ECE 3318
Applied Electricity and Magnetism

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Notes 33
Mutual Inductance

Mutual Inductance

Two coils are in proximity of each other.

Note: Each coil has a set of output terminals, but this is not shown.

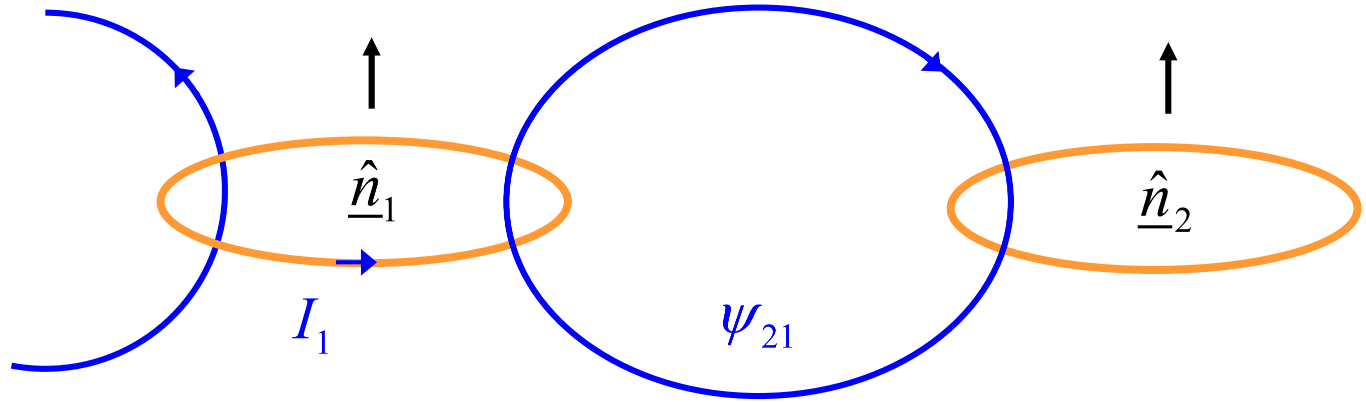
Current reference directions and unit normal vectors are defined on <u>both</u> coils.

(The unit normal vectors are each determined from the corresponding current reference directions, by the right-hand rule for inductor flux.)

Reminder:
Right-hand rule for inductor flux: Fingers are in the direction of the current I in the coil, and the thumb gives the direction of the unit normal (the reference direction for the flux).

Mutual Inductance (cont.)

M_{21}



Coil 1 is energized
Coil 2 is left open-circuited

$$\psi_{21} \equiv \int_{S_2} \underline{B}_1 \cdot \hat{\underline{n}}_2 \, dS$$

Meaning of subscripts:
"the flux through coil 2 due to coil 1"

These two numbers go together.

Define mutual inductance: $M_{21} \equiv \dfrac{\psi_{21}}{I_1}$

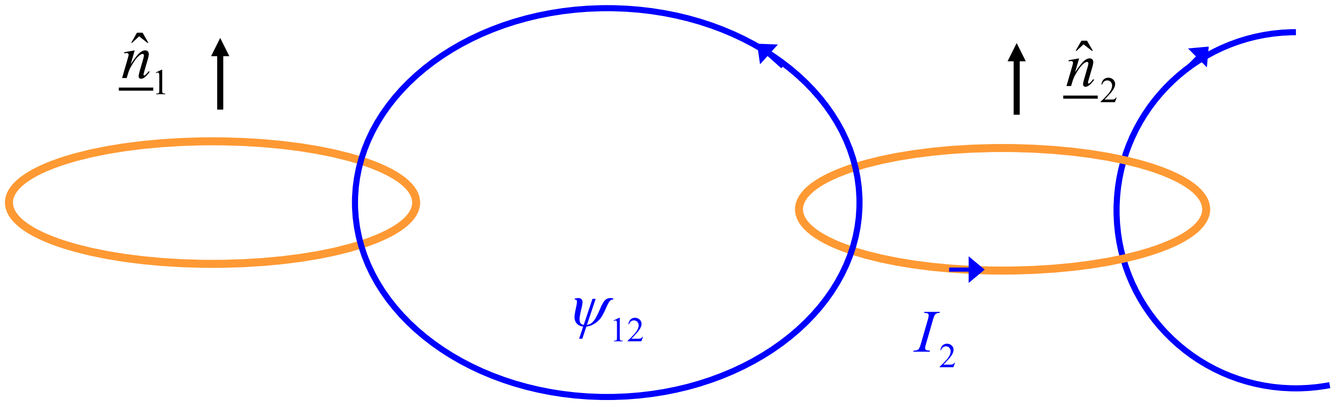
In general, if coil 2 has <u>multiple</u> turns:

