ECE 3318
Applied Electricity and Magnetism

Spring 2023

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Notes 3
Current

Notes prepared by the EM Group
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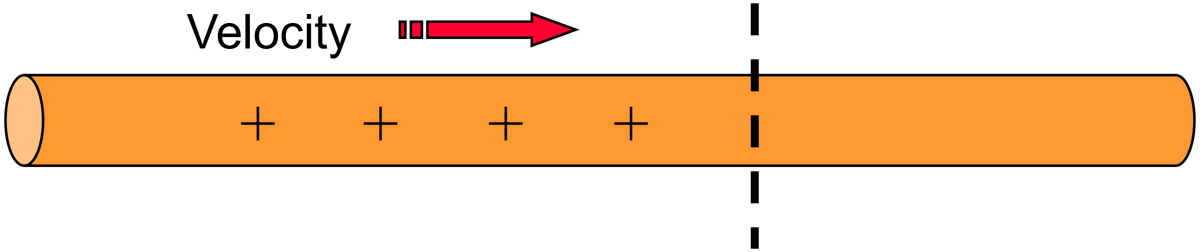
Current

Current is the flow of charge: the unit is the ampere (amp).

1 amp ≡ 1 [C/s]

Ampere

Convention (Ben Franklin): Current flows in the direction that positive charges move in.



Current flows from left to right.

Current (cont.)

History of the Ampere

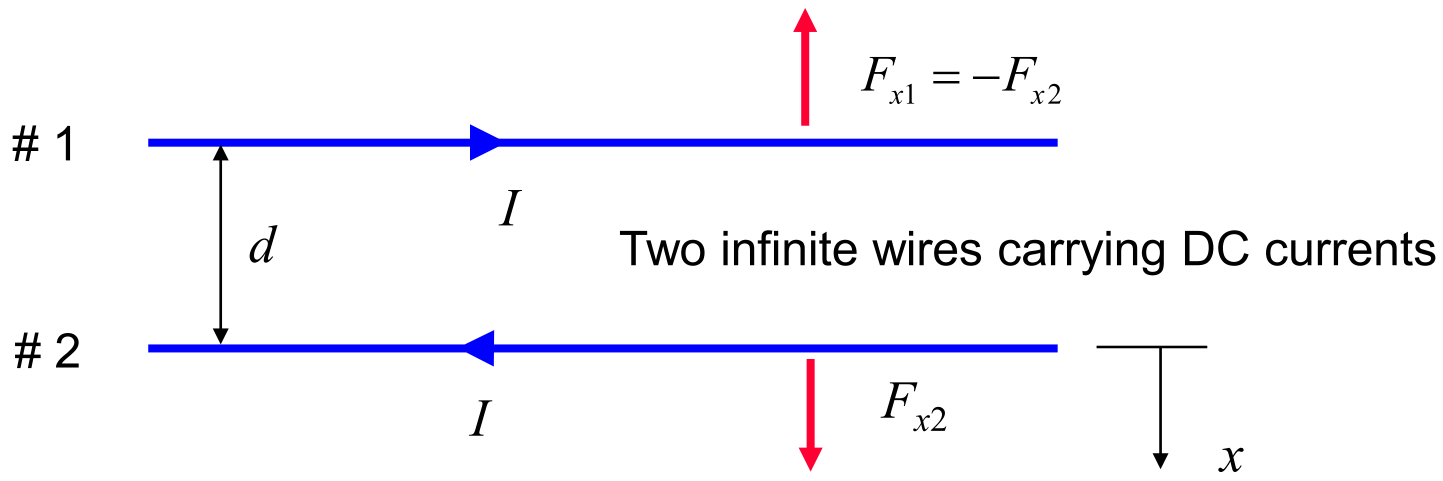
- Before May 20, 2019:
  - The amp was defined first, from the force between two parallel wires.
  - The coulomb was then defined from the amount of charge flowing in one second.
- After May 20, 2019:
  - The coulomb was defined first.
  - The amp was then defined from the flow rate of one coulomb per second.

Ampere

1 amp $\equiv$ 1 [C/s]

Current (cont.)

Previous definition of Amp (before May 20, 2019):



From later in the semester: $$F_{x2} = \frac{I^2 \mu_0}{2\pi d}$$

$$\mu_0 = 4\pi \times 10^{-7} \;\; [\text{H/m}]$$

Exact value!

Definition of $I = 1$ Amp:

$$F_{x2} = 2 \times 10^{-7} \quad [\text{N/m}] \quad \text{when} \quad d = 1 \quad [\text{m}]$$

Note: $\mu_0 =$ permeability of free space.

Current (cont.)

Present definition of Amp (after May 20, 2019):

$$1 \text{ amp} \equiv 1 \; [\mathrm{C/s}]$$

Present definition of coulomb:

$$\text{Proton}: \quad q = e \equiv 1.602176634 \times 10^{-19} \; [\mathrm{C}]$$

↑ Definition

← Exact defined value

$$\mu_0 \doteq 12.5663706212 \times 10^{-7} \; [\mathrm{A/m}]$$

(derivation omitted)

(no longer an exact value)

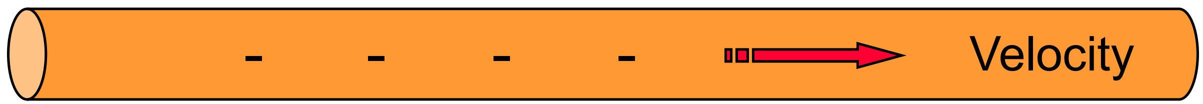
$$\text{Note}: \quad 4\pi \doteq 12.566370614$$

Note:
Although the values of both ε_0 and μ_0 changed in 2019, the speed of light did not. It is still defined the way it was in 1983:
$c \equiv 2.99792458 \times 10^{8}$ [m/s]

Current (cont.)

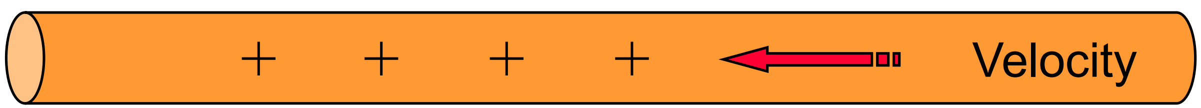
Postulate: Positive charges moving one way is equivalent to negative charges moving the other way (in terms of most measurable physical electromagnetic effects).

