# ECE 3318
# Applied Electricity and Magnetism

**Spring 2023**

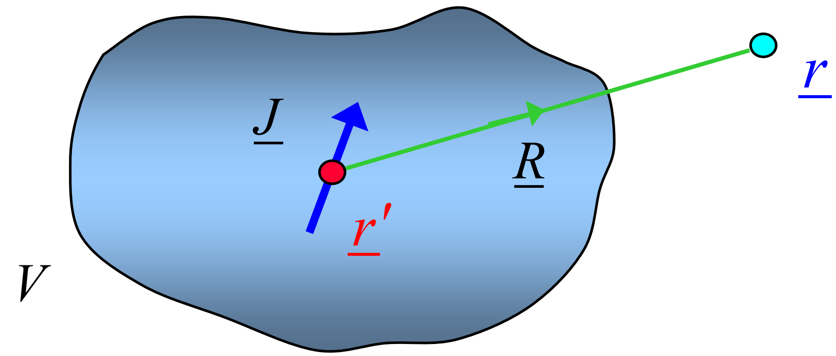
Prof. David R. Jackson
Dept. of ECE

## Notes 29
### Biot-Savart Law

# Biot-Savart Law

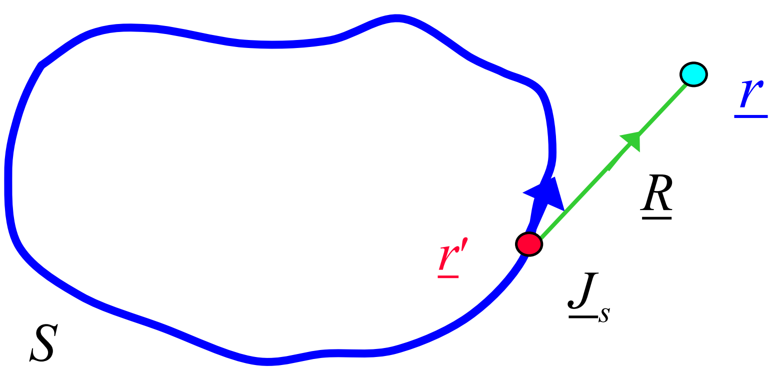
Please see the textbook for a derivation.

## a) Volume current




$$\underline{H} = \int_V \frac{\underline{J}(\underline{r}') \times \hat{\underline{R}}}{4\pi R^2} \, dV'$$

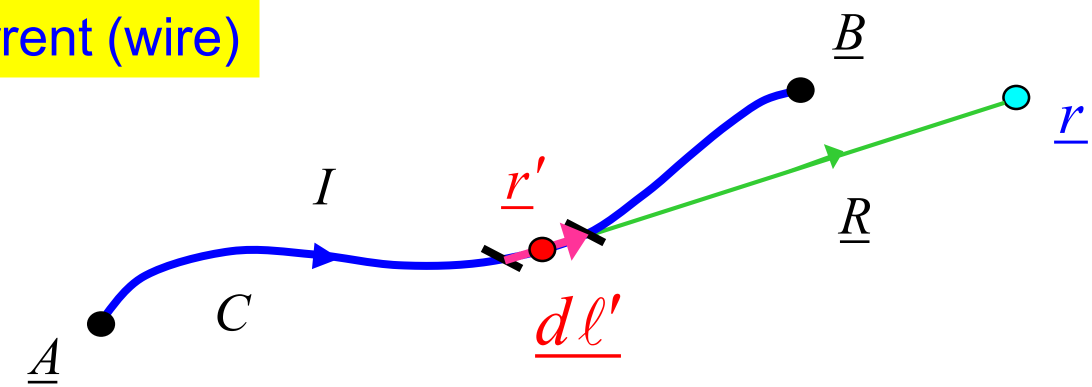
## b) Surface current




$$\underline{H} = \int_S \frac{\underline{J}_s(\underline{r}') \times \hat{\underline{R}}}{4\pi R^2} \, dS'$$

# Biot-Savart Law (cont.)

c) Line current (wire)




$$\underline{H} = \int_C \frac{I\, d\underline{\ell}' \times \hat{\underline{R}}}{4\pi R^2}$$