$$M_{21} \equiv \frac{\Lambda_{21}}{I_1} = \frac{N_2 \psi_{21}}{I_1}$$

Note:
For the figure shown, $\psi_{21} < 0$ if $I_1 > 0$.
Hence, $M_{21} < 0$.

Mutual Inductance (cont.)

M_{12}



Coil 2 is energized
Coil 1 is left open-circuited

$$\psi_{12} = \int_{S_1} \underline{B}_2 \cdot \hat{\underline{n}}_1 \, dS$$

These two numbers go together.

Meaning of subscripts:
"the flux through coil 1 due to coil 2"

Define mutual inductance: $$M_{12} \equiv \frac{\psi_{12}}{I_2}$$

In general, if coil 1 has multiple turns: $$M_{12} \equiv \frac{\Lambda_{12}}{I_2} = \frac{N_1 \psi_{12}}{I_2}$$

Note:
For the figure shown, $\psi_{12} < 0$ if $I_2 > 0$.
Hence, $M_{12} < 0$.

Mutual Inductance (cont.)

A general property (proof omitted) is that both mutual inductance components are always equal:

$$M_{12} = M_{21} = M$$

Note: The units of M are Henrys.

Circuit Law for Coupled Coils

Total flux through coil 1:

$$\Lambda_1 = \Lambda_{11} + \Lambda_{12} = L_1 i_1 + M_{12} i_2$$

Faraday's law:

$$v_1 = \frac{d\Lambda_1}{dt}$$

$$v_1 = L_1 \frac{di_1}{dt} + M_{12} \frac{di_2}{dt}$$

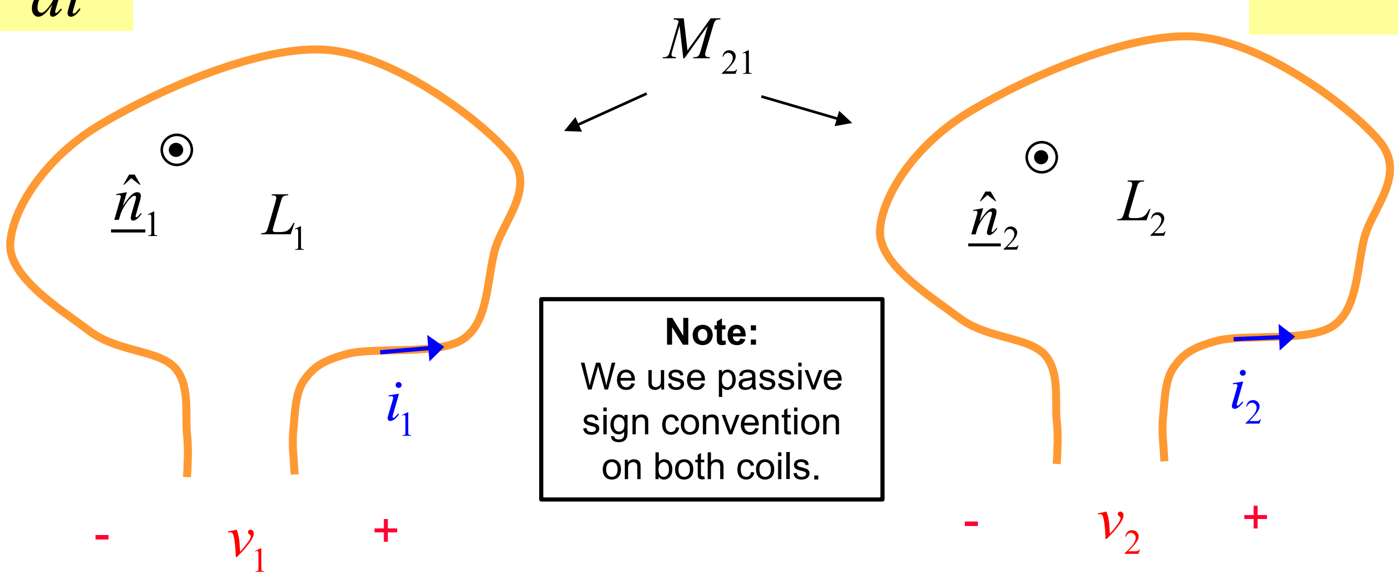
$$v_2 = L_2 \frac{di_2}{dt} + M_{21} \frac{di_1}{dt}$$