Flow rate is 1 [C] per second



This is equivalent to:

Flow rate is 1 [C] per second

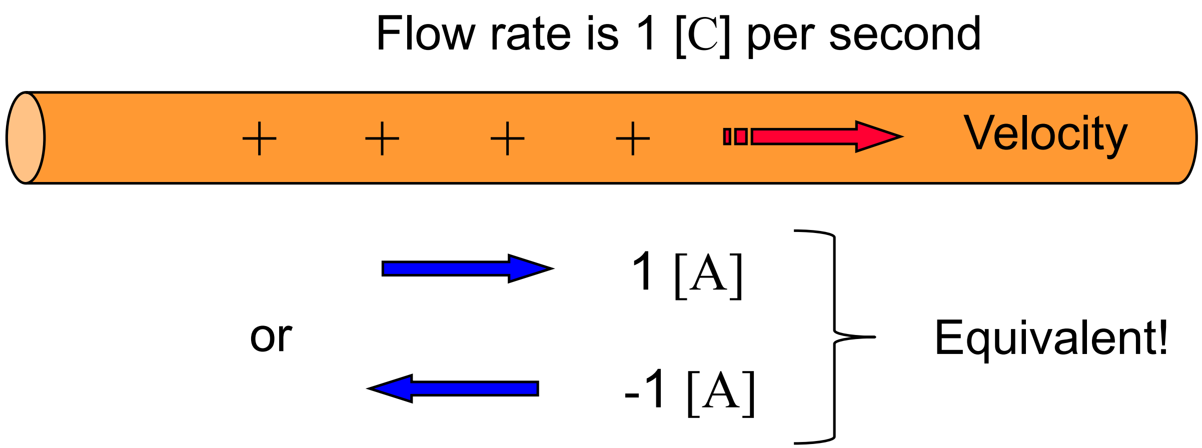


In both cases:  1 [A]

Current flows from right to left.

Current (cont.)

Sign convention: A positive current flowing one way is equivalent to a negative current flowing the other way.



Note:
The blue arrow is called the *reference direction arrow*. It tells us the direction we measure the current in.

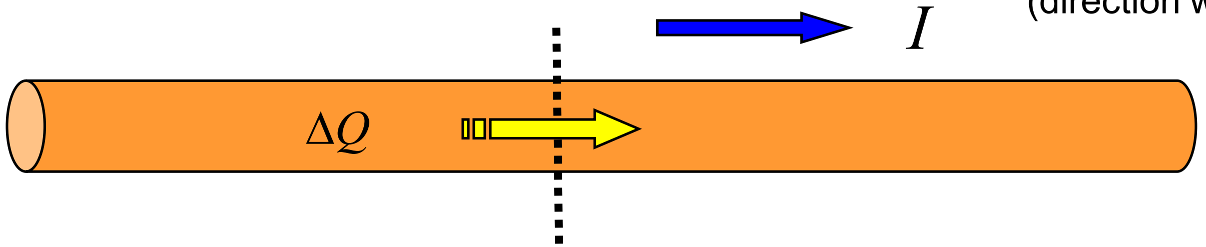
It is very useful in circuit theory to assume *reference directions* and allow for negative current.

Current (cont.)

Mathematical definition of current

Reference direction arrow

(direction we are measuring I)



ΔQ = amount of charge (positive or negative) that crosses the plane in the direction of the reference arrow in time Δt.

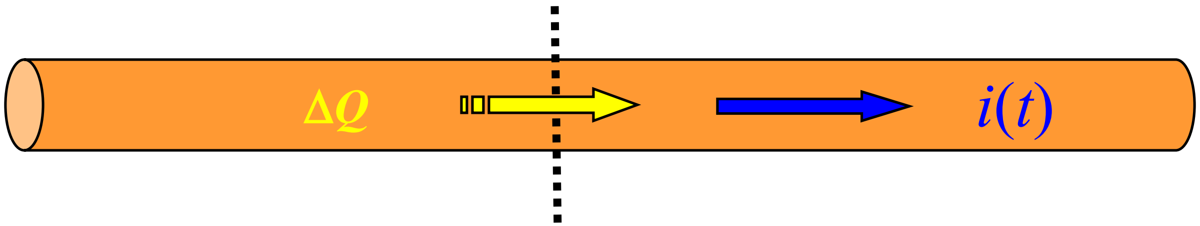
$$I = \frac{\Delta Q}{\Delta t}$$

Uniform current

More generally, $$i(t) = \frac{dQ}{dt}$$

Non-uniform current

Example



$$i(t) = 10\sqrt{2}\cos(2\pi 60t) \ [\mathrm{A}]$$

Find the charge $Q(t)$ that crosses the dashed line going from left to right in the time interval $(0, t)$ [s].

$$i(t) = \frac{dQ}{dt} \quad \Rightarrow \quad Q(t) = \int i(t)\,dt + C_1$$

$$= \int_0^t i(t)\,dt + \cancel{C_2} \quad (Q(0) = 0)$$

$$= \int_0^t i(t)\,dt$$

Example (cont.)

$$Q(t) = \int_0^t i(t)\,dt$$

$$= \int_0^t 10\sqrt{2}\cos(2\pi 60t)\,dt$$

$$= 10\sqrt{2}\left[\frac{1}{2\pi 60}\sin(2\pi 60t)\right]_0^t$$

$$Q(t) = \frac{10\sqrt{2}}{120\pi}\sin(2\pi 60t) \quad [\mathrm{C}]$$

Note on Vector Notation

Review of vector notation:

$\underline{J}$: vector

J or $\left|\underline{J}\right|$: magnitude of $\underline{J}$ vector

J_x : x component of $\underline{J}$ vector

$$\underline{J} = \underline{\hat{x}}\, J_x + \underline{\hat{y}}\, J_y + \underline{\hat{z}}\, J_z$$