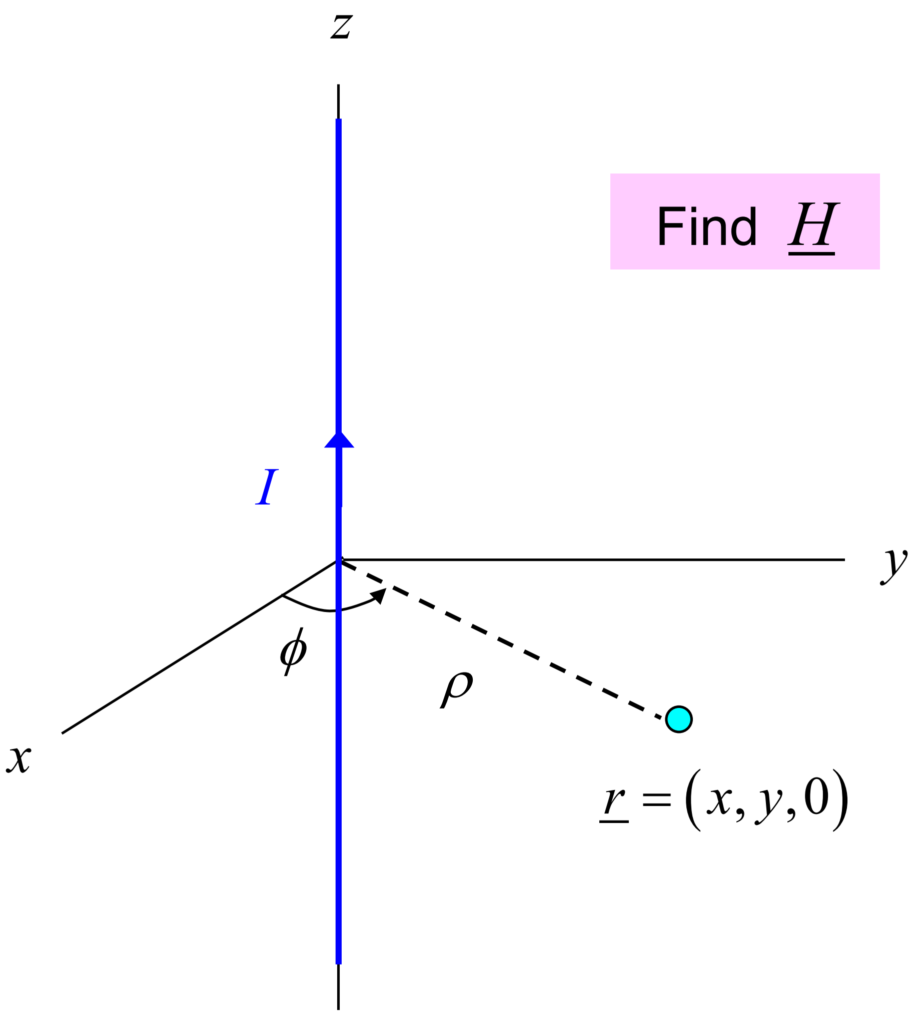
Note on notation:

$$d\underline{\ell}' = d\underline{r}'$$

**Rule:**
The contour $C$ is in direction of the $I$ arrow (i.e., the reference direction for the current).

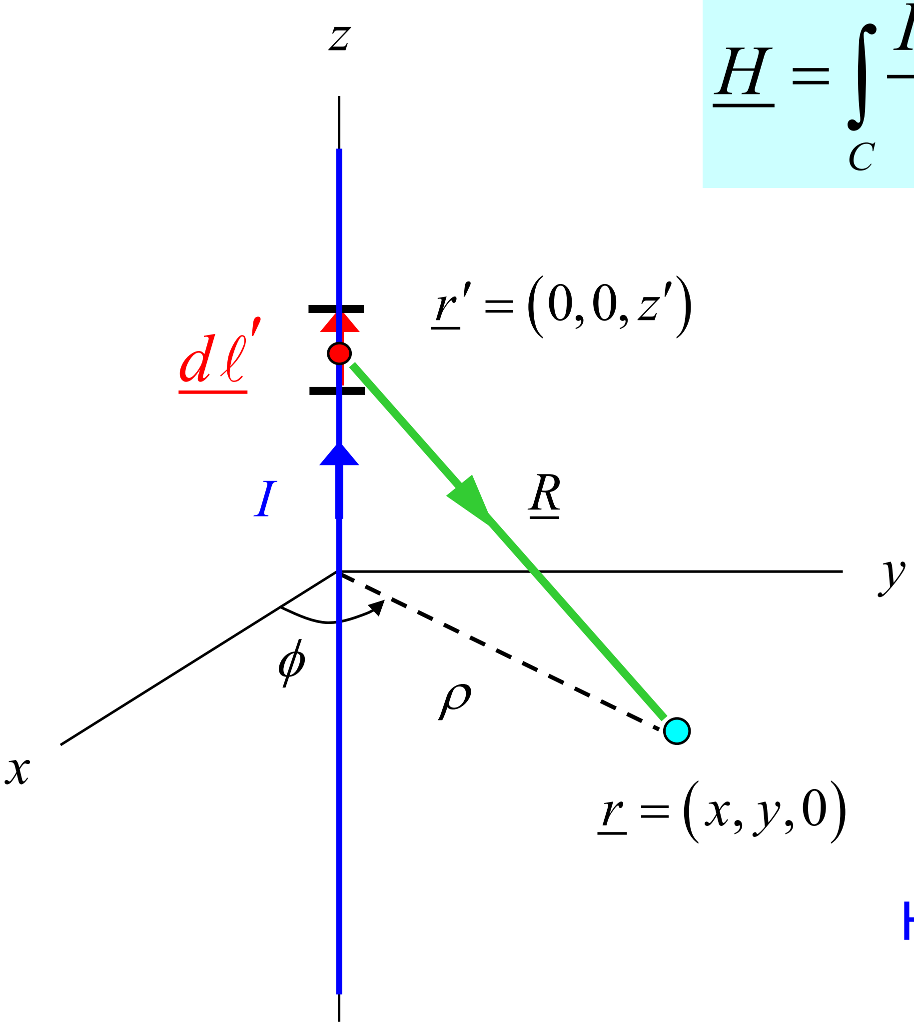
(This determines the starting point $\underline{A}$ and the ending point $\underline{B}$.)