Total flux through coil 2:

$$\Lambda_2 = \Lambda_{22} + \Lambda_{21} = L_2 i_2 + M_{21} i_1$$

Faraday's law:

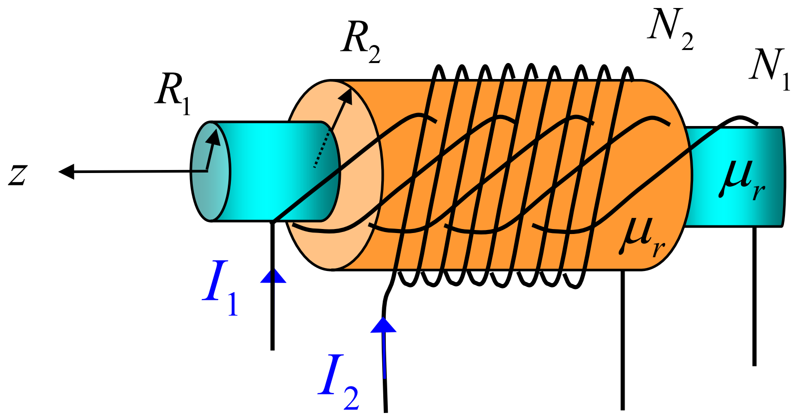
$$v_2 = \frac{d\Lambda_2}{dt}$$



Note:
We use passive sign convention on both coils.

Example

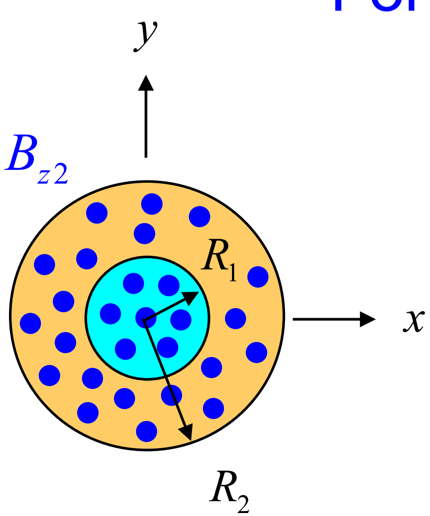
Find M_{12}, M_{21}



L_s = length

Right-hand rule for inductor flux:

$$\underline{\hat{n}}_1 = \underline{\hat{n}}_2 = \underline{\hat{z}}$$



For M_{12}:

$$M_{12} = \frac{\Lambda_{12}}{I_2} = \frac{N_1 \psi_{12}}{I_2}$$

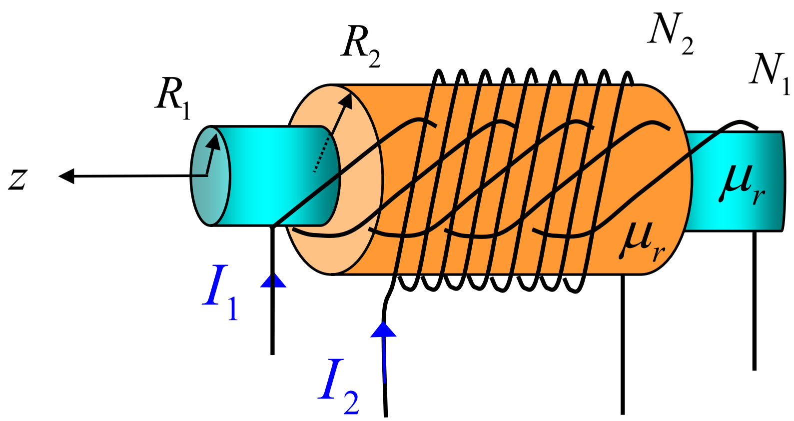
$$\psi_{12} = \int_{S_1} \underline{B}_2 \cdot \underline{\hat{n}}_1 \, dS = \int_{S_1} B_{z2} \, dS$$