$$\underline{J} = \underline{\hat{J}}\left|\underline{J}\right|$$

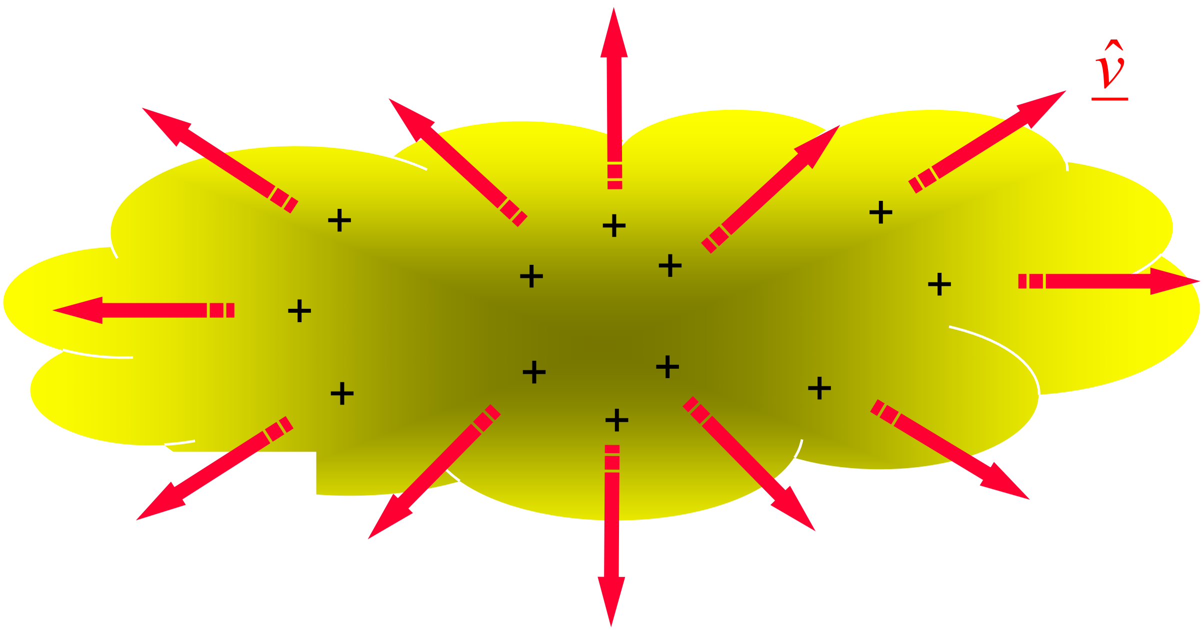
$$\left|\underline{J}\right| \equiv \sqrt{\underline{J} \cdot \underline{J}} = \sqrt{J_x^2 + J_y^2 + J_z^2}$$

Note: For complex vectors we have

$$\left|\underline{J}\right| \equiv \sqrt{\underline{J} \cdot \underline{J}^*} = \sqrt{\left|J_x\right|^2 + \left|J_y\right|^2 + \left|J_z\right|^2}$$

Current Density Vector $\underline{J}$

Consider a general "cloud" of charge density, where the charge density as well as the velocity of the charges may be different at each point.

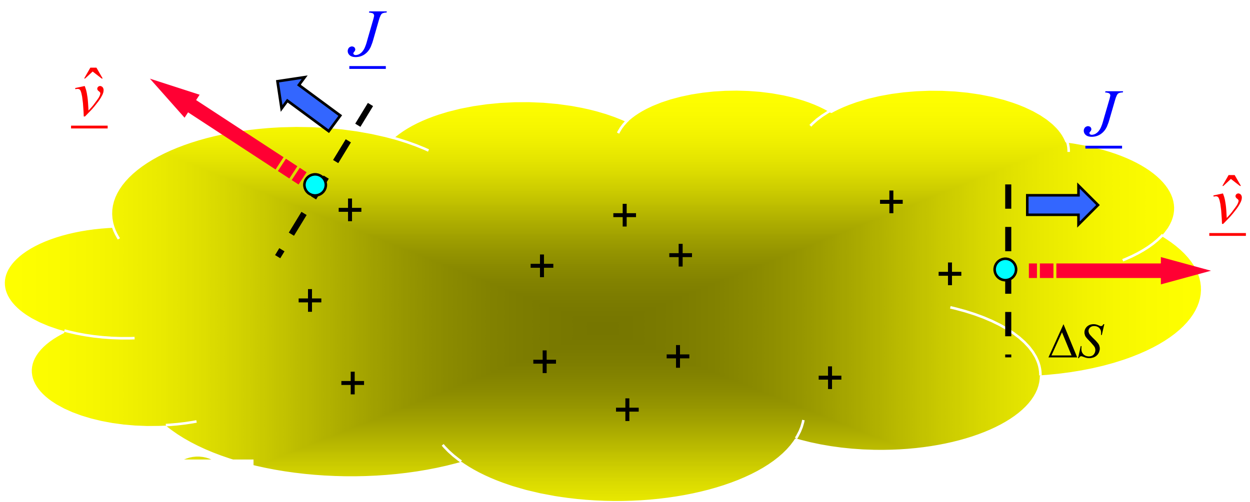


$$\hat{\underline{J}} = \hat{\underline{v}}$$

The current-density vector points in the direction of current flow (the direction of positive charge motion).

Current Density Vector (cont.)

The magnitude of the current-density vector $\underline{J}$ tells us the current density (current per square meter) that is crossing a small surface that is perpendicular to the current-density vector.



I = the current crossing the surface ΔS in the direction of the velocity vector.