# Example




Infinite line current

# Example (cont.)




$$\underline{H} = \int_C \frac{I\, d\underline{\ell}' \times \hat{\underline{R}}}{4\pi R^2}$$

$$d\underline{\ell}' = \hat{\underline{z}}\, dz'$$

$$\begin{aligned}
\underline{R} = \underline{r} - \underline{r}' &= \left(\hat{\underline{x}}\, x + \hat{\underline{y}}\, y + \hat{\underline{z}}(0)\right) - \hat{\underline{z}}\, z' \\
&= \left(\hat{\underline{x}}\, \rho\cos\phi + \hat{\underline{y}}\, \rho\sin\phi\right) - \hat{\underline{z}}\, z' \\
&= \rho\left(\hat{\underline{x}}\cos\phi + \hat{\underline{y}}\sin\phi\right) - \hat{\underline{z}}\, z' \\
&= \rho\hat{\underline{\rho}} - \hat{\underline{z}}\, z'
\end{aligned}$$

Hence we have:

$$\underline{R} = \rho\hat{\underline{\rho}} - \hat{\underline{z}}\, z'$$

$$R = \sqrt{\rho^2 + z'^2}$$

$$\hat{\underline{R}} = \frac{\rho\hat{\underline{\rho}} - z'\hat{\underline{z}}}{\sqrt{\rho^2 + z'^2}}$$

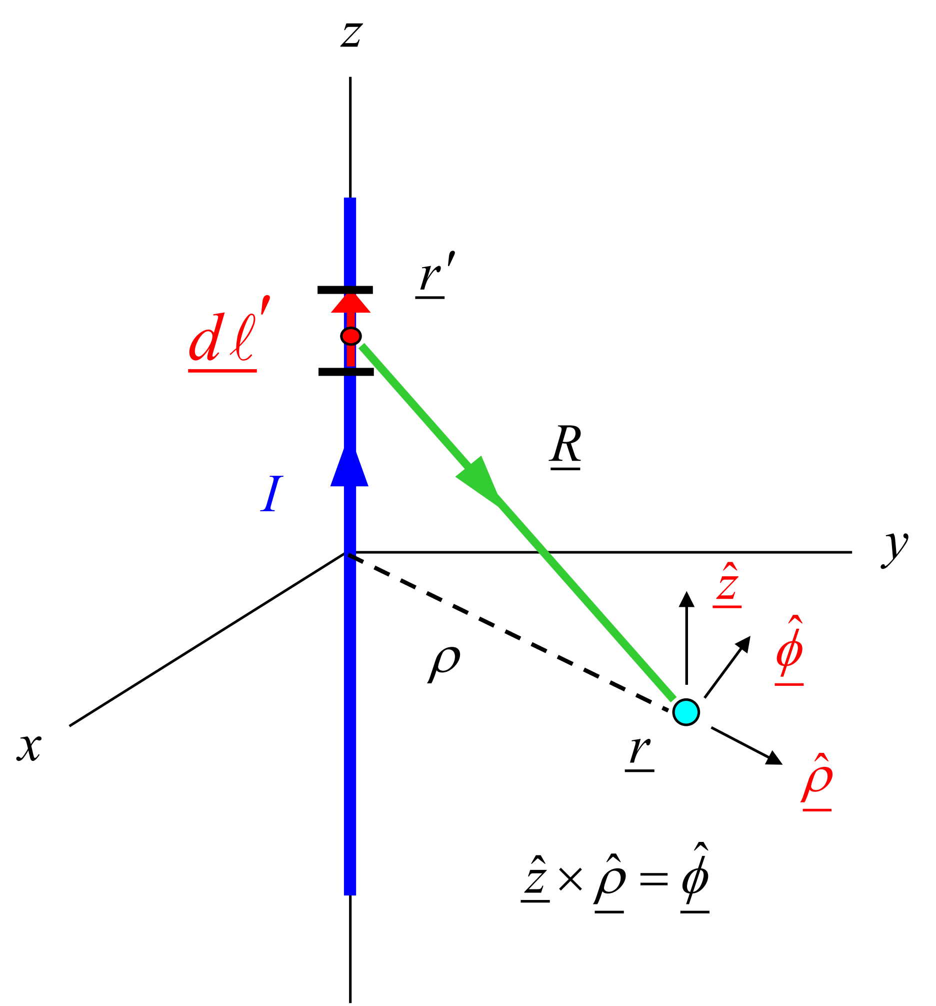
# Example (cont.)

$$\underline{H} = \int_C \frac{I\, d\underline{\ell}' \times \hat{\underline{R}}}{4\pi R^2}$$

$$\underline{H} = \int_C \frac{I\left(\hat{\underline{z}}\, dz'\right) \times \left[\dfrac{\rho\hat{\underline{\rho}} - z'\,\hat{\underline{z}}}{\sqrt{\rho^2 + z'^2}}\right]}{4\pi\left(\rho^2 + z'^2\right)}$$

$$= \frac{I}{4\pi} \int_{-\infty}^{+\infty} \frac{\rho\hat{\underline{\phi}}\, dz'}{\left(\rho^2 + z'^2\right)^{3/2}}$$

$$= \frac{I\rho}{4\pi}\hat{\underline{\phi}} \int_{-\infty}^{+\infty} \frac{dz'}{\left(\rho^2 + z'^2\right)^{3/2}}$$




$$\hat{\underline{z}} \times \hat{\underline{\rho}} = \hat{\underline{\phi}}$$

**Note:**
The contour $C$ runs up (in the direction of the current reference direction).
(This is why the lower limit is minus infinity, not plus infinity.)