$$= B_{z2} \left(\pi R_1^2 \right)$$

$$= \mu_0 \mu_r \left(n_2 I_2 \right) \left(\pi R_1^2 \right) = \mu_0 \mu_r \left(\frac{N_2}{L_s} \right) I_2 \left(\pi R_1^2 \right)$$

$$M_{12} = N_1 \mu_0 \mu_r \left(\frac{N_2}{L_s} \right) \pi R_1^2 \quad [\mathrm{H}]$$

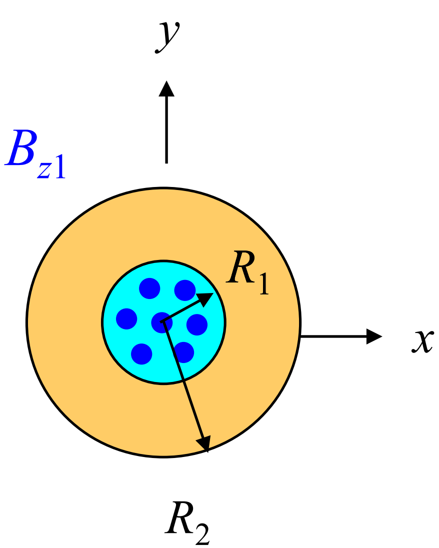
Example (cont.)



L_s = length

$\underline{\hat{n}}_1 = \underline{\hat{n}}_2 = \underline{\hat{z}}$

For M_{21}:



$$M_{21} = \frac{\Lambda_{21}}{I_1} = \frac{N_2 \psi_{21}}{I_1}$$

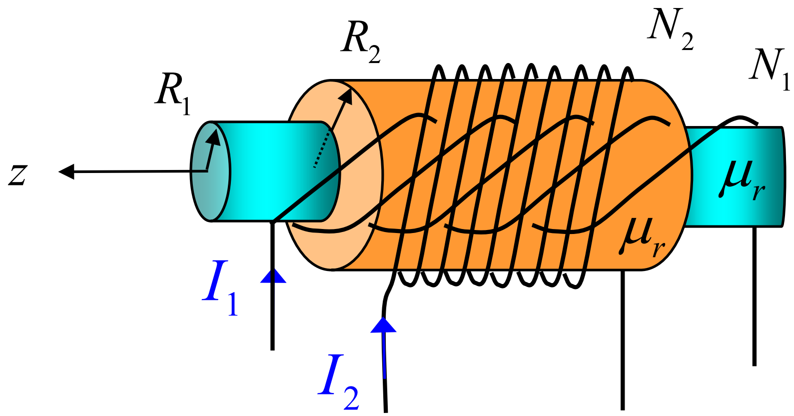
$$\psi_{21} = \int_{S_2} \underline{B}_1 \cdot \underline{\hat{n}}_2 \, dS = \int_{S_2} B_{z1} \, dS = B_{z1}\left(\pi R_1^2\right)$$

$$= \mu_0 \mu_r \left(n_1 I_1\right)\left(\pi R_1^2\right) = \mu_0 \mu_r \left(\frac{N_1}{L_s}\right) I_1 \left(\pi R_1^2\right)$$

$$M_{21} = N_2 \mu_0 \mu_r \left(\frac{N_1}{L_s}\right) \pi R_1^2 \;\; [\mathrm{H}]$$

Note: $B_{z1} = 0$ for $\rho > R_1$.

Example (cont.)



