# ECE 3318
# Applied Electricity and Magnetism

**Spring 2023**

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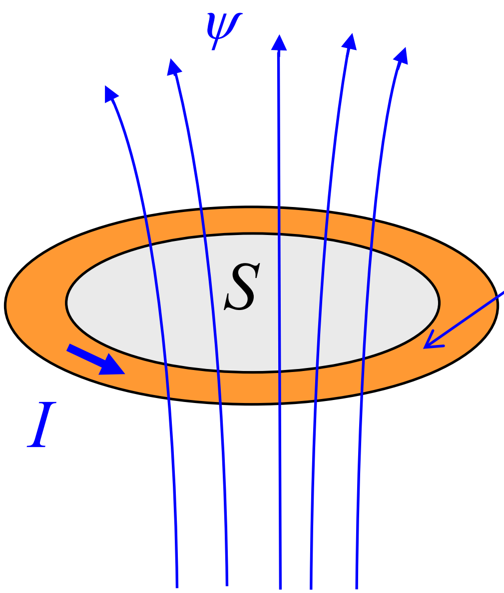
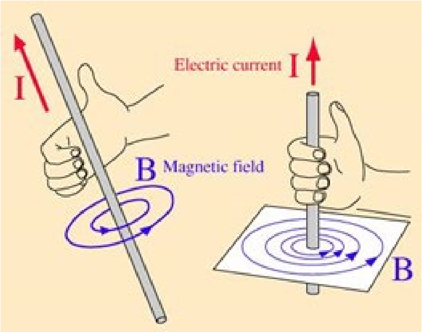
## Notes 31
### Inductance

# Inductance

Consider a single loop carrying a current:




Single turn coil

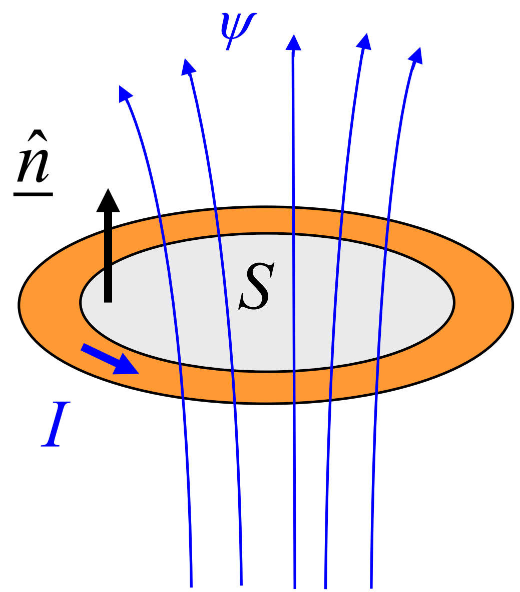



Right-hand rule for determining the direction of the magnetic field from a current.

Note that the current shown here (flowing counterclockwise) produces a magnetic flux that goes up through the loop.

The current $I$ produces a flux $\psi$ though the loop.

# Inductance




Single turn coil

Magnetic flux through loop:

$$\psi \equiv \int_S \underline{B} \cdot \hat{\underline{n}} \, dS \quad [\text{Wb}]$$

$\hat{\underline{n}}$ = unit normal to loop

The unit normal is chosen from a
"right hand rule for the inductor flux":

The fingers are in the direction of the current reference direction, and the thumb then gives the direction of the unit normal for the flux calculation.

(In this picture, $\psi$ is the flux going up.)

Definition of inductance:

$$L \equiv \frac{\psi}{I} \quad [\text{H}]$$

**Note:** $L$ is always positive

# Right-Hand Rules

**Note**:
Please see the Appendix for a summary of right-hand rules.

1) RH rule for Stokes's theorem

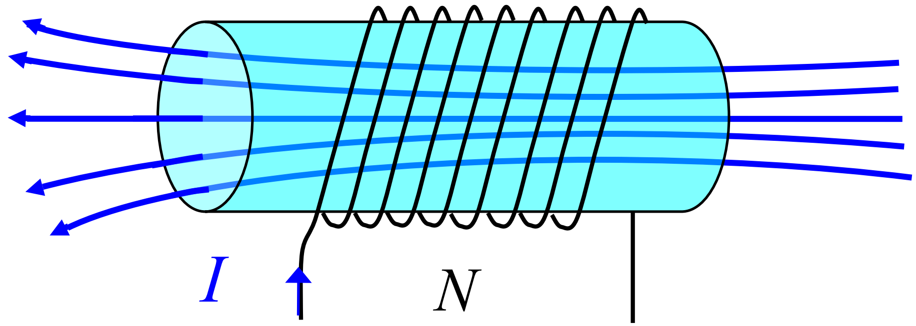
2) RH rule for Faraday's law

3) RH rule for Ampere's law

4) RH rule for the direction of the magnetic field

5) RH rule for the inductor flux

# N-Turn Solenoid




$$\Lambda = \text{total flux}$$
$$= N\psi$$

We assume here that the same flux cuts though each of the $N$ turns.

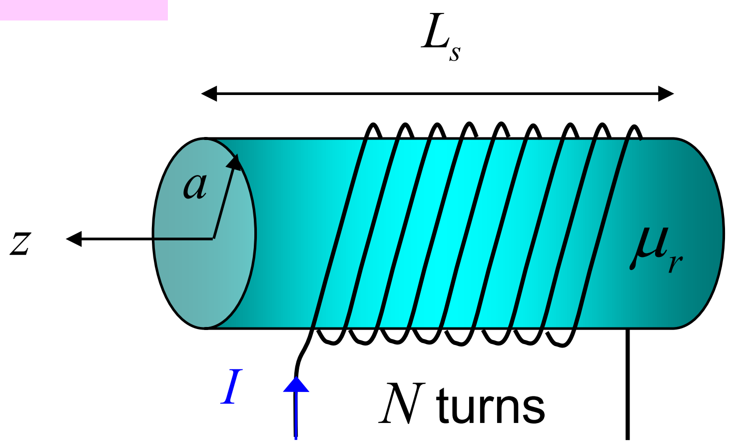
$$\psi = \text{flux through one turn}$$

The definition of inductance is now

$$L \equiv \frac{\Lambda}{I} = \frac{N\psi}{I}$$

# Example

Find $L$




$$L = N\frac{\psi}{I}$$

**Note:**
We neglect "fringing" here and assume that we have the same magnetic field in the core as if the solenoid were infinite.