# Example (cont.)

Note:

$$\int \frac{dx}{\left(x^2+a^2\right)^{3/2}} = \frac{x}{a^2\sqrt{x^2+a^2}}$$

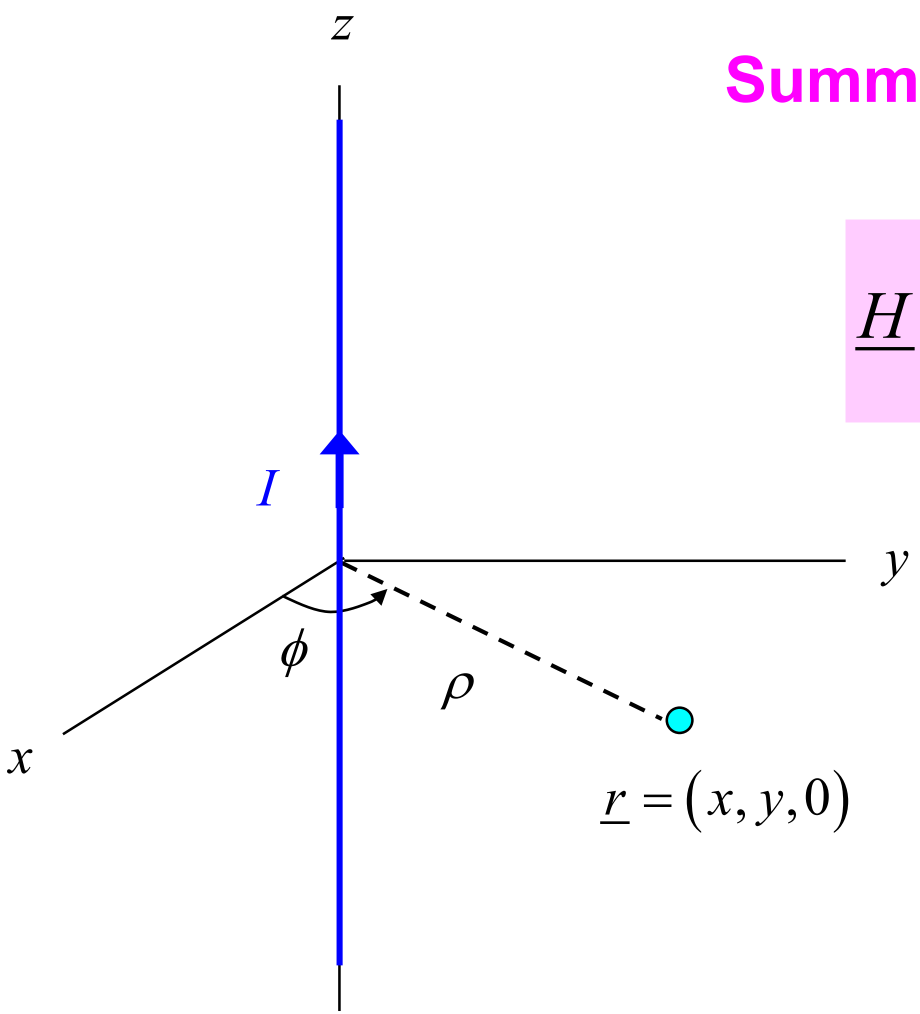
$$\underline{H} = \frac{I\rho}{4\pi}\underline{\hat{\phi}} \int_{-\infty}^{+\infty} \frac{dz'}{\left(\rho^2+z'^2\right)^{3/2}}$$

$$= \frac{I\rho}{4\pi}\underline{\hat{\phi}}\,(2)\int_{0}^{+\infty} \frac{dz'}{\left(z'^2+\rho^2\right)^{3/2}} = \frac{I\rho}{4\pi}\underline{\hat{\phi}}\,(2)\left[\frac{z'}{\rho^2\sqrt{z'^2+\rho^2}}\right]_0^{\infty}$$

$$= \frac{I\rho}{4\pi}\underline{\hat{\phi}}\,(2)\left[\frac{z'}{\rho^2 z'\sqrt{1+\rho^2/z'^2}}\right]_0^{\infty} = \frac{I\rho}{4\pi}\underline{\hat{\phi}}\,(2)\left[\frac{1}{\rho^2\sqrt{1+\rho^2/z'^2}}\right]_0^{\infty} = \frac{I\rho}{4\pi}\underline{\hat{\phi}}\,(2)\left[\frac{1}{\rho^2}\right]$$

