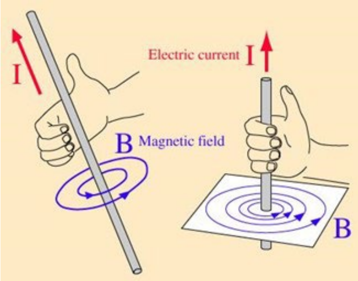
$$\int_{back} \underline{H} \cdot \underline{dr} = \int_{-w/2}^{w/2} H_x\, dx = H_x^- w$$

Example (cont.)

$$-H_x^+ w + H_x^- w = J_{sz} w$$

$$-H_x^+ - H_x^+ = J_{sz}$$

$$H_x^+ = -\frac{1}{2} J_{sz}$$



$$\underline{H} = \underline{\hat{x}}\left(-\frac{J_{sz}}{2}\right) \; [\mathrm{A/m}], \quad y > 0$$

$$\underline{H} = \underline{\hat{x}}\left(+\frac{J_{sz}}{2}\right) \; [\mathrm{A/m}], \quad y < 0$$

Note:
We can use a right hand-rule to quickly determine the direction of the magnetic field:

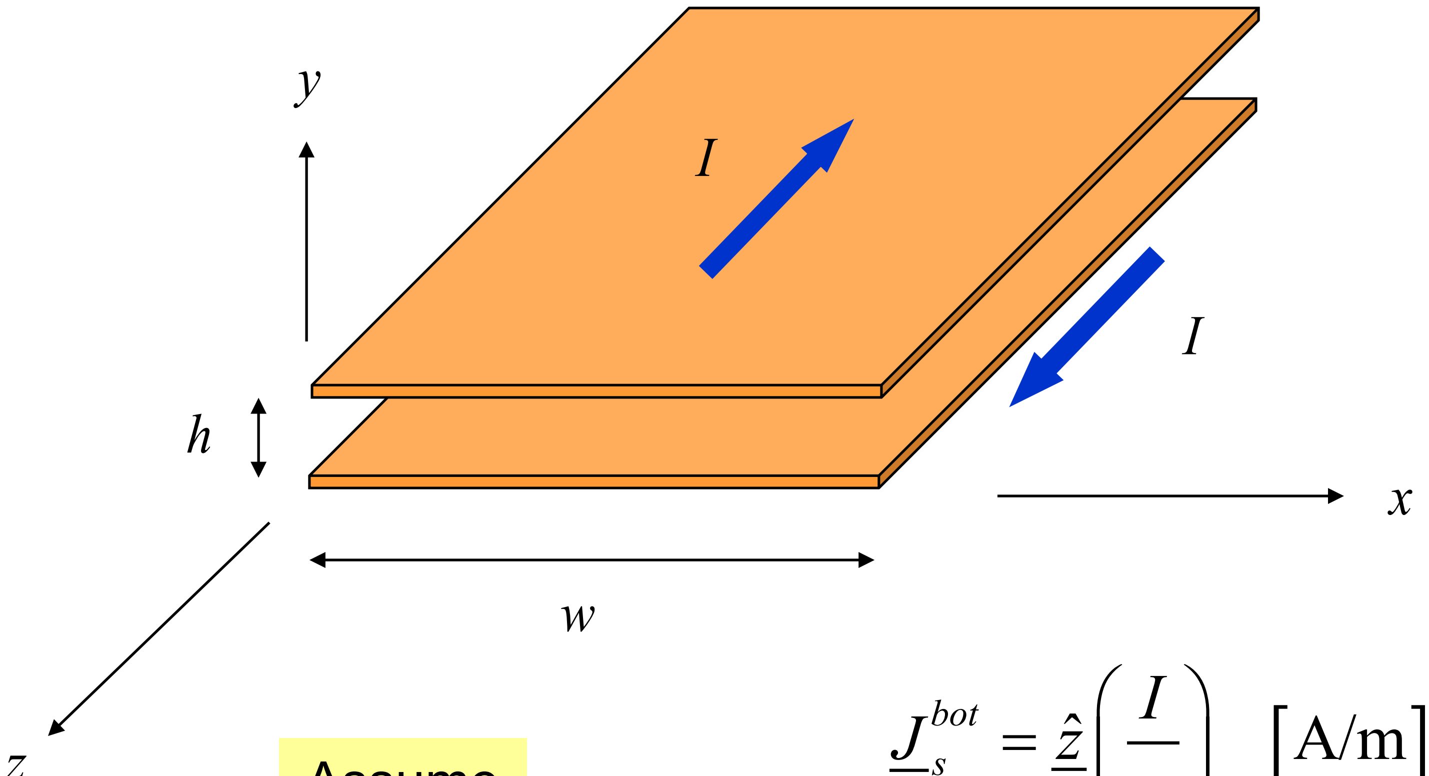
Put your thumb is in the direction of the current, and your fingers will give the overall direction of the magnetic field.