L_s = length

Summary:

$$M_{12} = N_1 \mu_0 \mu_r \left(\frac{N_2}{L_s} \right) \pi R_1^2$$

$$M_{21} = N_2 \mu_0 \mu_r \left(\frac{N_1}{L_s} \right) \pi R_1^2$$

$$M_{12} = M_{21} = M = \mu_0 \mu_r \left(\frac{\pi R_1^2}{L_s} \right) (N_1 N_2) \quad [\mathrm{H}]$$

Dot Convention

The dot convention allows us to use mutual inductance M without having to visually inspect how the coils are wound.

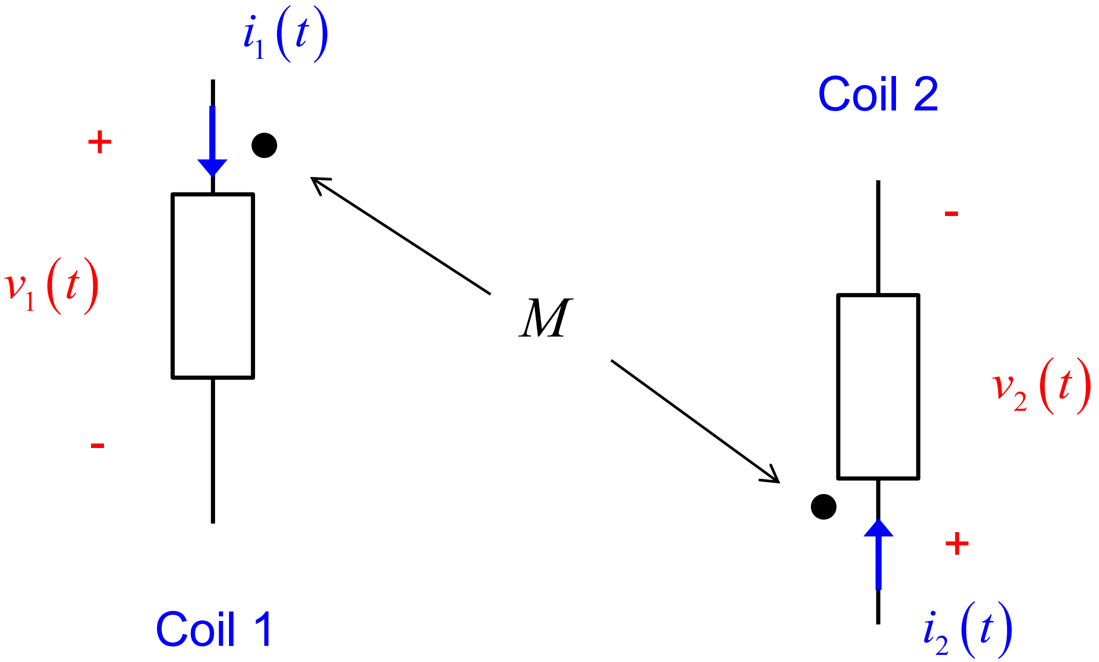
$$v_1 = L_1 \frac{di_1}{dt} + M \frac{di_2}{dt} \qquad v_2 = L_2 \frac{di_2}{dt} + M \frac{di_1}{dt}$$

The dots tell us where to put the positive sign for the voltage on one coil, and where the current enters the other coil.

Note:
We also label voltages and currents with the passive sign convention, to be consistent with the self inductance.

Dot Convention (cont.)