$$n = N / L_s$$

$$\begin{aligned}\psi &= \left(\pi a^2\right)\underline{B}\cdot\underline{\hat{n}} \\ &= \left(\pi a^2\right)B_z \\ &= \left(\pi a^2\right)\mu_0\mu_r H_z \\ &= \left(\pi a^2\right)\mu_0\mu_r\left(nI\right)\end{aligned}$$

RH rule for inductor flux: $\underline{\hat{n}} = \underline{\hat{z}}$

From previous notes:

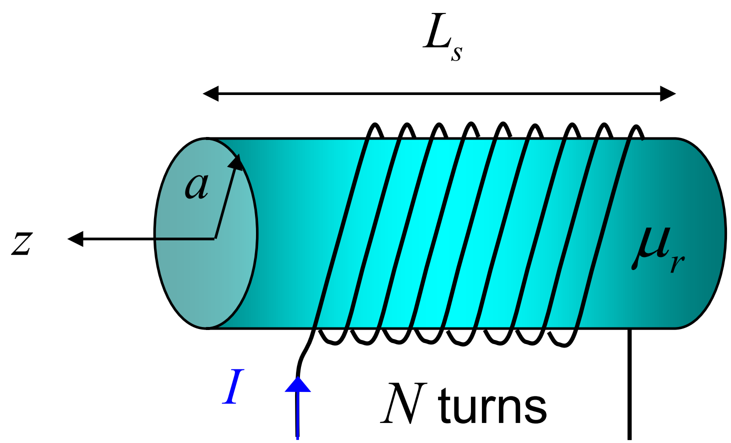
$$\begin{aligned}\underline{H} &= \underline{\hat{z}}\left(nI\right), \quad \rho < a \\ &= \underline{0}, \quad \rho > a\end{aligned}$$

so

$$L = N\left[\left(\pi a^2\right)\mu_0\mu_r\left(nI\right)\right]/I$$

**Note :** $n = N / L_s$

# Example (cont.)




Final result:

$$L = \frac{N^2}{L_s}\left(\pi a^2\right)\mu_0 \mu_r \quad [\mathrm{H}]$$

**Note:** $L$ is increased by using a high-permeability core!

# Toroidal Inductor

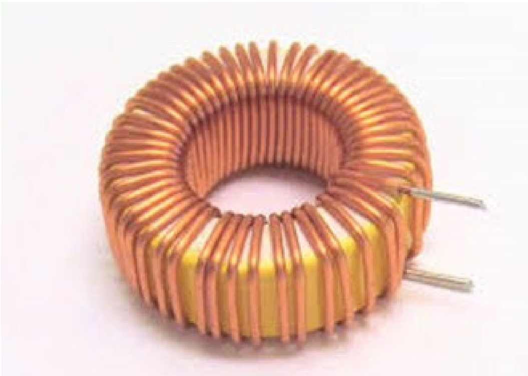
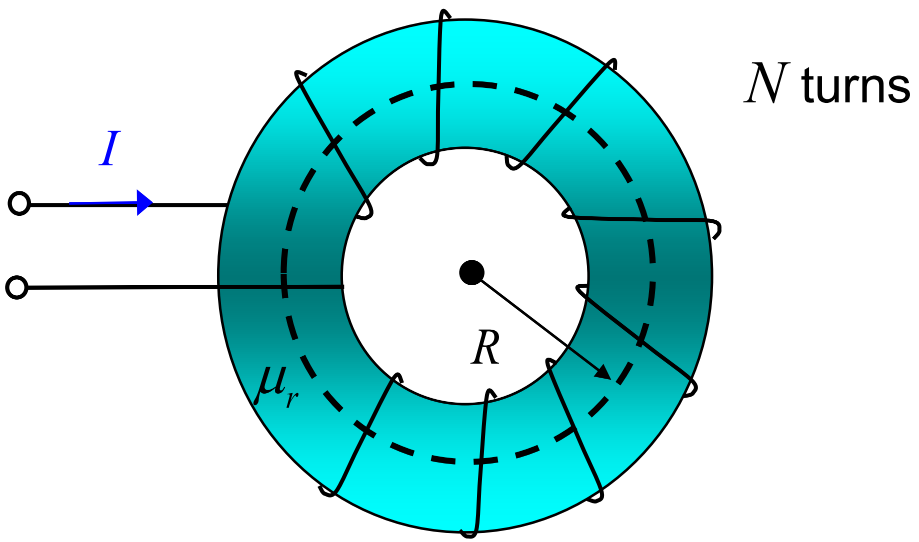




Fig. 2 Toroidal Inductor with improving mounting

# Example

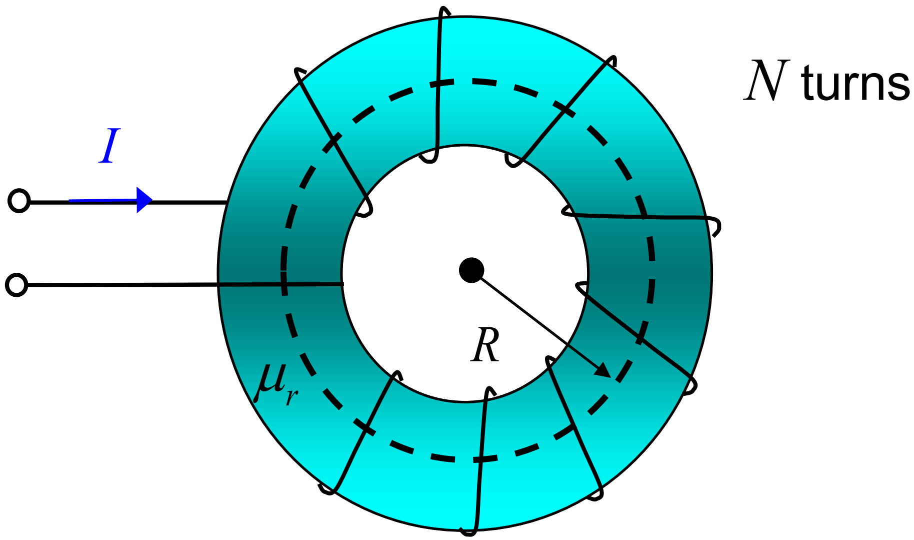
## Toroidal Inductor



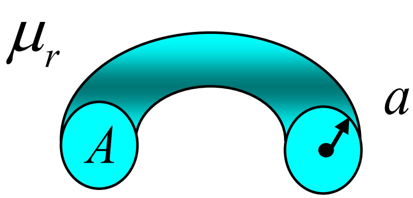

- Find $\underline{H}$ inside toroid
- Find $L$

**Note:**
This is a practical structure, in which we do not have to neglect fringing and assume that the core length is very large in order for the answer to be accurate.

