ECE 3318
Applied Electricity and Magnetism

Spring 2023

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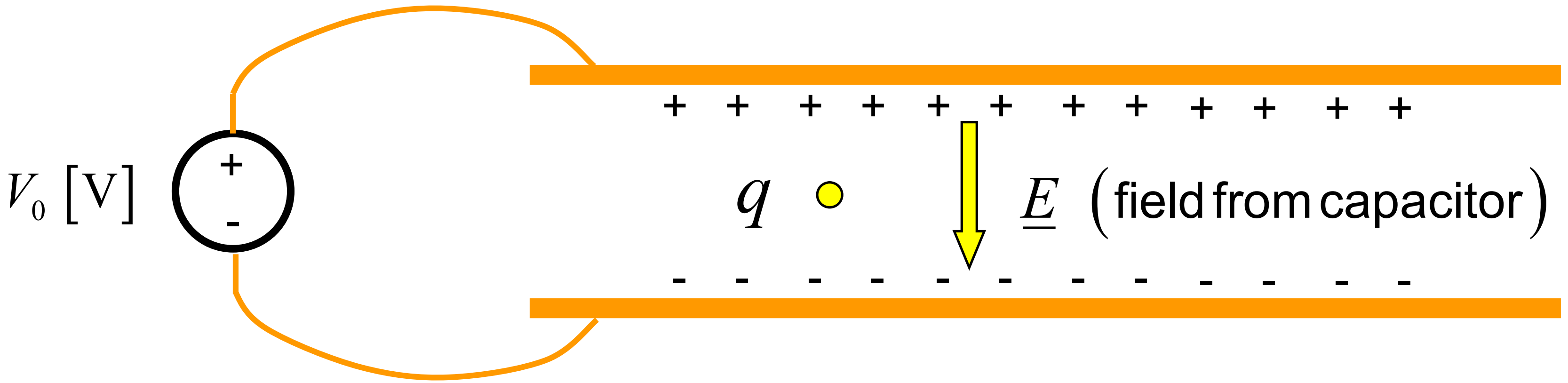
Dept. of ECE

Notes 4
Electric Field and Voltage

Notes prepared by the EM Group
University of Houston

Electric Field



$$\underline{E} \equiv \frac{1}{q} \underline{F}$$

Note:
The "test" charge is small enough so that it does not disturb the charge on the capacitor and hence the field from the capacitor.

so

$$\underline{F} = q\, \underline{E}$$

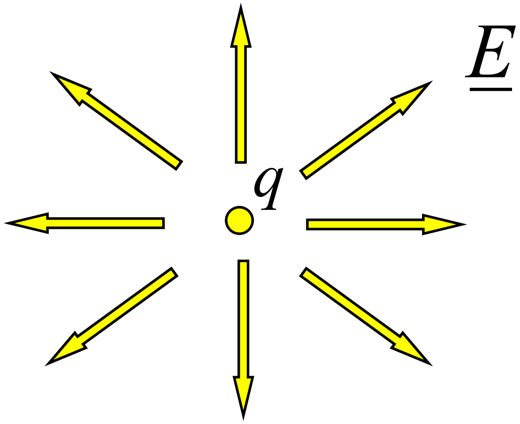
The electric-field vector is *the force vector for a unit charge*.

Units of electric field: [V/m]

Electric Field (cont.)

Note: The electric-field vector may be non-uniform.

Point charge q in free space:

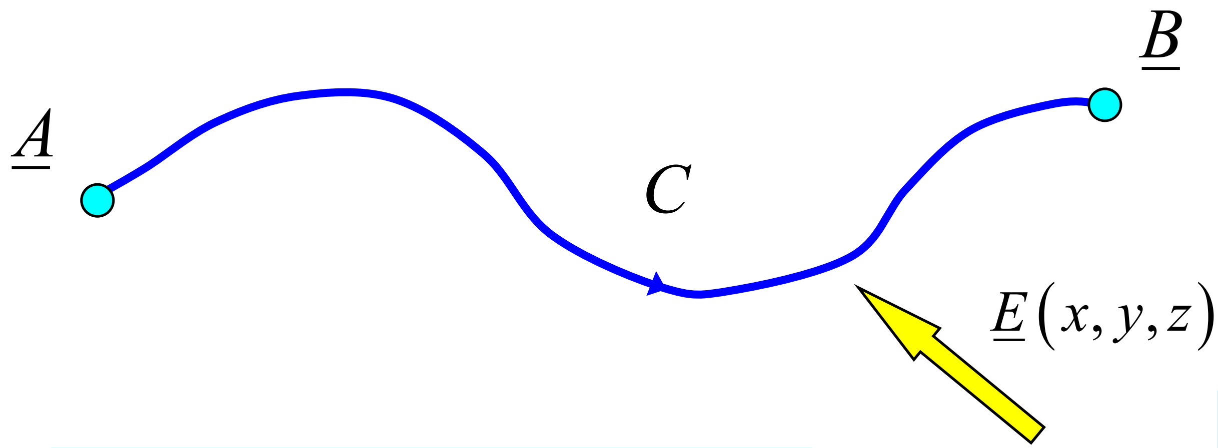


$$\underline{E} = \underline{\hat{r}} \left(\frac{q}{4\pi\varepsilon_0 r^2} \right) \quad \text{(nonuniform)}$$