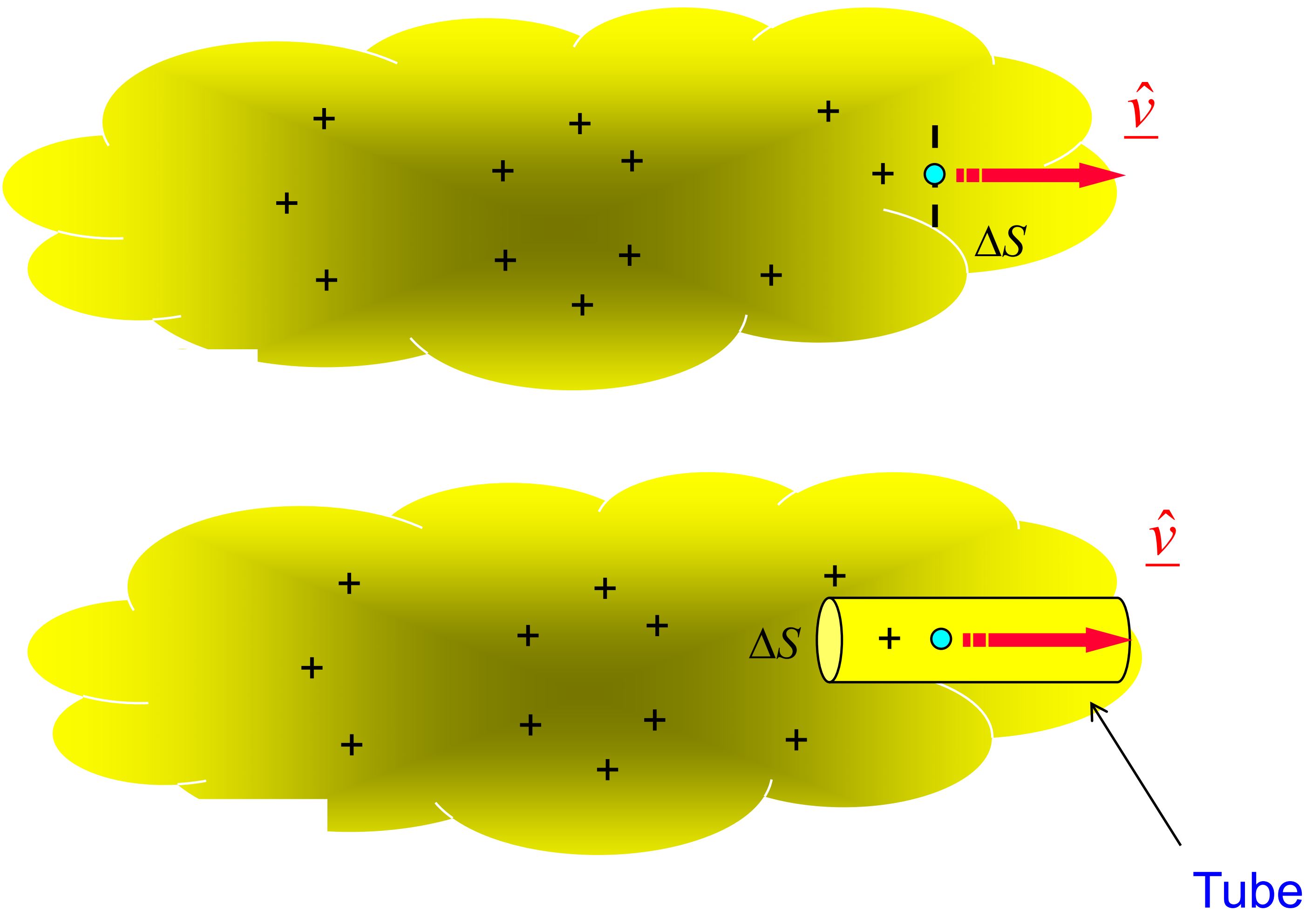
$$J = \left|\underline{J}\right| \equiv \frac{I}{\Delta S} \quad [\text{A/m}^2]$$

Hence

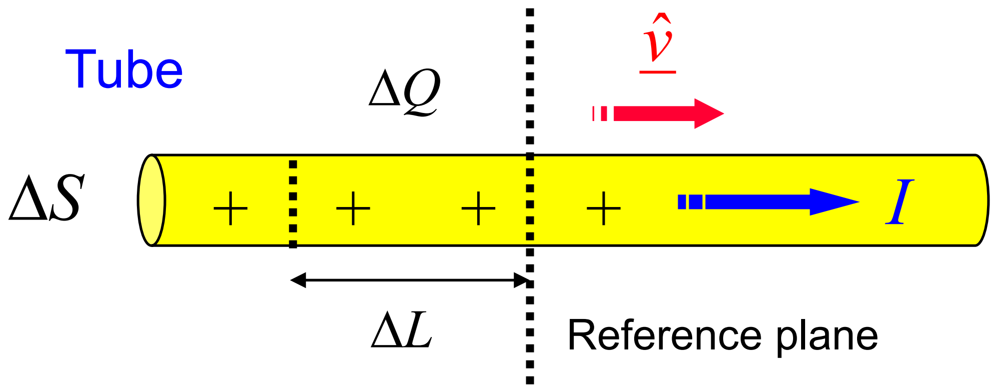
$$\underline{J} \equiv \left(\frac{I}{\Delta S}\right) \hat{\underline{v}} \quad [\text{A/m}^2]$$

Current Density Vector (cont.)

Consider a small tube of moving charges inside the cloud:



Current Density Vector (cont.)



ΔL = distance traveled by charges in time Δt.

$$J = \frac{I}{\Delta S} = \frac{\Delta Q / \Delta t}{\Delta S} = \frac{(\Delta Q / \Delta t)\Delta L}{\Delta S\, \Delta L} = \left(\frac{\Delta Q}{\Delta V}\right)\left(\frac{\Delta L}{\Delta t}\right)$$

or $J = \rho_v v$ so $\underline{J} = J\,\hat{\underline{v}} = (\rho_v v)\,\hat{\underline{v}}$

Hence $$\underline{J} = \rho_v \underline{v}$$

Current Density Vector (cont.)

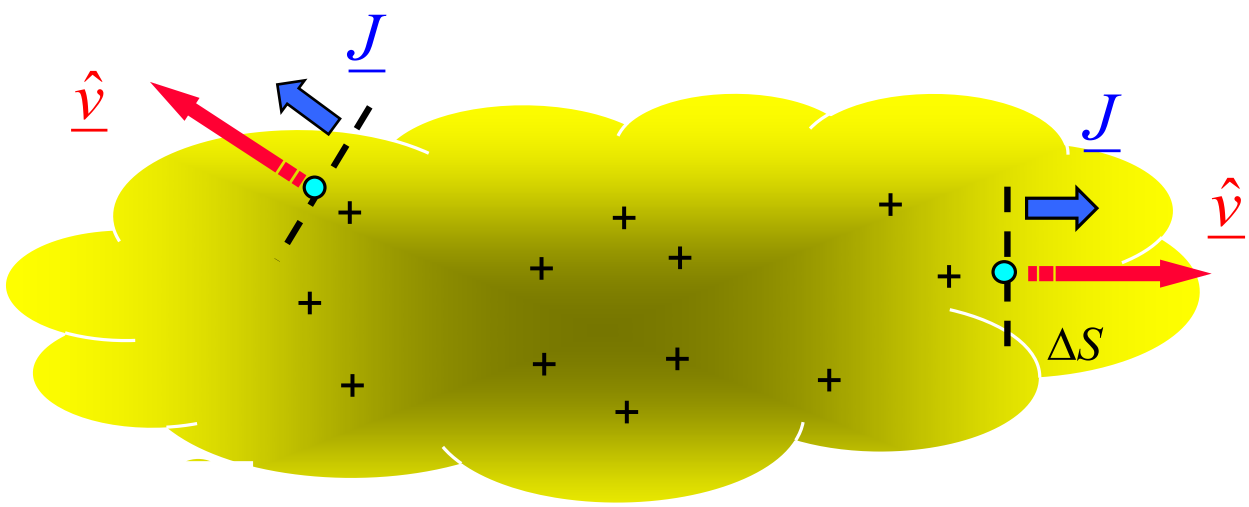
Summary

$$\underline{J} \equiv \left(\frac{I}{\Delta S} \right) \underline{\hat{v}} \quad [\mathrm{A/m^2}]$$