# Example (cont.)




The radius $R$ is the *average radius* (measured to the center of the toroid).

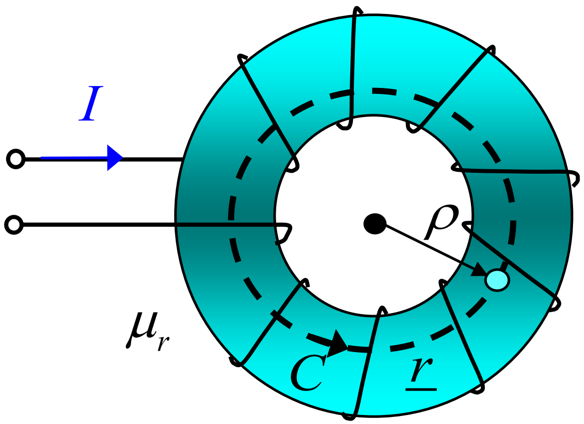



Assume

$$\underline{H} = \underline{\hat{\phi}}\, H_\phi$$

$A$ is the cross-sectional area: $A = \pi a^2$

# Example (cont.)




$N$ turns

$$\oint_C \underline{H} \cdot \underline{dr} = I_{encl}$$

$$\oint_C \underline{H} \cdot \left( \underline{\hat{\phi}} \, \rho \, d\phi \right) = I_{encl}$$

so

$$\int_0^{2\pi} H_\phi \, \rho \, d\phi = I_{encl}$$

$$\Rightarrow \quad H_\phi \left( 2\pi\rho \right) = I_{encl}$$

Hence

$$H_\phi = \frac{I_{encl}}{2\pi\rho}$$

We then have

RH rule in Ampere's law :

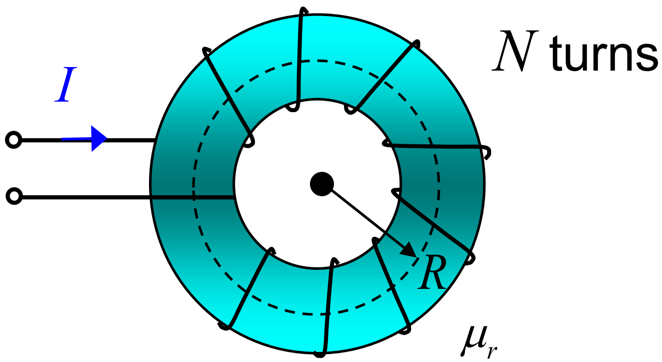
$$I_{encl} = +NI$$

$$\underline{H} = \underline{\hat{\phi}} \left( \frac{NI}{2\pi\rho} \right) \quad [\text{A/m}]$$

# Example (cont.)

RH rule for the inductor flux :

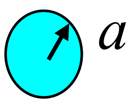
$$\underline{\hat{n}} = +\underline{\hat{\phi}}$$




$$\underline{H} = \underline{\hat{\phi}}\left(\frac{NI}{2\pi\rho}\right) \quad [\text{A/m}]$$

$$L = \frac{N\psi}{I}$$

$$A = \pi a^2$$




$$L \approx \frac{N}{I}\left(\underline{B}\cdot\underline{\hat{n}}\right)A$$

$$\frac{N}{I}\left(B_\phi\Big|_{\rho=R} A\right)$$

$$= \frac{N}{I}\left(\mu_0\,\mu_r H_\phi\Big|_{\rho=R} A\right)$$

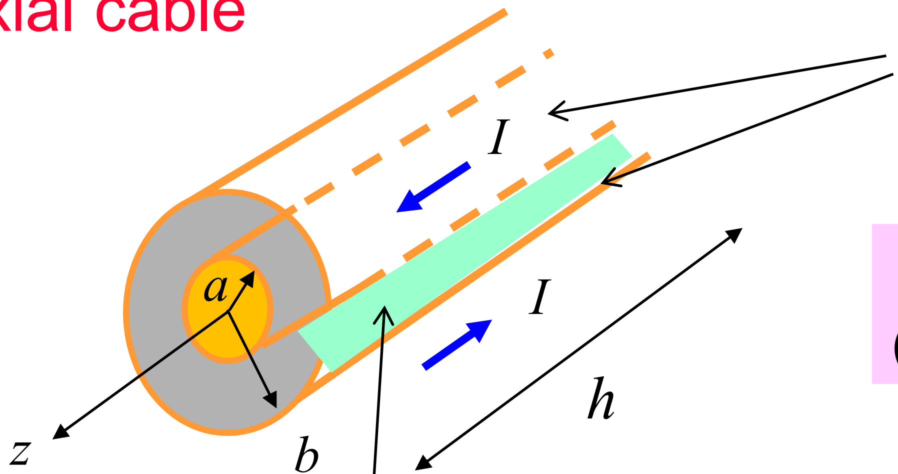
$$= \frac{N}{I}\mu_0\mu_r\left(\frac{NI}{2\pi R}\right)A$$