$$= \underline{\hat{\phi}}\left(\frac{I}{2\pi\rho}\right)$$

# Example (cont.)

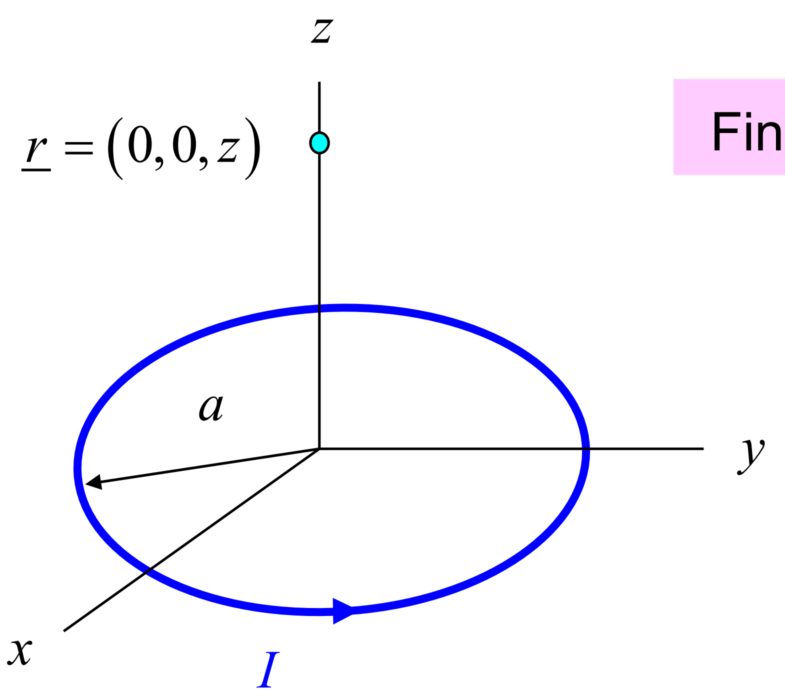



**Summary**

$$\underline{H} = \underline{\hat{\phi}} \left( \frac{I}{2\pi\rho} \right) \quad [\text{A/m}]$$

**Note:** This agrees with the answer obtained by Ampere's law.

# Example

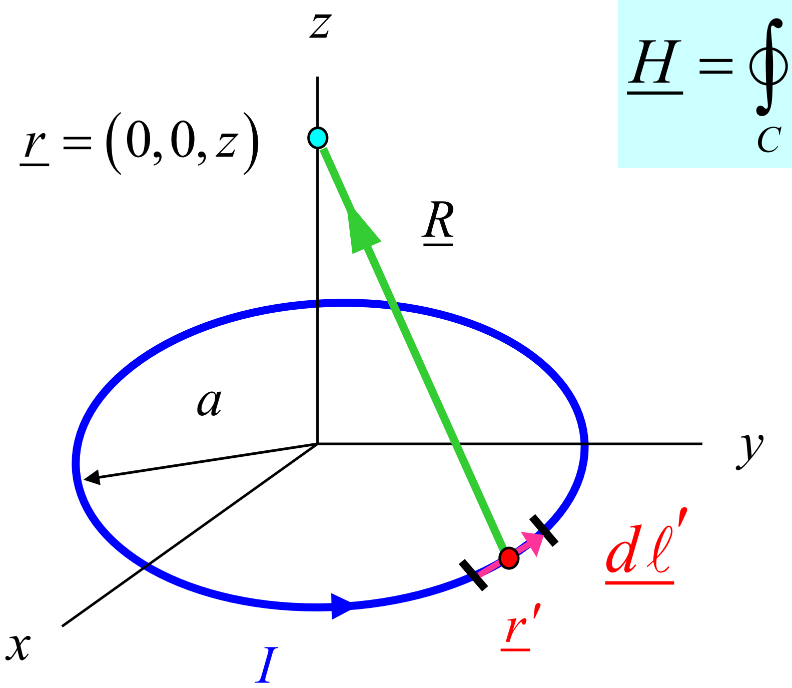



Find $\underline{H}$

Circular loop of current

**Note:** This problem cannot be solved using Ampere's law!

# Example (cont.)




$$\underline{H} = \oint_C \frac{I\, d\underline{\ell}' \times \underline{\hat{R}}}{4\pi R^2}$$

$$d\underline{\ell}' = \underline{\hat{\phi}}' \, a \, d\phi'$$

$$\underline{R} = \underline{\hat{z}}\, z + \left(-\underline{\hat{\rho}}'\right) a$$

$$R = \sqrt{z^2 + a^2}$$

$$\underline{\hat{R}} = \frac{\underline{\hat{z}}\, z - \underline{\hat{\rho}}' a}{\sqrt{z^2 + a^2}}$$

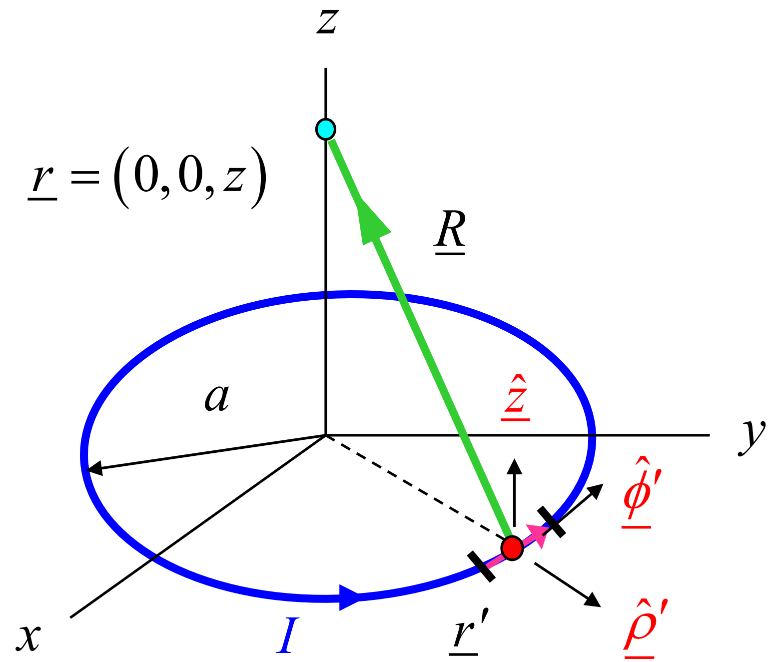
**Note:**
The path $C$ runs counterclockwise (in the direction of the current reference direction), and hence the integral in $\phi$ is from $0$ to $2\pi$ (not $2\pi$ to $0$).