Note: The magnetic field does not depend on y.

Example

Find $\underline{H}$ everywhere

Parallel-plate transmission line



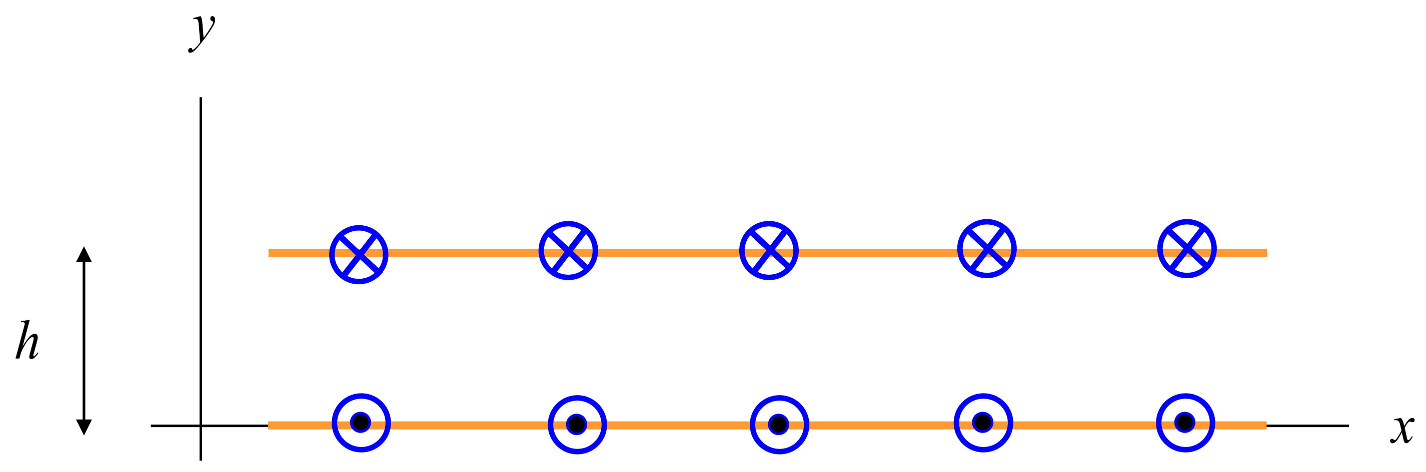
Assume
$w >> h$

(We neglect fringing here.)

$$\underline{J}_s^{bot} = \underline{\hat{z}}\left(\frac{I}{w}\right) \quad [\mathrm{A/m}]$$

$$\underline{J}_s^{top} = \underline{\hat{z}}\left(\frac{-I}{w}\right) \quad [\mathrm{A/m}]$$

Example (cont.)