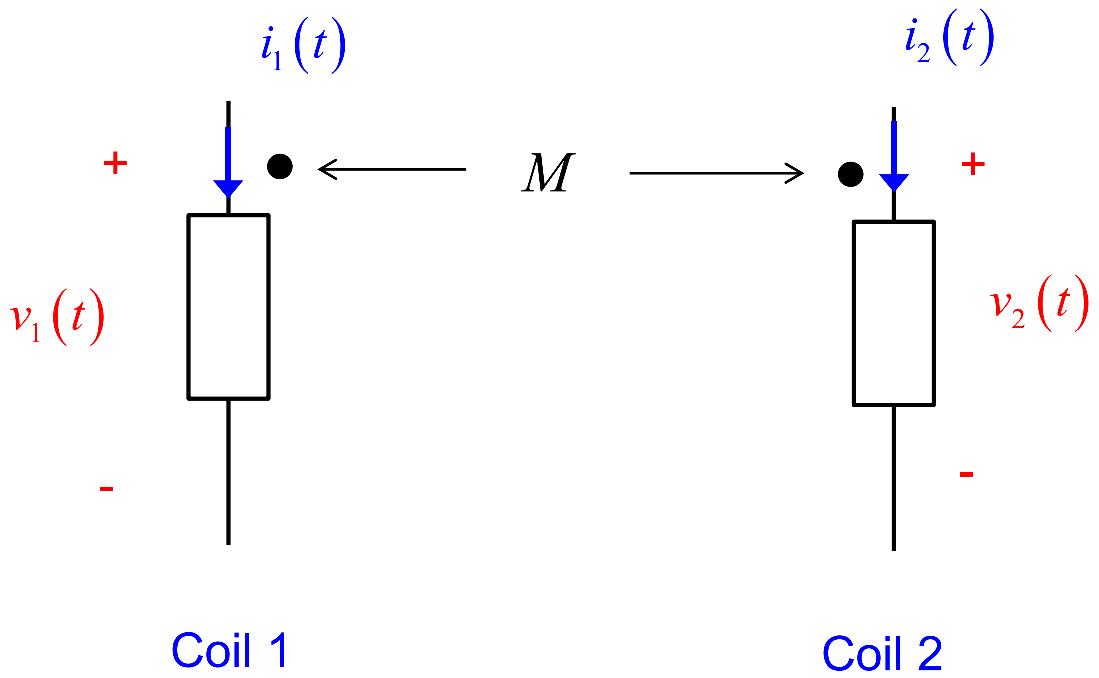
Here is one possible dot arrangement:



$$v_1 = L_1 \frac{di_1}{dt} + M \frac{di_2}{dt} \qquad v_2 = L_2 \frac{di_2}{dt} + M \frac{di_1}{dt}$$

Dot Convention (cont.)

Here is another possible dot arrangement:



Note:
The M here is the negative of the M on the last slide (if the coils are the same).

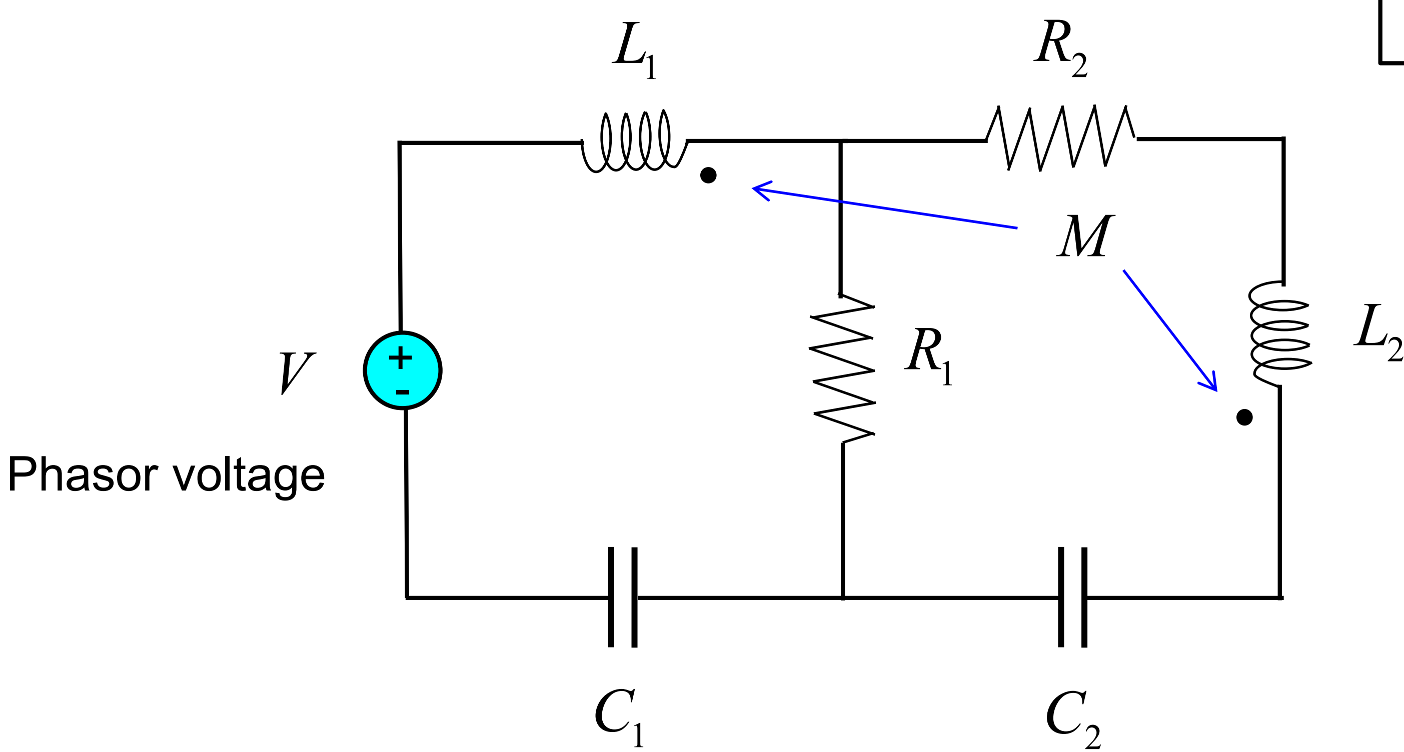
We can always choose the dots to make M positive if we wish.