**so**

$$\underline{H} = \frac{I}{4\pi} \int_C \frac{a\underline{\hat{\phi}}' \times \left(\underline{\hat{z}}\, z - \underline{\hat{\rho}}' a\right)}{\left(z^2 + a^2\right)^{3/2}} \, d\phi'$$

# Example (cont.)

$$\underline{H} = \frac{Ia}{4\pi}\int_C \frac{\hat{\underline{\phi}}' \times \left(\hat{\underline{z}}\, z - \hat{\underline{\rho}}' a\right)}{\left(z^2 + a^2\right)^{3/2}}\, d\phi'$$




**Note:**
A prime is not necessary on unit vector $z$.

$$\underline{H} = \frac{Ia}{4\pi}\int_0^{2\pi} \frac{\hat{\underline{\rho}}'\, z - a\left(-\hat{\underline{z}}\right)}{\left(z^2 + a^2\right)^{3/2}}\, d\phi'$$

$$= \frac{Ia}{4\pi\left(z^2 + a^2\right)^{3/2}}\left[\cancel{z\int_0^{2\pi} \hat{\underline{\rho}}'\, d\phi'} + a\,\hat{\underline{z}}\int_0^{2\pi} d\phi'\right]$$

$$\hat{\underline{\phi}}' \times \hat{\underline{z}} = \hat{\underline{\rho}}'$$

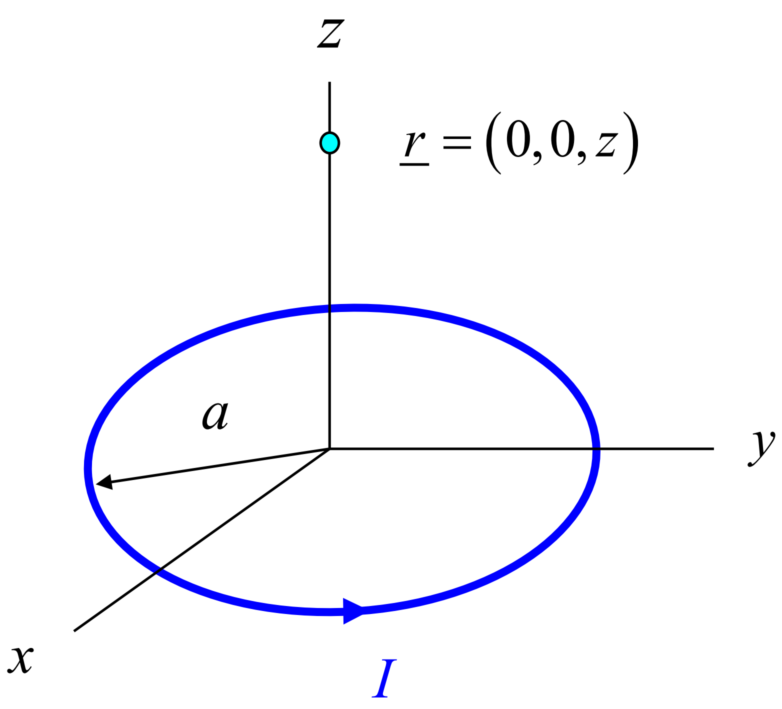
$$\hat{\underline{\phi}}' \times \hat{\underline{\rho}}' = -\hat{\underline{z}}$$

$$\underline{H} = \frac{Ia}{4\pi\left(z^2 + a^2\right)^{3/2}}\left(a\,\hat{\underline{z}}\, 2\pi\right)$$

so $$\underline{H} = \hat{\underline{z}}\frac{Ia^2}{2\left(z^2 + a^2\right)^{3/2}} \quad [\mathrm{A/m}]$$