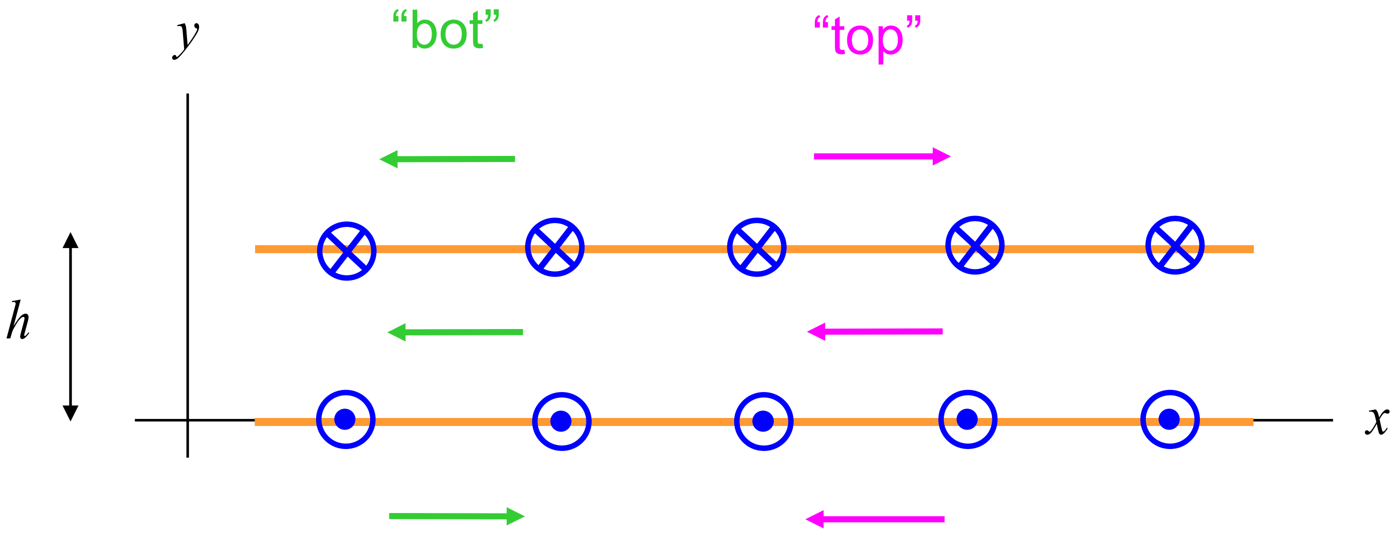


Two parallel sheets (plates) of opposite surface current

$$J_{sz}^{bot} = \left(\frac{I}{w}\right) \quad [\mathrm{A/m}]$$

$$J_{sz}^{top} = \left(\frac{-I}{w}\right) \quad [\mathrm{A/m}]$$

Example (cont.)



Magnetic field due to bottom plate

Magnetic field due to top plate

$$\underline{H} = \underline{H}^{bot} + \underline{H}^{top}$$

Example (cont.)

We then have

$$\underline{H} = \begin{cases} 2\left[-\underline{\hat{x}}\left(\dfrac{J_{sz}^{bot}}{2}\right)\right], & 0 < y < h \\ \underline{0}, & \text{otherwise} \end{cases}$$

Recall that

$$J_{sz}^{bot} = \left(\frac{I}{w}\right) \quad [\text{A/m}]$$

Hence

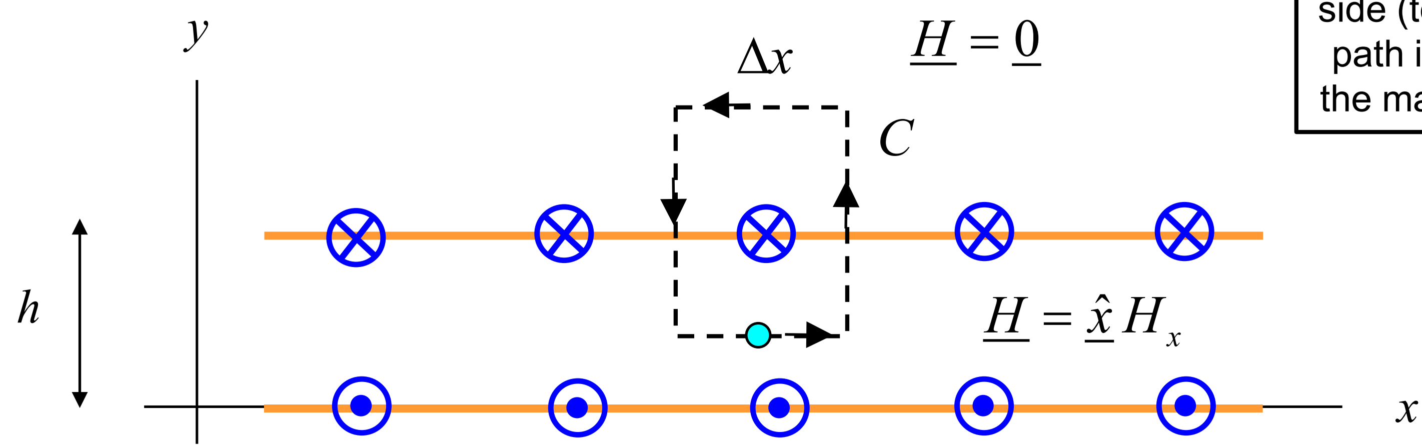
$$\underline{H} = -\underline{\hat{x}}\left(\frac{I}{w}\right) \quad [\text{A/m}], \quad 0 < y < h$$