$$v_1 = L_1 \frac{di_1}{dt} + M \frac{di_2}{dt} \qquad v_2 = L_2 \frac{di_2}{dt} + M \frac{di_1}{dt}$$

Example

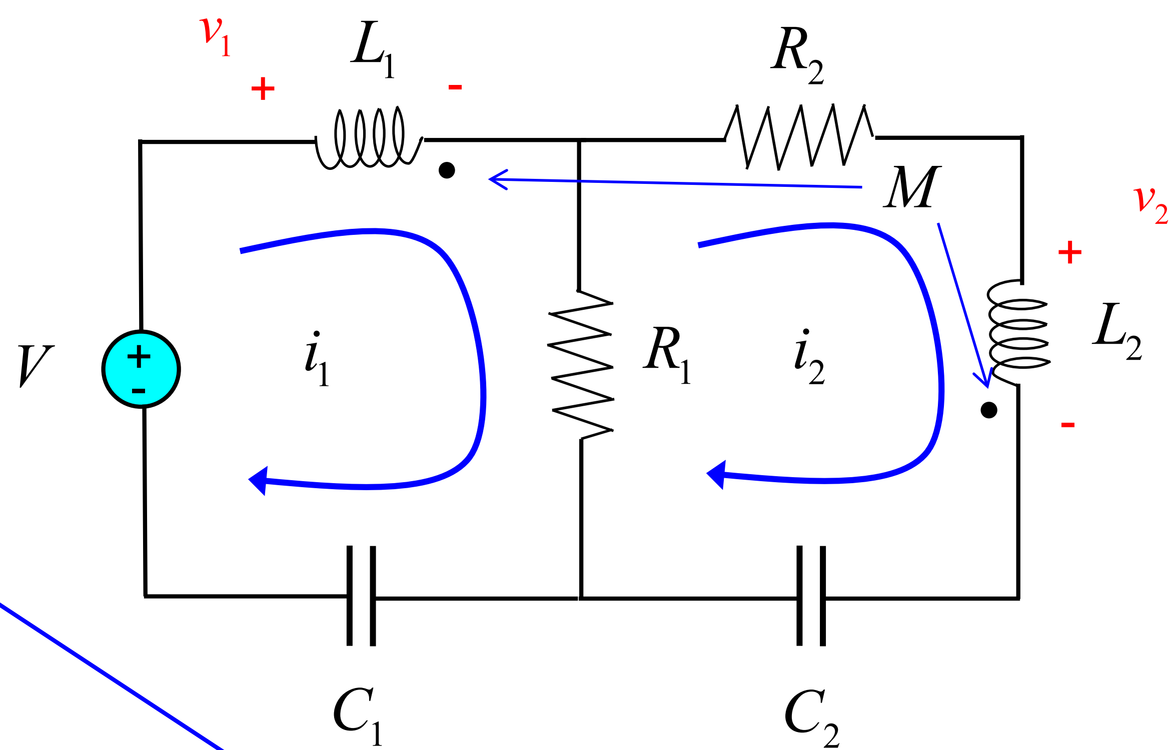
Write down KVL phasor-domain mesh-current equations to describe this circuit.