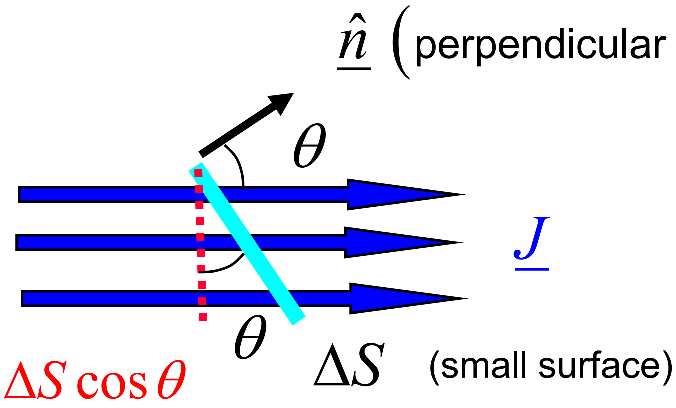
Definition of $\underline{J}$

$$\underline{J} = \rho_v \underline{v}$$

Charge-current equation



Current Crossing a Surface



Note:
The surface ΔS does not have to be perpendicular to the current density vector.

$$\Delta I = J\left(\Delta S \cos\theta\right) = \left(J \cos\theta\right)\Delta S = \left(\underline{J} \cdot \underline{\hat{n}}\right)\Delta S$$

$$\underline{J} \cdot \underline{\hat{n}} = \left|\underline{J}\right|\left|\underline{\hat{n}}\right| \cos\theta = \left|\underline{J}\right| \cos\theta = J \cos\theta$$

This is the current crossing the surface ΔS in the direction of the unit normal.

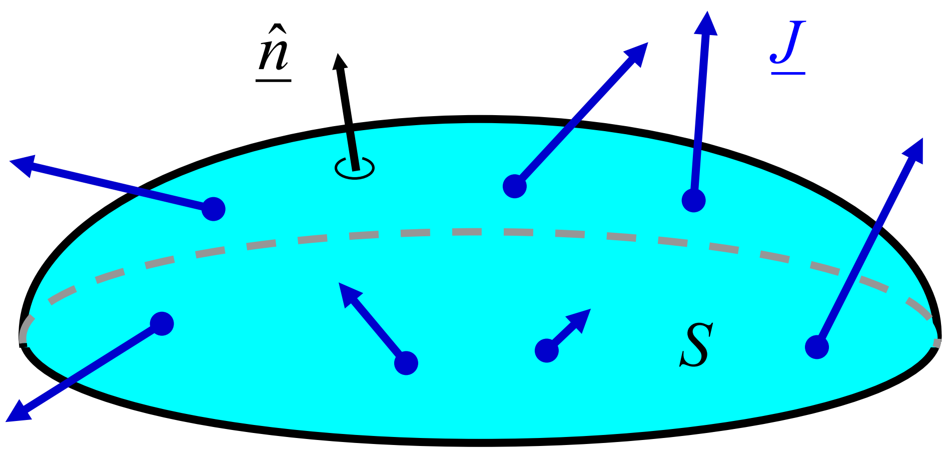
Current Crossing Surface

$$\Delta I = \left(\underline{J} \cdot \underline{\hat{n}}\right) \Delta S$$

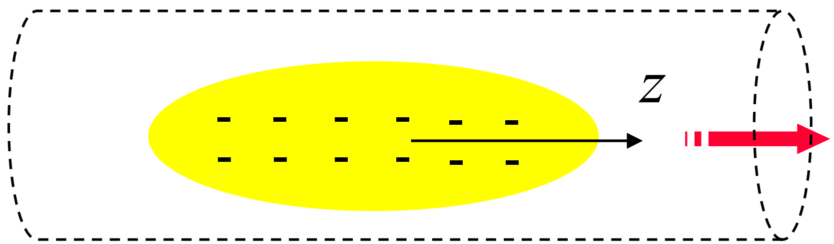
Integrating over a surface,

$$I = \int_S \underline{J} \cdot \underline{\hat{n}} \; dS$$

Note:
The direction of the unit normal vector determines whether the current is measured going in or out.



Example



$$\underline{v} = \underline{\hat{z}}\left(1.0\times10^{-4}\right)\ [\text{m/s}]$$

Cloud of electrons

$N_e = 8.47\times10^{28}$ [electrons / m^3]

The cloud of electrons is inside of a copper wire.

(a) Find: current density vector inside the wire

$$\underline{J} = \rho_v \underline{v}$$

Notes:

1) There are 8.47×10^{28} atoms / m^3.

2) There is one electron/atom in the conduction band.

$$\rho_v = N_e\, q_e = \left(8.47\times10^{28}\,[\text{electrons / m}^3]\right)\left(-1.602\times10^{-19}\,[\text{C/electron}]\right) = -1.36\times10^{10}\,[\text{C/m}^3]$$

$$\underline{J} = \left(-1.36\times10^{10}\right)\left(\underline{\hat{z}}\left(1.0\times10^{-4}\right)\right)$$

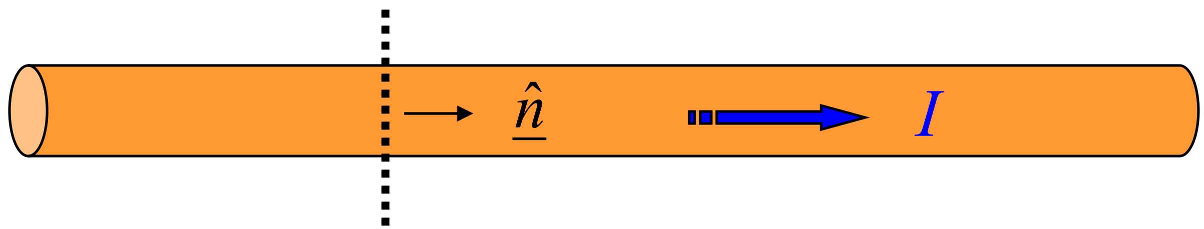
Hence $\underline{J} = \underline{\hat{z}}\left(-1.36\times10^{6}\right)[\text{A/m}^2]$

Example (cont.)