$$\underline{H} = 0, \quad \text{otherwise}$$

Example (cont.)

We could also apply Ampere's law directly (without using superposition):

Note:
It is convenient to put one side (top) of the Amperian path in the region where the magnetic field is zero.



$$\underline{H} = \underline{0}$$

$$\oint_C \underline{H} \cdot \underline{dr} = I_{encl}$$

$$H_x \Delta x = J_{sz}^{top} \Delta x = \left(-\frac{I}{w} \right) \Delta x \quad \Rightarrow \quad H_x = -\frac{I}{w}$$

Hence

$$\underline{H} = -\underline{\hat{x}} \left(\frac{I}{w} \right) \quad [\mathrm{A/m}], \quad 0 < y < h$$

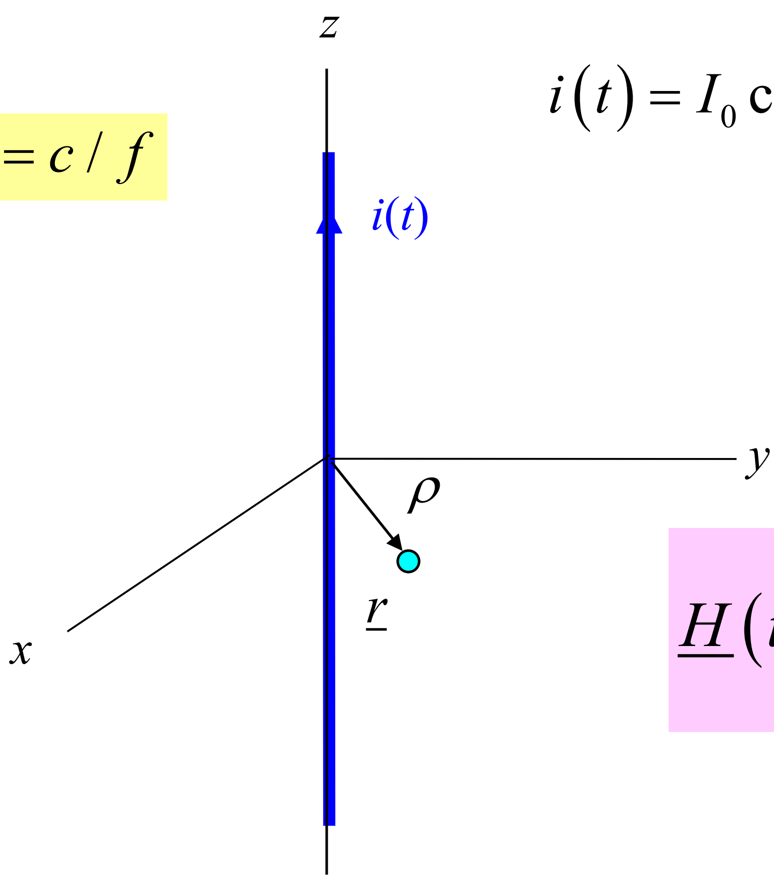
Low Frequency Calculations

At low frequency, the DC formulas should be accurate,
as long as we account for the time variation in the results.