Hence

$$L = \mu_0\mu_r A\left(\frac{N^2}{2\pi R}\right) \quad [\text{H}]$$

# Example

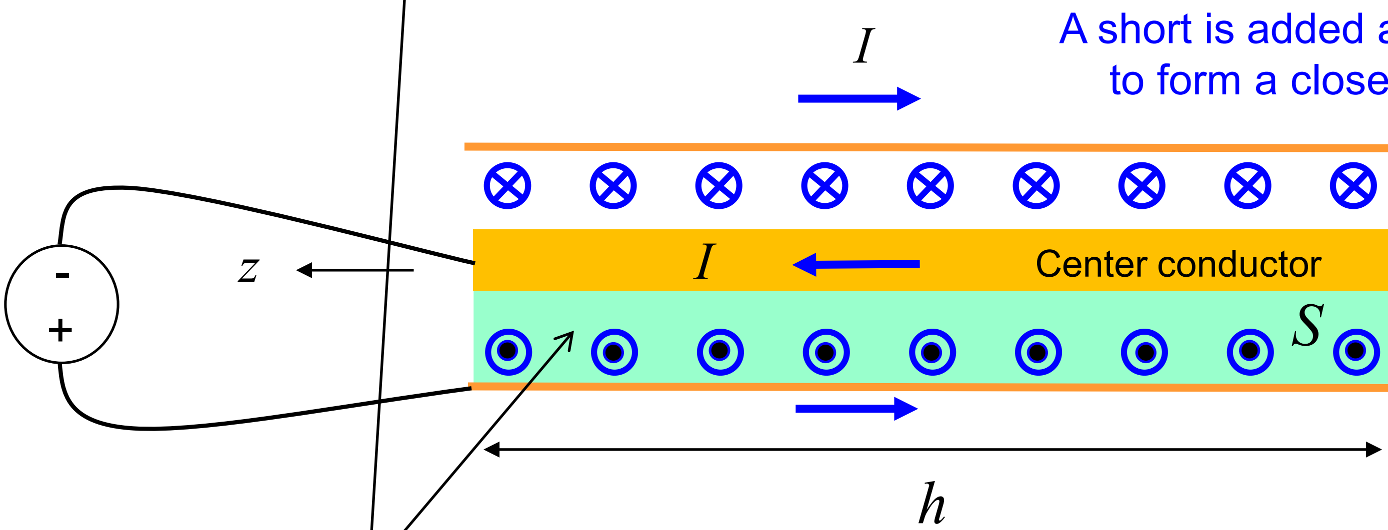
Coaxial cable




Ignore flux inside wire and shield
(PEC conductors)

Find $L_l$
(inductance per unit length)

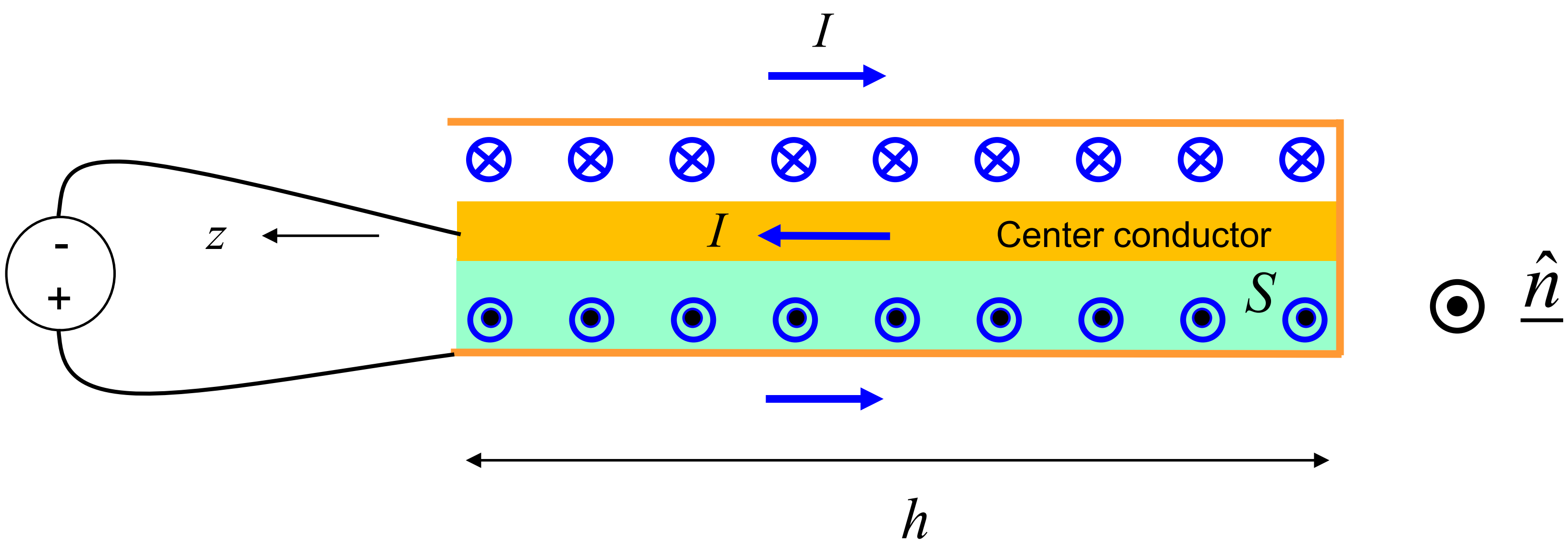
A short is added at the end
to form a closed loop.