(b) Find: current I in the wire for the given reference direction

$$\underline{J} = \underline{\hat{z}}\left(-1.36\times 10^{6}\right)\ [\mathrm{A/m^2}]$$

Radius $a = 1$ [mm] **Note:** The wire is neutral, but the positive nuclei do not move.



$$I = \int_S \underline{J}\cdot\underline{\hat{n}}\ dS$$

$$= \int_S \left(-\underline{\hat{z}}\ 1.36\times 10^{6}\right)\cdot\underline{\hat{z}}\ dS$$

$$= \int_S -\ 1.36\times 10^{6}\ dS$$

$$= -\ 1.36\times 10^{6}\left(\pi a^{2}\right)$$

$$I = -\ 1.36\times 10^{6}\left(\pi a^{2}\right)$$

$$= -1.36\times 10^{6}\left(\pi\left(0.001\right)^{2}\right)$$

$$I = -4.26\ [\mathrm{A}]$$

Example (cont.)

Properties of copper (Cu)

Using knowledge of chemistry, calculate the value of N_e (that is used in the previous example).

Density of Cu: 8.94 $[\mathrm{g/cm^3}]$

Atomic weight of Cu: 63.546 (atomic number is 29, but this is not needed)

Avogadro's constant: $6.0221417930\times10^{23}$ atoms/mol

1 mol $=$ amount of material in grams equal to the atomic weight of atom

1 electron per atom in the conduction band ($N_e = \mathrm{atoms/m^3}$)

$$\text{Atomic weight} \Rightarrow 63.546\,[\mathrm{g/mol}] \Rightarrow 63.546\times10^{-3}\,[\mathrm{kg/mol}]$$

$$N_e = \left(8.94\times10^3\,[\mathrm{kg/m^3}]\right)\left(\frac{1}{63.546\times10^{-3}}\,[\mathrm{mol/kg}]\right)\left(6.0221417930\times10^{23}\,[\mathrm{atoms/mol}]\right)$$

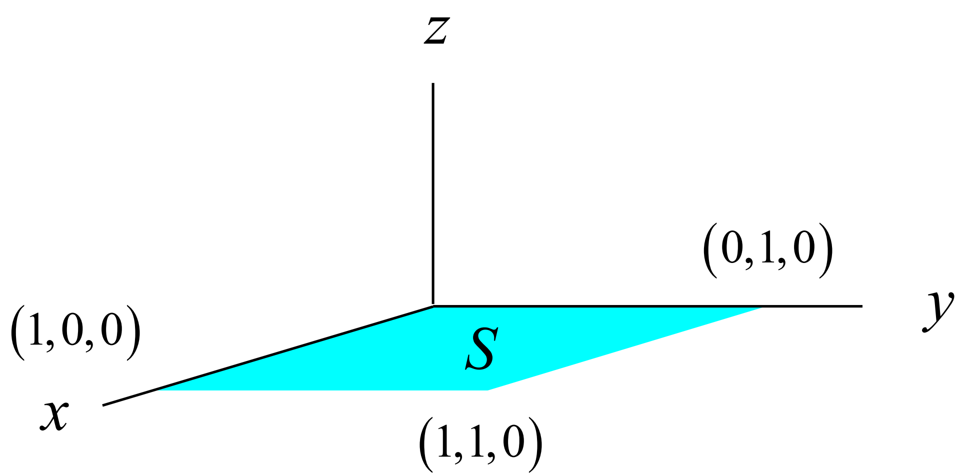
$$= 8.47\times10^{28}\,[\mathrm{atoms/m^3}]$$

$$= 8.47\times10^{28}\,[\mathrm{electrons/m^3}]$$

$$N_e = 8.47\times10^{28}\,[\mathrm{electrons/m^3}]$$

Example

$$\underline{J} = \underline{\hat{x}}\left(3x^2 y\right) + \underline{\hat{y}}\left(3z^3 y^2\right) + \underline{\hat{z}}\left(3xy\right) \ [\mathrm{A/m^2}]$$



Find the current I crossing the surface S in the *upward* direction.

$\underline{\hat{n}} = +\underline{\hat{z}}$

$$I = \int_S \left(\underline{J} \cdot \underline{\hat{n}}\right) dS = \int_S \left(\underline{J} \cdot \underline{\hat{z}}\right) dS = \int_S 3xy \, dS$$

Example (cont.)

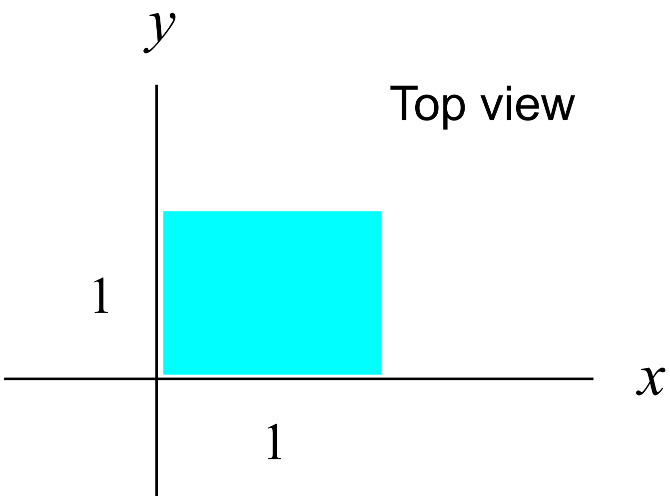
$$I = \int_S 3xy\, dS$$

$$= \int_0^1 \int_0^1 3xy\, dx\, dy$$

$$= 3 \int_0^1 x\, dx \int_0^1 y\, dy$$

$$= 3\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$$

$$= \frac{3}{4}$$

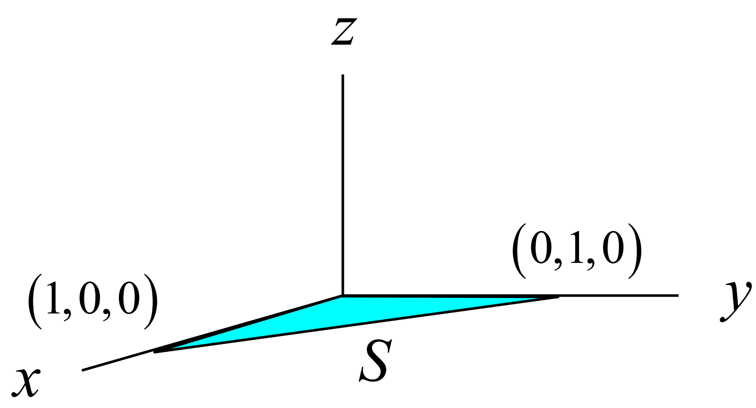


Note:
The integrand is separable, and the limits of integration are fixed numbers. Hence, we can split this into a product of two one-dimensional integrals.

$$I = 0.75\ [\mathrm{A}]$$

Example

$$\underline{J} = \underline{\hat{x}}\left(3x^2 y\right) + \underline{\hat{y}}\left(3z^3 y^2\right) + \underline{\hat{z}}\left(3xy\right) \;\; [\mathrm{A/m^2}]$$



Find the current I crossing the surface S in the *upward* direction.