Example (line current in time domain)

$$\rho << \lambda_0 = c / f$$

$$i(t) = I_0 \cos(\omega t + \phi)$$



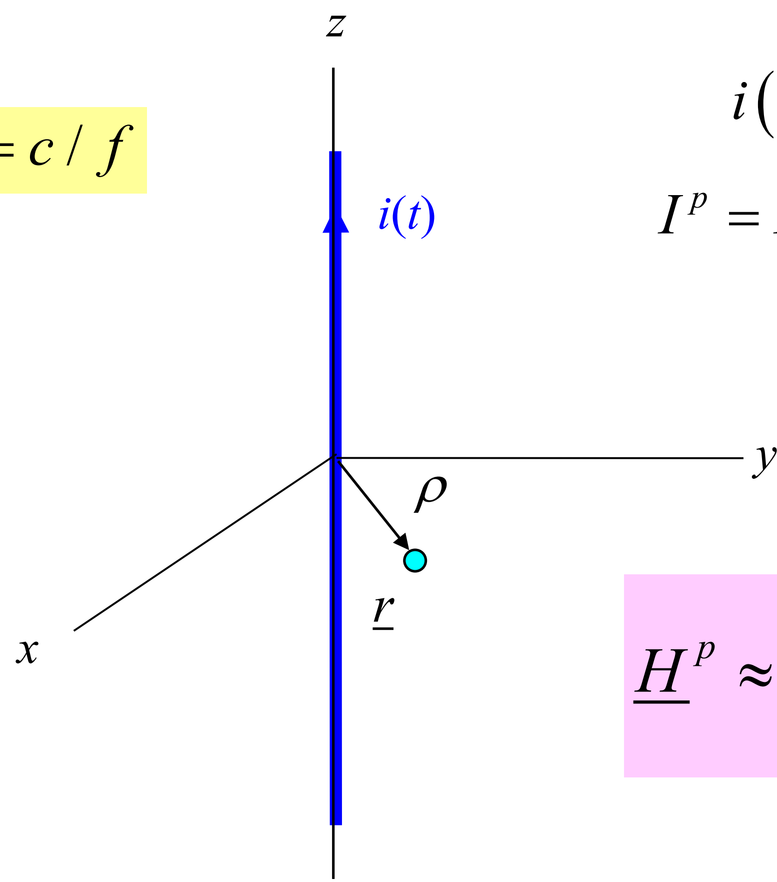
$$\underline{H}(t) \approx \underline{\hat{\phi}} \left(\frac{i(t)}{2\pi\rho} \right) \quad [\text{A/m}]$$

Low Frequency Calculations (cont.)

We can also used the DC formulas in the phasor domain.

Example (line current in the phasor domain)

$$\rho << \lambda_0 = c / f$$



$$i(t) = I_0 \cos(\omega t + \phi)$$

$$I^p = I_0 e^{j\phi}$$ (phasor current)

$$\underline{H}^p \approx \underline{\hat{\phi}} \left(\frac{I^p}{2\pi\rho} \right) \quad [\mathrm{A/m}]$$

(phasor $\underline{H}$ field)

Low Frequency Calculations (cont.)

Example (cont.)

Converting from phasor domain to the time domain:

$$\underline{H}^{p} \approx \underline{\hat{\phi}}\left(\frac{I^{p}}{2\pi\rho}\right) \quad \text{(phasor } \underline{H} \text{ field)}$$

$$\underline{H}(t) = \mathrm{Re}\left(\underline{H}^{p} e^{j\omega t}\right) \approx \mathrm{Re}\left(\underline{\hat{\phi}}\left(\frac{I^{p}}{2\pi\rho}\right)e^{j\omega t}\right) = \underline{\hat{\phi}}\left(\frac{1}{2\pi\rho}\right)\mathrm{Re}\left(I^{p} e^{j\omega t}\right)$$

$$= \underline{\hat{\phi}}\left(\frac{1}{2\pi\rho}\right)\mathrm{Re}\left(\left(I_0 e^{j\phi}\right)e^{j\omega t}\right)$$

$$= \underline{\hat{\phi}}\left(\frac{1}{2\pi\rho}\right)I_0 \cos\left(\omega t + \phi\right)$$

Hence

$$\boxed{\underline{H}(t) \approx \underline{\hat{\phi}}\left(\frac{i(t)}{2\pi\rho}\right) \quad [\mathrm{A/m}]}$$