**Note:** We calculate the flux through the surface $S$ shaded in green.

# Example (cont.)

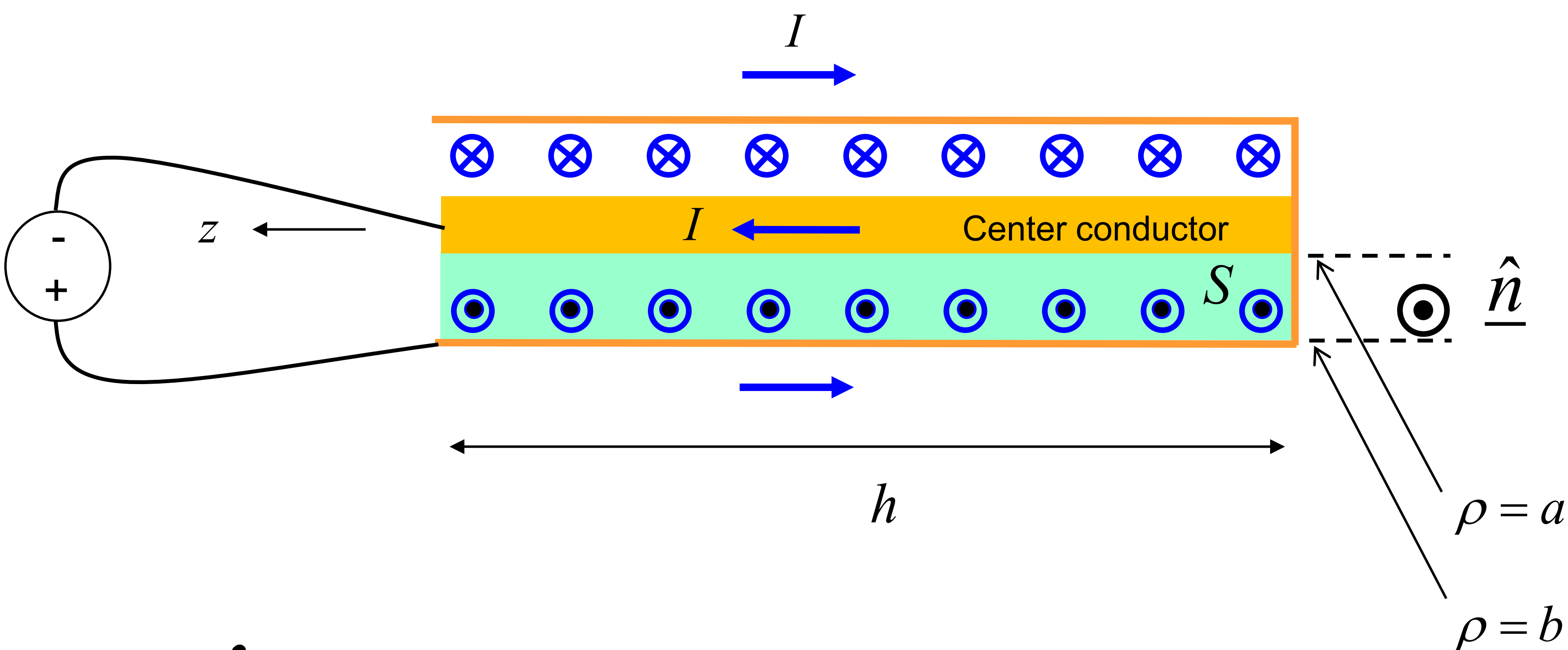
RH rule for the inductor flux : $\underline{\hat{n}} = +\underline{\hat{\phi}}$




$$L = \frac{\psi}{I}$$

$$\psi = \int_S \underline{B} \cdot \underline{\hat{n}}\, dS = \int_S B_\phi \, dS = \mu \int_S H_\phi \, dS$$

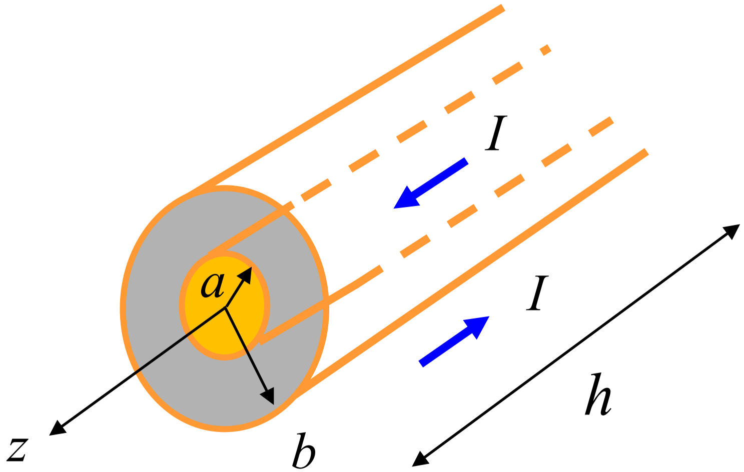
# Example (cont.)




$$\psi = \mu \int_S H_\phi \, dS$$

$$\psi = \mu\, h \int_a^b H_\phi \, d\rho = \mu\, h \int_a^b \frac{I}{2\pi\rho} d\rho = \mu\, h \frac{I}{2\pi} \left[\ln \rho\right]_a^b$$

# Example (cont.)




$$\psi = \mu h \frac{I}{2\pi} \left[ \ln \rho \right]_a^b$$

$$= \mu h \frac{I}{2\pi} \ln\left(\frac{b}{a}\right)$$

Hence $\quad L = \dfrac{\psi}{I} = \mu h \dfrac{1}{2\pi} \ln\left(\dfrac{b}{a}\right)$

Per-unit-length: $\quad L_l = \dfrac{\mu_0 \mu_r}{2\pi} \ln\left(\dfrac{b}{a}\right) \quad [\mathrm{H/m}]$

# Example (cont.)

Recall:

$$Z_0 = \sqrt{\frac{L_l}{C_l}}$$

$$L_l = \frac{\mu_0 \mu_r}{2\pi} \ln\left(\frac{b}{a}\right) \quad [\text{H/m}]$$

**Note**: $\mu_r = 1$ for most practical coaxial cables.

From previous notes:

$$C_l = \frac{2\pi \varepsilon_0 \varepsilon_r}{\ln\left(\frac{b}{a}\right)} \quad [\text{F/m}]$$

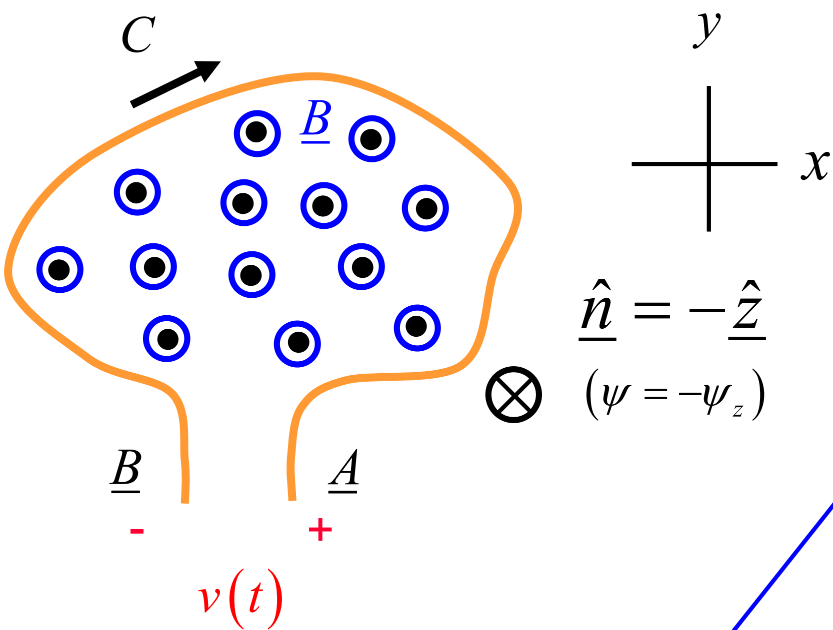
Hence:

$$Z_0 = \frac{\eta_0}{2\pi} \sqrt{\frac{\mu_r}{\varepsilon_r}} \ln\left(\frac{b}{a}\right) \quad [\Omega]$$

$$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 376.7603 \; [\Omega]$$

(intrinsic impedance of free space)

# Voltage Law for Inductor




$\underline{\hat{n}} = -\underline{\hat{z}}$

$(\psi = -\psi_z)$

Apply Faraday's law:

$$\oint_C \underline{E} \cdot d\,\underline{r} = -\frac{d\psi}{dt}$$

$$\psi = \int_S \underline{B} \cdot \underline{\hat{n}}\, dS$$

**Reminder:**
The unit normal is chosen from the right-hand rule in Faraday's law, according to the direction of the path $C$. Here, $\psi = -\psi_z$.