$\underline{\hat{n}} = +\underline{\hat{z}}$

$$I = \int_S \left(\underline{J} \cdot \underline{\hat{n}}\right) dS = \int_S \left(\underline{J} \cdot \underline{\hat{z}}\right) dS = \int_S 3xy \, dS$$

Example (cont.)

$$I = \int_S 3xy \, dS$$

$$= \int_0^1 \left(\int_0^{y(x)} 3xy \, dy \right) dx$$

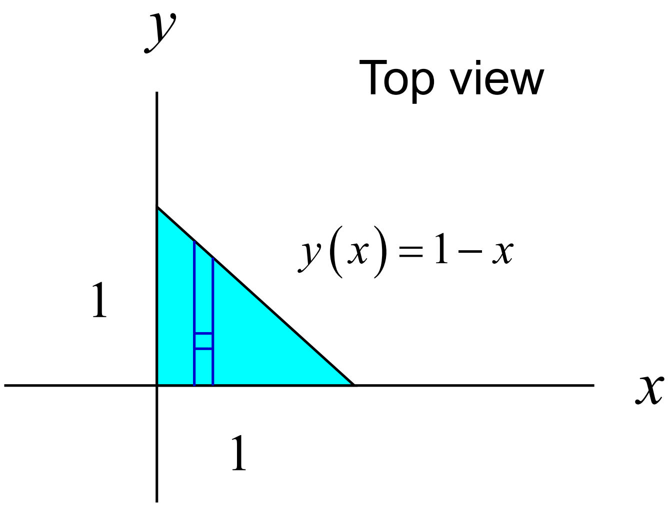
$$= \int_0^1 3x \left(\int_0^{y(x)} y \, dy \right) dx$$

$$= \int_0^1 3x \left(\frac{1}{2} y^2 \right)_0^{1-x} dx$$

$$= \int_0^1 \left[3x \left(\frac{1}{2} (1-x)^2 \right) \right] dx$$

$$= \frac{3}{2} \left[\frac{1}{4} x^4 - \frac{2}{3} x^3 + \frac{1}{2} x^2 \right]_0^1$$

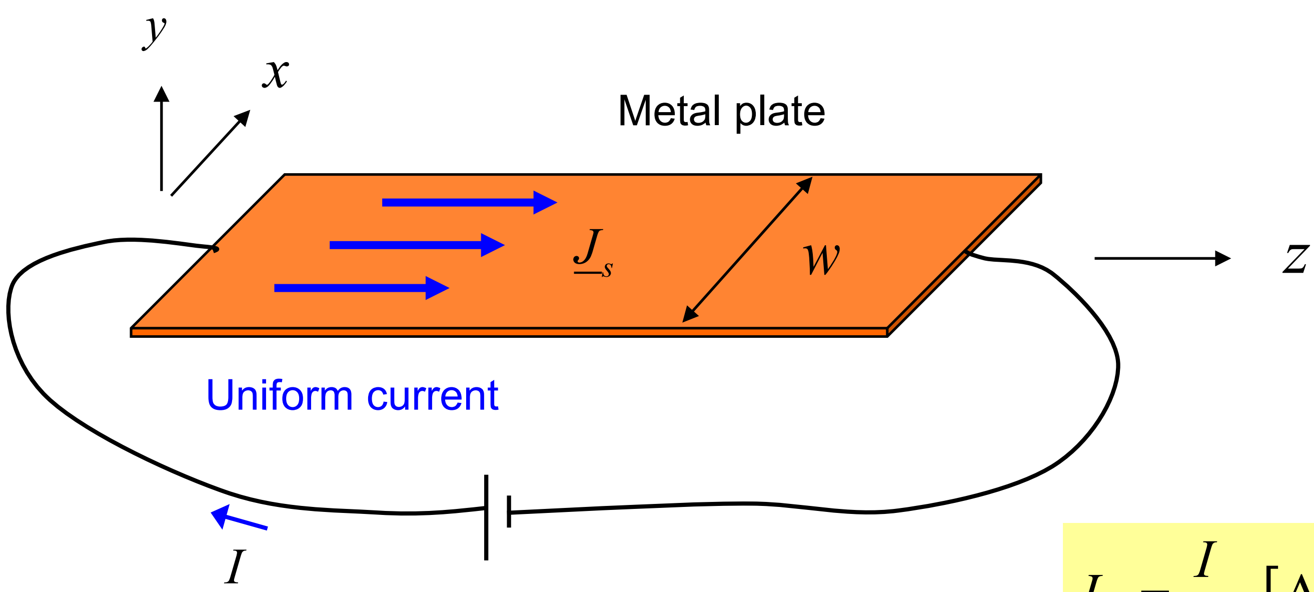
Note:
The integrand is separable, but the limits of integration are <u>not</u> fixed numbers. Hence, we cannot split this into a product of two one-dimensional integrals.



$$I = 0.125 \text{ [A]}$$

Surface Current

This is a useful concept for thin currents!



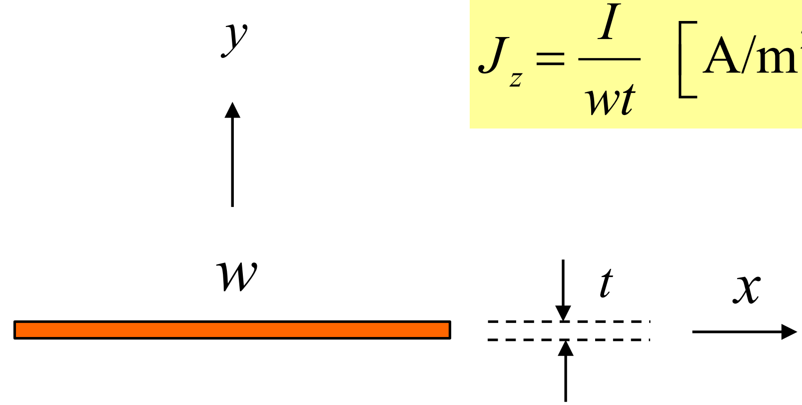
$$J_{sz} = \frac{I}{w} \quad [\mathrm{A/m}]$$

$$J_z = \frac{I}{wt} \quad [\mathrm{A/m^2}]$$

$$\underline{J}_s = \underline{\hat{z}}\left(\frac{I}{w}\right) \quad [\mathrm{A/m}]$$