Voltage Drop

Volta

The voltage drop ***between two points*** is now defined.



$$V_{AB} \equiv V\left(\underline{A}\right) - V\left(\underline{B}\right) \equiv \int_{\underline{A}}^{\underline{B}} \underline{E} \cdot d\underline{r}$$

Notation:

$$\int_{\underline{A}}^{\underline{B}} \underline{E} \cdot d\underline{r} = \int_{C} \underline{E} \cdot d\underline{r}$$

$C =$ path from $\underline{A}$ to $\underline{B}$.

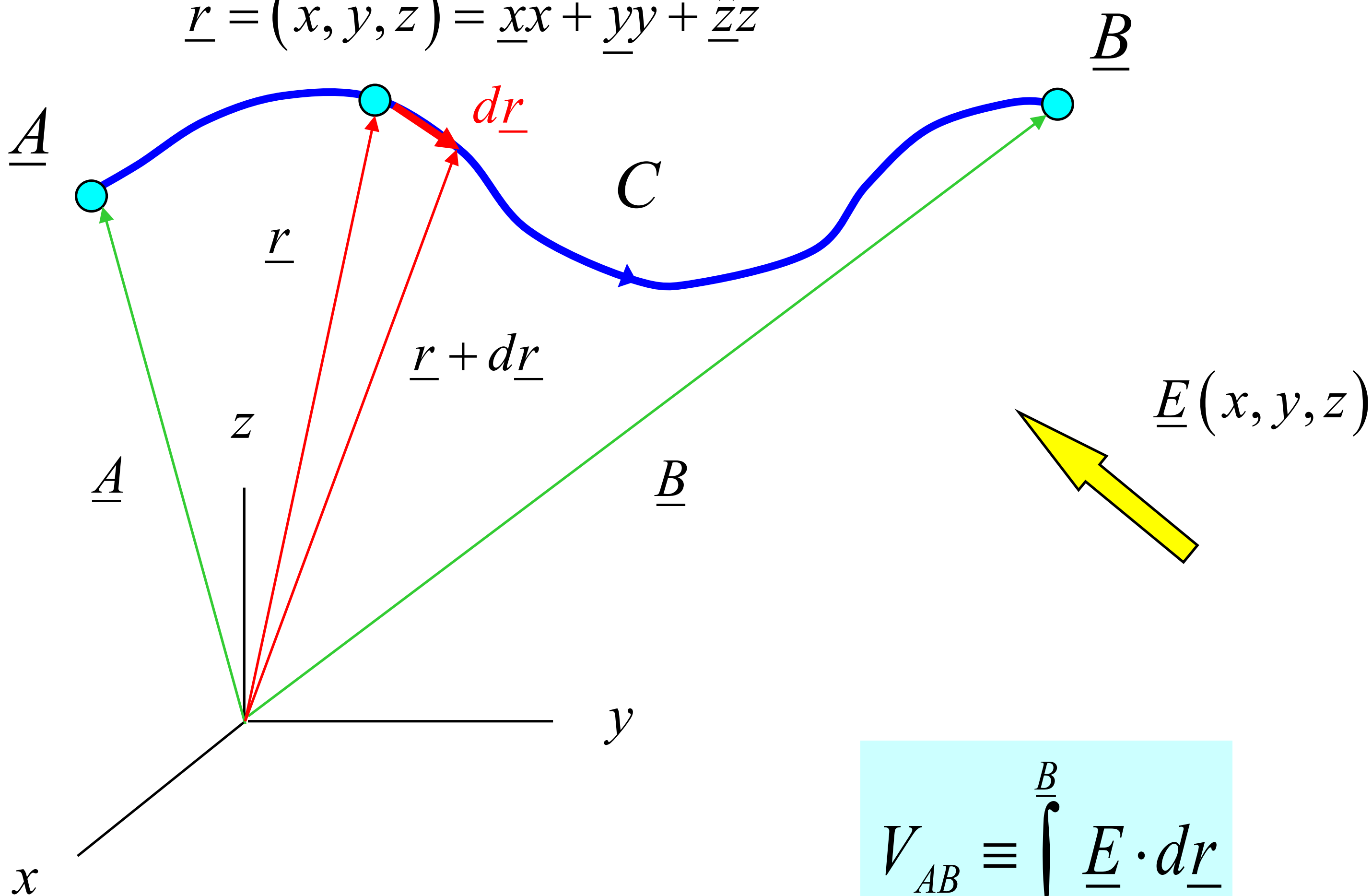
Comment: In *statics*, the line integral is independent of the shape of the path.

(This will be proven later after we talk about the curl.)

Voltage Drop

Here all of the mathematical terms are defined.