PEC wire:

$$v(t) = \int_{\underline{A}}^{\underline{B}} \underline{E} \cdot d\,\underline{r} = \oint_C \underline{E} \cdot d\,\underline{r}$$

**Note:** There is no electric field inside the wire.

Hence

$$v(t) = -\left(-\frac{d\psi_z}{dt}\right)$$

Here $\psi$ is the flux coming out of the page.

# Voltage Law for Inductor (cont.)

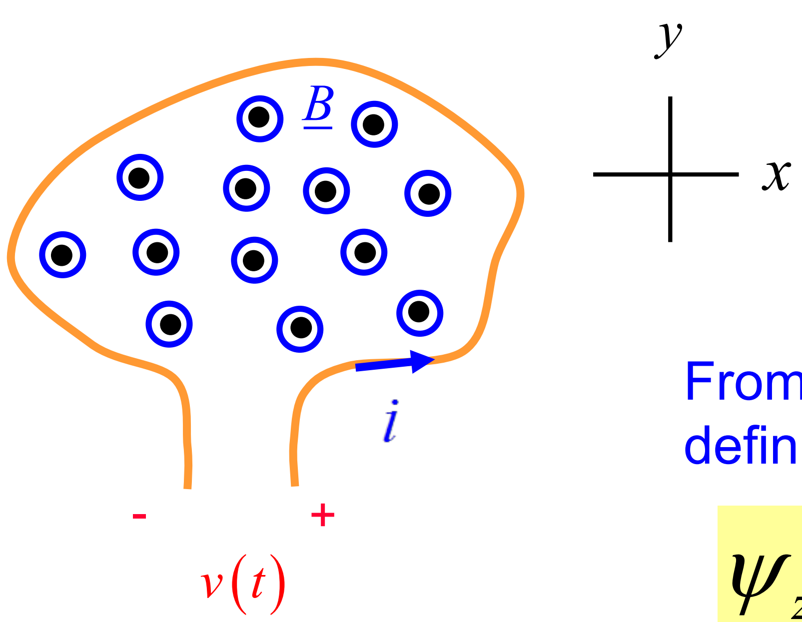
$$v(t) = \frac{d\psi_z}{dt}$$

so

$$v(t) = \frac{d(Li)}{dt}$$

or

$$v(t) = L\frac{di}{dt}$$




From the inductor definition we have:

$$\psi_z = +Li$$

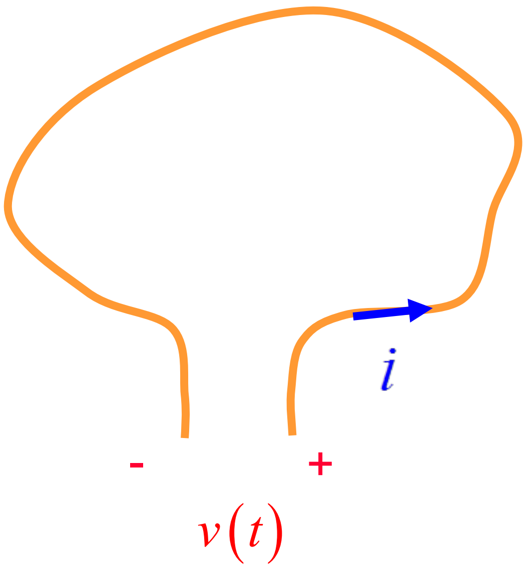
**Note:**
The direction of the current is chosen from the right-hand rule for the inductor flux.

**Note:** We are using the “passive” sign convention here.

# Voltage Law for Inductor (cont.)

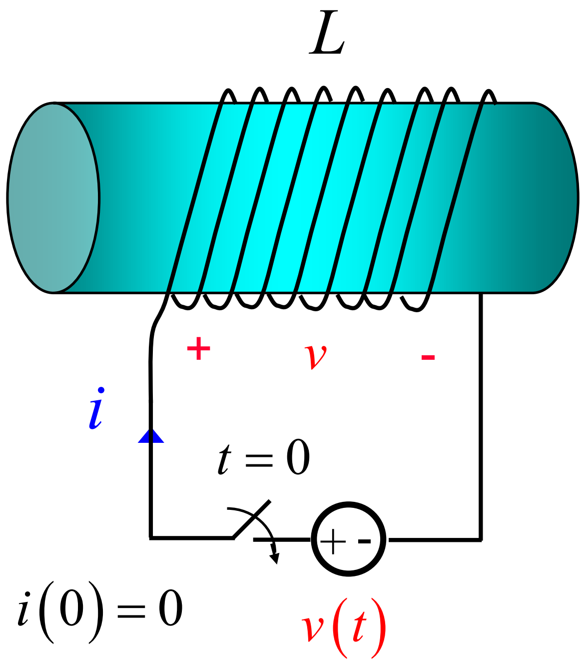
**Summary of Inductor Law**

$$v(t) = L\frac{di}{dt}$$




**Note:** We are using the “passive” sign convention here.

# Energy Stored in Inductor




$$v(t) = L\frac{di}{dt}$$

$$p(t) = vi = \left(L\frac{di}{dt}\right)i$$

$$W(t) = \int_0^t p(t)\,dt = L\int_0^t i\frac{di}{dt}dt$$

$$= L\int_{i(0)}^{i(t)} i\,di = L\frac{1}{2}i^2\Big|_{i(0)}^{i(t)} = \frac{1}{2}Li^2$$