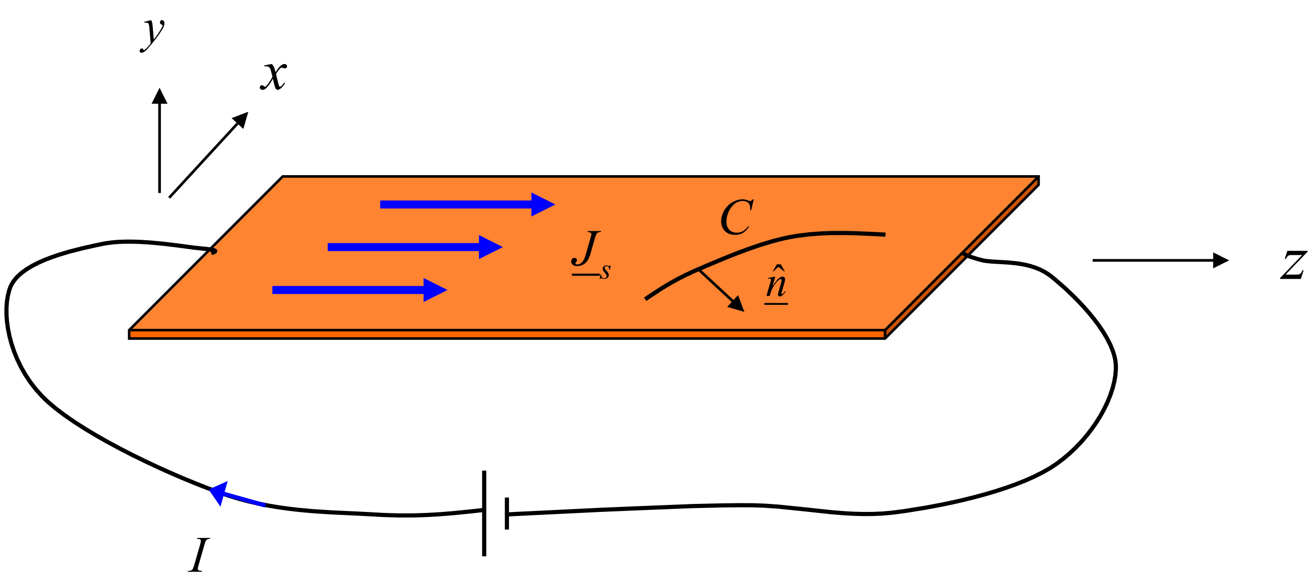
Inside the actual conductor:

$$\underline{J} = \underline{\hat{z}}\left(\frac{I}{wt}\right) \quad [\mathrm{A/m^2}]$$



Surface Current (cont.)

Current I_C flowing across a path C:



$$I_C = \int_C \underline{J}_s \cdot \hat{\underline{n}}\, dl$$

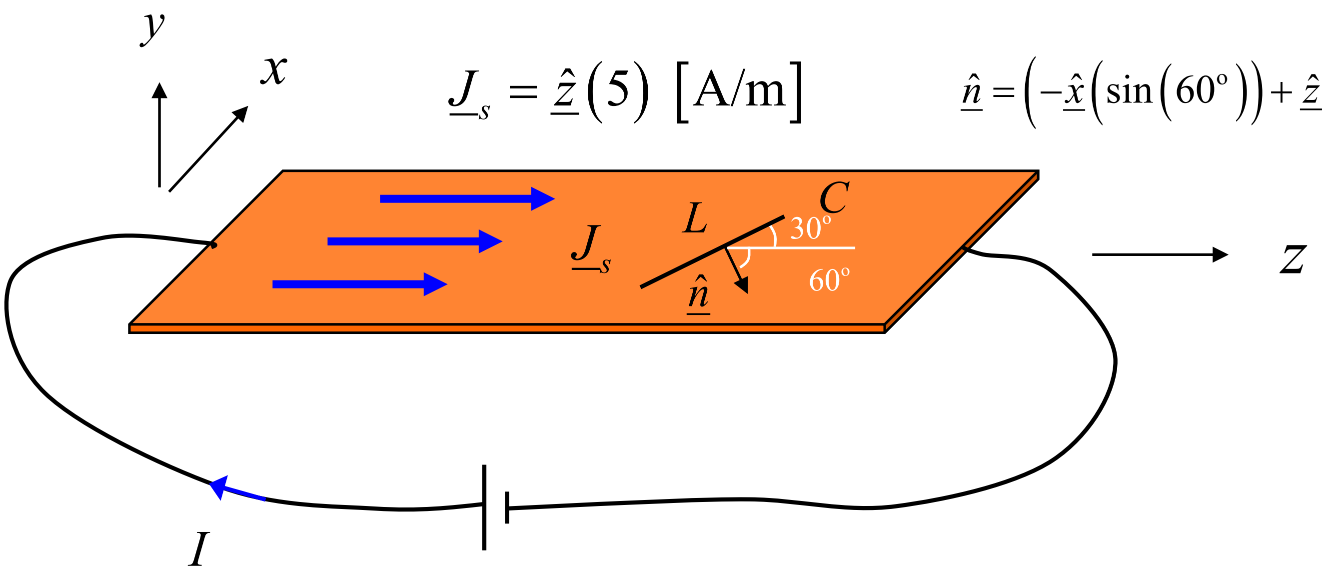
Note:
For the unit normal pictured above, this will give us the current crossing the curve from left to right.

Example

Find the current I_C flowing across the path C (from left to right).

C = straight-line path of length $L = 3$ meters, making an angle of 30° from the z axis.

$$\underline{J}_s = \underline{\hat{z}}(5) \; [\text{A/m}] \qquad \underline{\hat{n}} = \left(-\underline{\hat{x}}\left(\sin\left(60^{\circ}\right)\right) + \underline{\hat{z}}\left(\cos\left(60^{\circ}\right)\right)\right)$$



$$I_C = \int_C \underline{J}_s \cdot \underline{\hat{n}}\, dl = \left(\underline{J}_s \cdot \underline{\hat{n}}\right)\int_C dl = \left(\underline{J}_s \cdot \underline{\hat{n}}\right) L = \left(\underline{\hat{z}}(5)\cdot\left(-\underline{\hat{x}}(0.86603) + \underline{\hat{z}}(+0.5)\right)\right)(3) = 7.5$$

Hence, we have: $I_C = 7.5 \; [\text{A}]$