# Example (cont.)

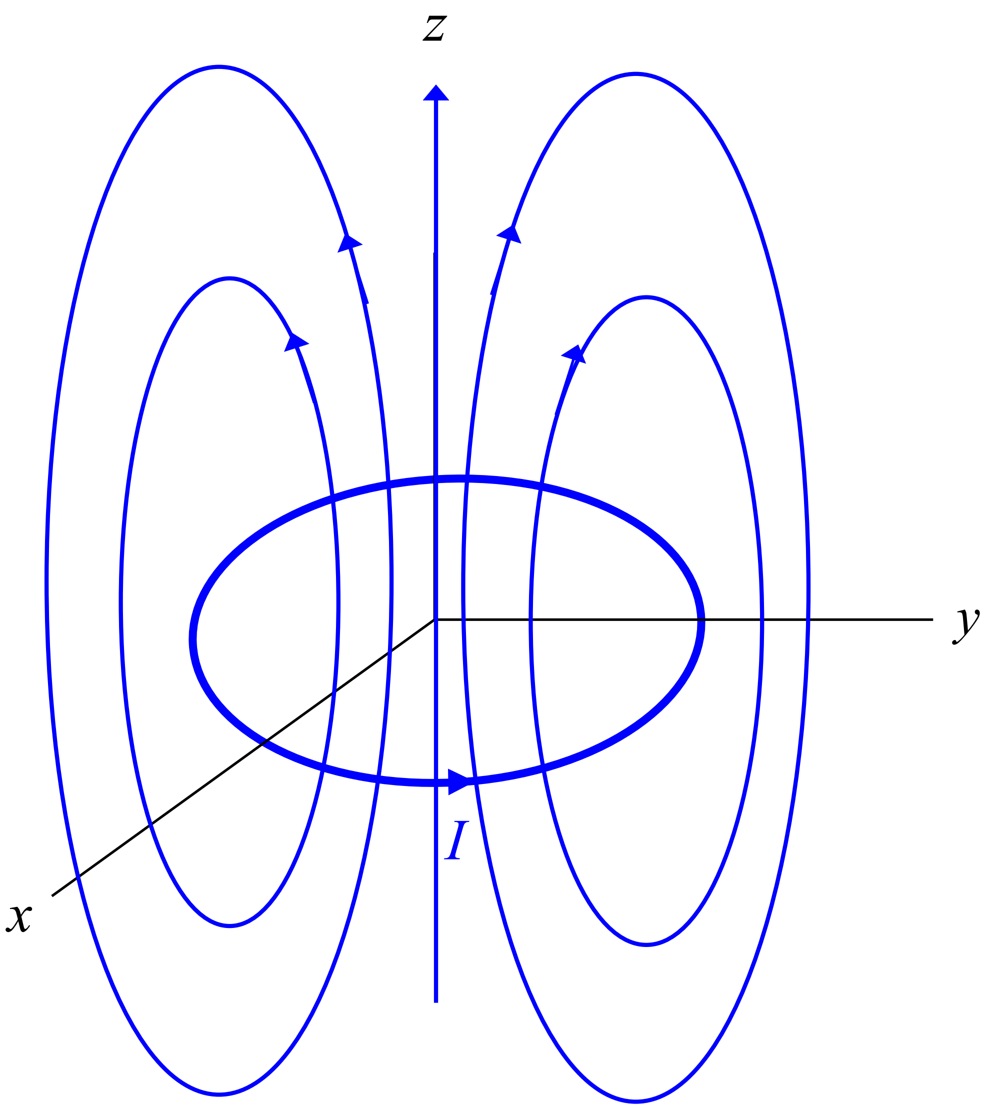
**Summary**



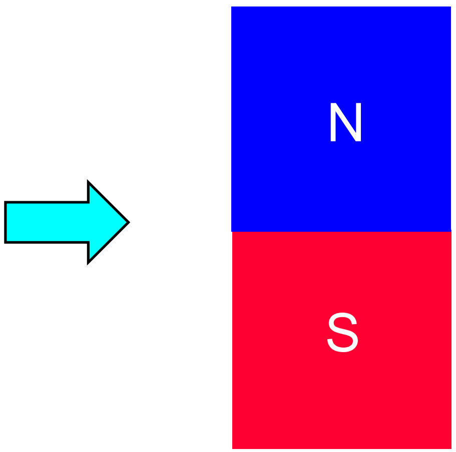

$$\underline{H} = \underline{\hat{z}} \frac{I a^2}{2\left(z^2 + a^2\right)^{3/2}} \quad [\mathrm{A/m}]$$

# Example (cont.)

Flux picture

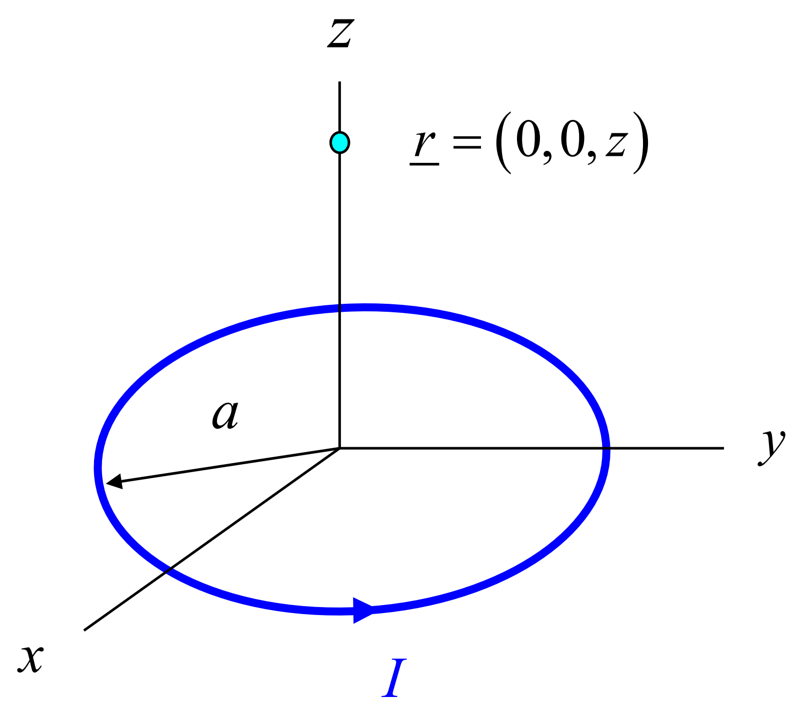



Far away, the loop acts like a bar magnet, with the north pole on top.



# Example (cont.)

$$\underline{H} = \underline{\hat{z}} \frac{Ia^2}{2\left(z^2 + a^2\right)^{3/2}} \quad [\text{A/m}]$$




For $z >> a$:

$$\underline{H} \approx \underline{\hat{z}} \left( \frac{Ia^2}{2} \right) \frac{1}{|z|^3} \quad [\text{A/m}]$$

The magnetic field falls off quickly with distance!