Hence we have:

$$U_H(t) = \frac{1}{2}Li^2(t)\ \text{[J]}$$

For DC we have:

$$U_H = \frac{1}{2}LI^2\ \text{[J]}$$

# Energy Formula for Inductor

We can write the inductance in terms of stored energy as:

$$L = \frac{2U_H}{I^2}$$

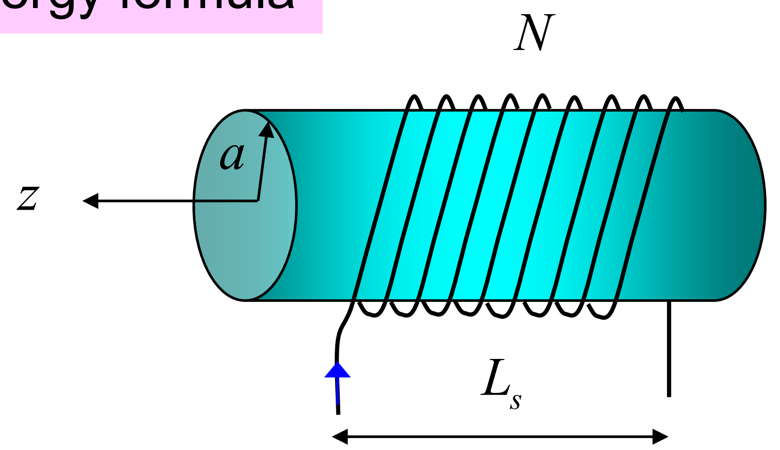
Next, we use $$U_H = \int_V \frac{1}{2} \underline{B} \cdot \underline{H} \, dV$$

We then have $$L = \frac{1}{I^2} \int_V \underline{B} \cdot \underline{H} \, dV$$

This gives us an alternative way to calculate inductance.

# Example

Find $L$ using the energy formula




Solenoid

From a previous example, we have $$U_H = \frac{1}{2}\mu_0\mu_r\pi a^2\left(\frac{N^2}{L_s}\right)I^2$$

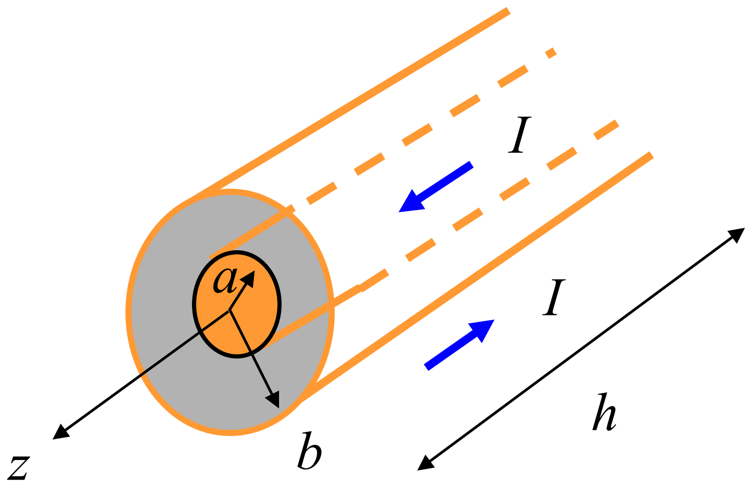
Energy formula: $$L = \frac{2U_H}{I^2}$$

Hence, we have $$L = \mu_0\mu_r\pi a^2\left(\frac{N^2}{L_s}\right) \quad [\mathrm{H}]$$

# Example

Find $L_l$ using the energy formula

$$L = \frac{1}{I^2}\int_V \underline{B}\cdot\underline{H}\,dV$$




Coaxial cable

$$L = \frac{1}{I^2}\mu_0\mu_r\int_V \left|\underline{H}\right|^2 dV$$

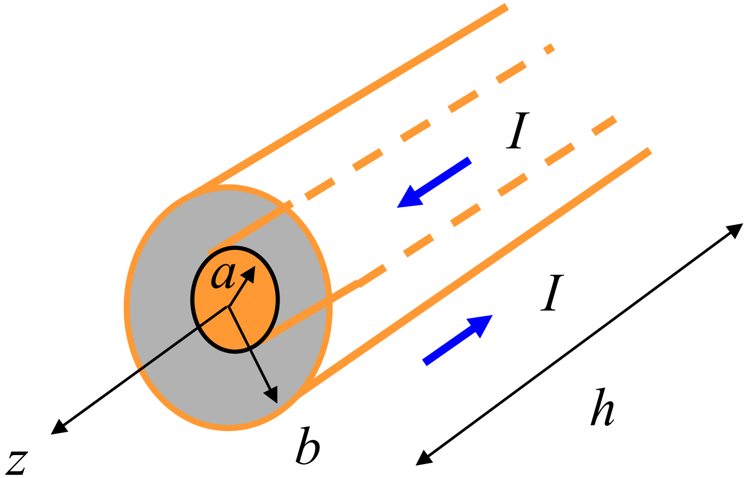
$$= \frac{1}{I^2}\mu_0\mu_r\int_V H_\phi^2\,dV$$

$$= \frac{1}{I^2}\mu_0\mu_r\int_0^h\int_0^{2\pi}\int_a^b\left(\frac{I}{2\pi\rho}\right)^2 \rho\,d\rho\,d\phi\,dz$$

$$= \mu_0\mu_r\left(\frac{1}{2\pi}\right)^2 (h)(2\pi)\int_a^b \frac{1}{\rho}\,d\rho$$

**Note:**
We ignore the magnetic stored energy from fields inside the good conductors.
(We would have fields inside the conductors at DC, but not at high frequency.)