Note:
M could be either positive or negative.



Example (cont.)

Inductor voltages



Note:
In these two equations the first minus sign is from the dot being on the opposite side as the "+" sign of the voltage drop. The second minus sign (the one on the current) is because we want the current going into the dot.

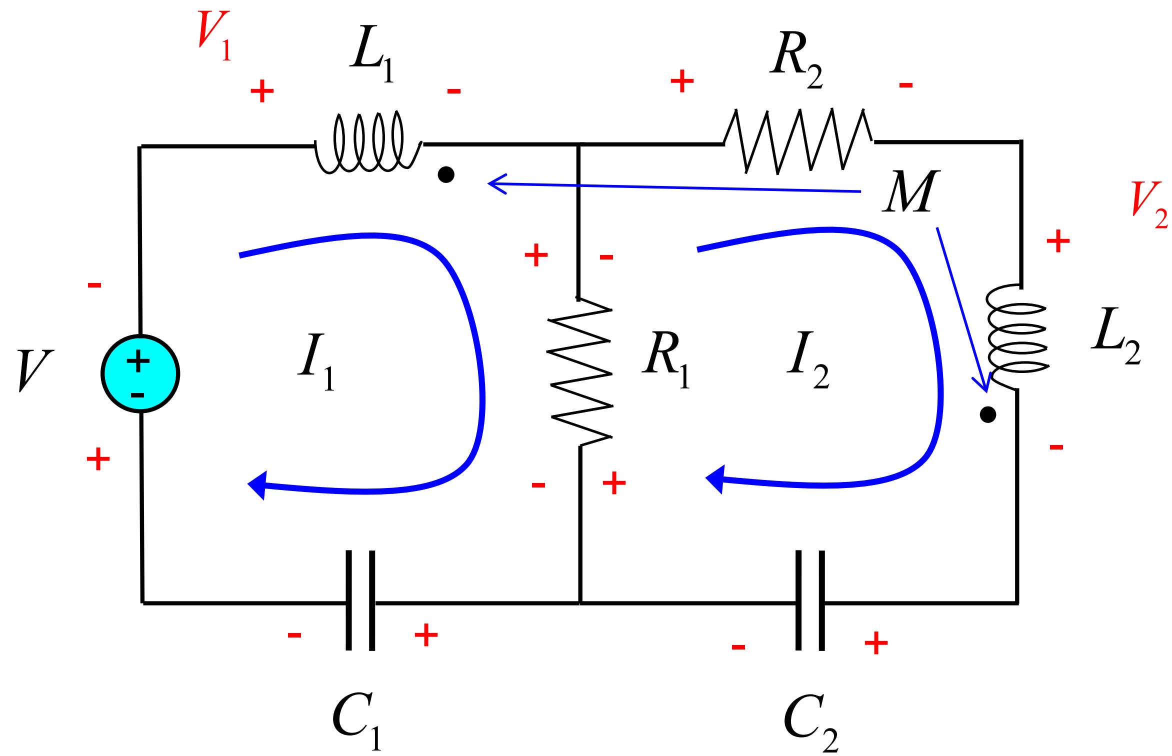
$$v_1 = L_1 \frac{di_1}{dt} - M \frac{d(-i_2)}{dt} \qquad v_2 = L_2 \frac{di_2}{dt} - M \frac{d(-i_1)}{dt}$$

Phasor domain

$$V_1 = j\omega L_1 I_1 + j\omega M I_2 \qquad V_2 = j\omega L_2 I_2 + j\omega M I_1$$

Example (cont.)

KVL



$$-V + \left[j\omega L_1 I_1 + j\omega M I_2 \right] + R_1 \left(I_1 - I_2 \right) + \left(\frac{1}{j\omega C_1} \right) I_1 = 0$$

$$R_1 \left(I_2 - I_1 \right) + R_2 I_2 + \left[j\omega L_2 I_2 + j\omega M I_1 \right] + \left(\frac{1}{j\omega C_2} \right) I_2 = 0$$