# Example (cont.)

**Wireless charging system:**

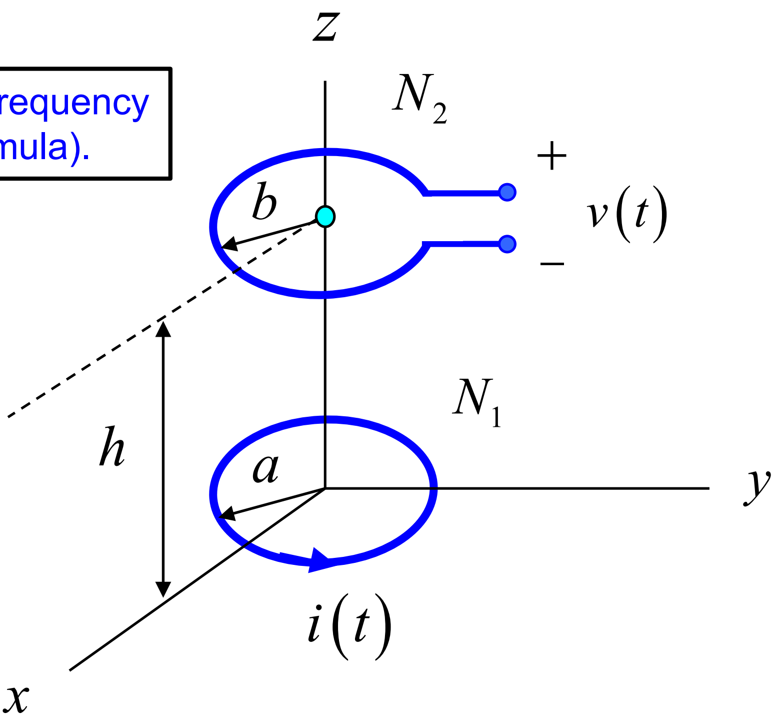
$$i(t) = I_0 \cos(\omega t)$$

Accurate at low frequency (uses DC formula).

$$H_z^a(t) \approx N_1 \left( \frac{i(t)\, a^2}{2} \right) \frac{1}{h^3}$$

$$\psi_z^b(t) \approx \mu_0 \left( \pi b^2 \right) \left[ N_1 \left( \frac{i(t)\, a^2}{2} \right) \frac{1}{h^3} \right]$$

$$v(t) = N_2 \frac{d\psi_z^b}{dt}$$




**Note:**
We assume that the magnetic field from coil 1 is approximately constant over the area of coil 2.

$$\Rightarrow \quad v(t) = -N_2 \mu_0 \left( \pi b^2 \right) N_1 \left( \frac{I_0 \omega a^2}{2} \right) \frac{1}{h^3} \sin(\omega t)$$

# Example (cont.)

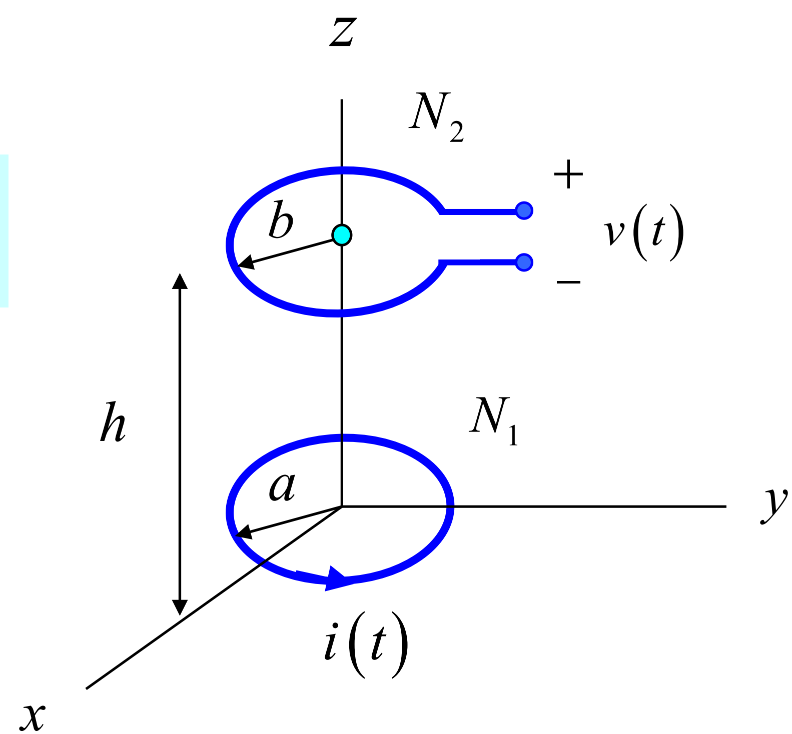
**Wireless charging system:**

$$i(t) = I_0 \cos(\omega t)$$

$$v(t) = -\omega N_1 N_2 \mu_0 I_0 (A_1 A_2) \frac{1}{h^3}\left(\frac{1}{2\pi}\right)\sin(\omega t)$$

$$A_1 = \pi a^2$$

$$A_2 = \pi b^2$$




Voltage on receive coil $\propto 1/h^3$

Power delivered by receive coil $\propto v^2(t)$

Power delivered by receive coil $\propto 1/h^6$