# Example (cont.)




$$L = \mu_0 \mu_r \left( \frac{1}{2\pi} \right)^2 (h)(2\pi) \int_a^b \frac{1}{\rho} d\rho$$

$$= \frac{\mu_0 \mu_r h}{2\pi} \ln\left( \frac{b}{a} \right)$$

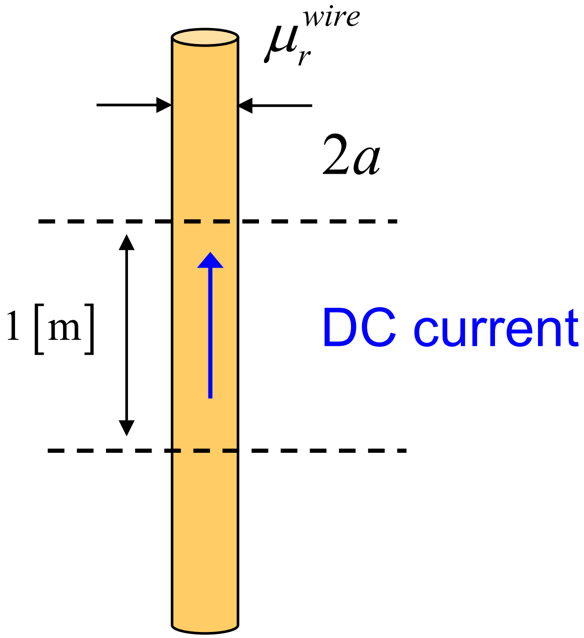
Per-unit-length: $L_l = \frac{\mu_0 \mu_r}{2\pi} \ln\left( \frac{b}{a} \right) \quad [\mathrm{H/m}]$

**Note:**
We could include the stored energy inside inner wire region and the shield region if we were at DC. These contributions would give us the "internal inductance" of the coax.
This would only be important at low frequency, not high frequency, for good conductors.

# Internal Inductance

**Internal inductance per unit length of straight infinite wire**

This comes from the energy stored inside the conductor.




Take one-meter length in $z$ direction:

$$U_H^{\text{int}} = \int_V \frac{1}{2} \underline{B} \cdot \underline{H}\, dV$$

$$= \frac{1}{2} \mu_0 \mu_r^{wire} (1) \int_0^{2\pi} \int_0^{a} H_\phi^2\, \rho\, d\rho\, d\phi$$

Ampere's law: (Use counterclockwise Amperian path.)

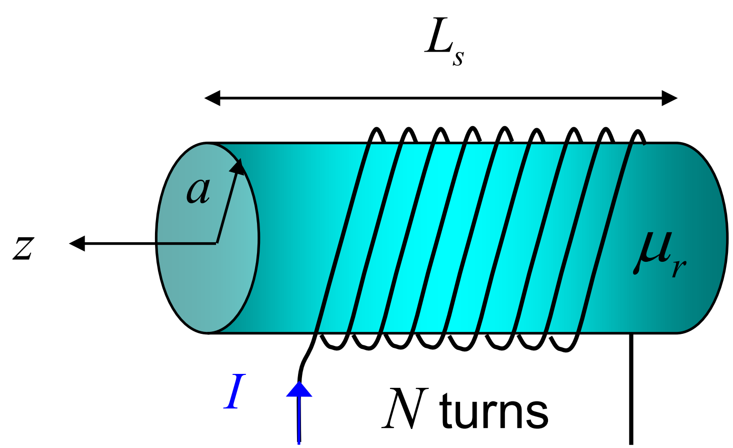
$$H_\phi = \frac{I_{encl}}{2\pi\rho} = \frac{I}{2\pi\rho}\left(\frac{\rho^2}{a^2}\right) = \frac{I}{2\pi}\left(\frac{\rho}{a^2}\right) \quad (\rho < a)$$

$$\rho < a : \; I_{encl} = I\left(\frac{\pi\rho^2}{\pi a^2}\right)$$

Final result:

$$L_l^{\text{int}} = \left(\frac{\mu_0 \mu_r^{wire}}{8\pi}\right) \; [\text{H/m}]$$

# Example




External inductance:

$$L^{ext} = \frac{N^2}{L_s}\left(\pi a^2\right)\mu_0 \mu_r \quad [\mathrm{H}]$$

(from previous calculation)

$$L_l^{\mathrm{int}} = \left(\frac{\mu_0 \mu_r^{wire}}{8\pi}\right) \quad [\mathrm{H/m}]$$

$\Rightarrow$

Internal inductance:

$$L^{\mathrm{int}} = \left(\frac{\mu_0 \mu_r^{wire}}{8\pi}\right) 2\pi a N \; [\mathrm{H}]$$

(from wire formula)

# Appendix

In this appendix we summarize the various right-hand rules that we have seen so far in electromagnetics.

# Summary of Right-Hand Rules

Stokes's theorem: $\oint_C \underline{V} \cdot d\underline{r} = \int_S \left(\nabla \times \underline{V}\right) \cdot \underline{\hat{n}}\ dS$

Fingers are in the direction of the path $C$, the thumb gives the direction of the unit normal.

Faraday's law:
(stationary path)

$$\int_C \underline{E} \cdot d\underline{r} = -\frac{d\psi}{dt}$$

$$\psi = \int_S \underline{B} \cdot \underline{\hat{n}}\ dS$$

Fingers are in the direction of the path $C$, the thumb gives the direction of the unit normal for calculating the flux.

Ampere's law:

$$\oint_C \underline{H} \cdot d\underline{r} = I_{encl}$$

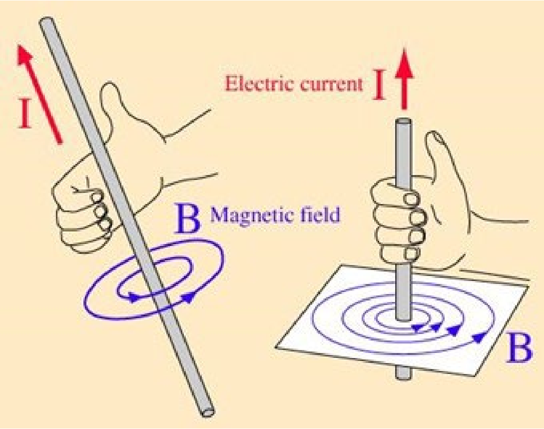
$$I_{encl} = \int_S \underline{J} \cdot \underline{\hat{n}}\, dS$$

Fingers are in the direction of the path $C$, the thumb gives the reference direction for the current enclosed.

# Summary of Right-Hand Rules (cont.)

## Magnetic field law:

For a wire or a current sheet or a solenoid, the thumb is in the direction of the current and the fingers give the direction of the magnetic field. (For a current sheet or solenoid the fingers are simply giving the overall sense of the magnetic field direction.)




## Inductor flux rule:

$$L \equiv \frac{\psi}{I}$$

$$\psi = \int_S \underline{B} \cdot \underline{\hat{n}} \; dS$$

Fingers are in the direction of the current $I$ and the thumb gives the direction of the unit normal for calculating the flux.