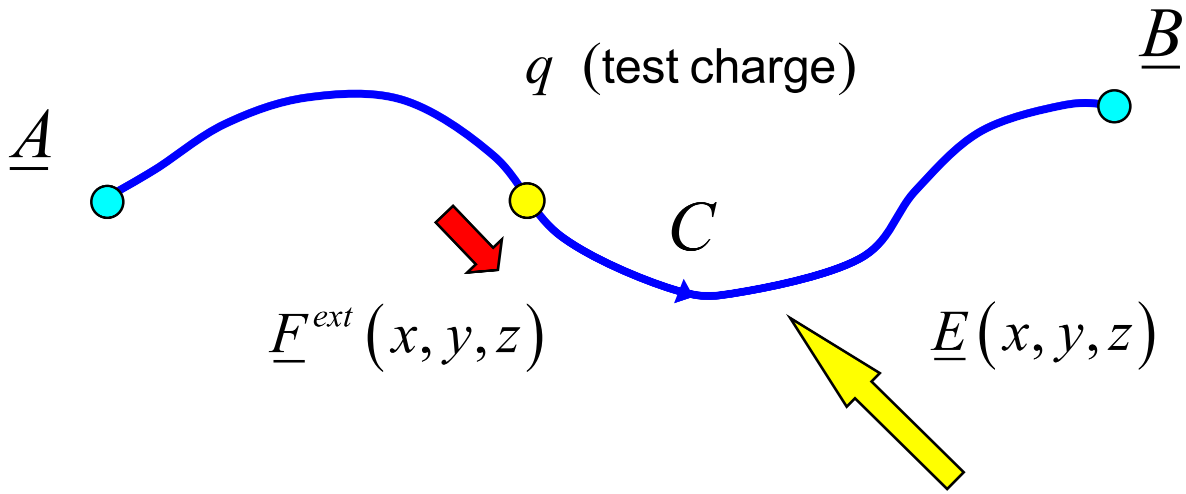
$$\underline{r} = (x, y, z) = \underline{\hat{x}}x + \underline{\hat{y}}y + \underline{\hat{z}}z$$



$$V_{AB} \equiv \int_{\underline{A}}^{\underline{B}} \underline{E} \cdot d\underline{r}$$

Voltage Drop (Cont.)

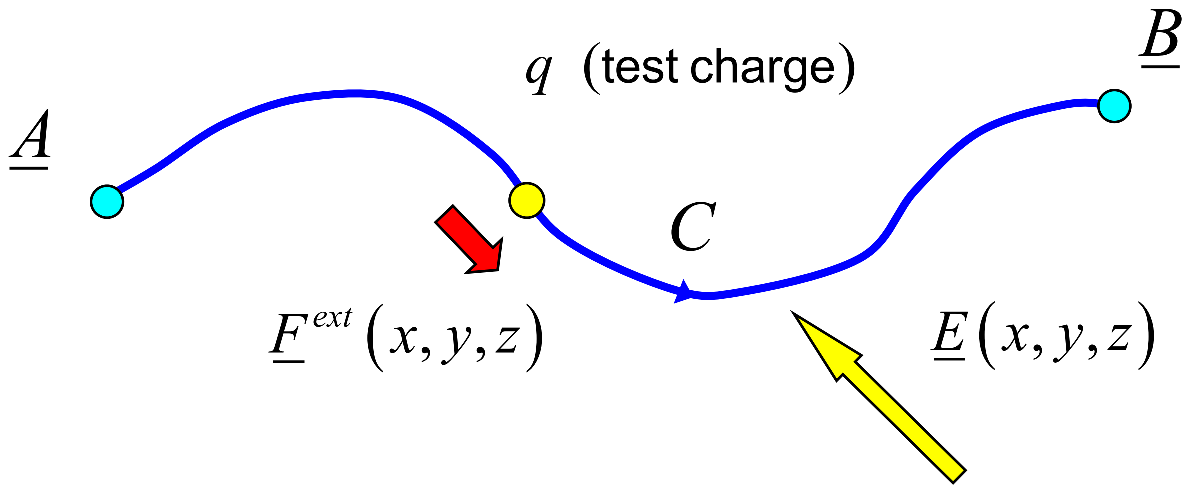
We now explore the *physical interpretation* of voltage drop.



A test charge is moved from point $\underline{A}$ to point $\underline{B}$ at a constant speed (no increase in kinetic energy).

$$\underline{F}^{E} = q\,\underline{E} \qquad \underline{F}^{ext} = -\underline{F}^{E} = -q\,\underline{E}$$

Voltage Drop (cont.)



Define:

W^{ext} = work done by observer in moving the charge.

$$W^{ext} = \int_{\underline{A}}^{\underline{B}} \underline{F}^{ext} \cdot d\underline{r} = \int_{\underline{A}}^{\underline{B}} -q\,\underline{E} \cdot d\underline{r} = -q \int_{\underline{A}}^{\underline{B}} \underline{E} \cdot d\underline{r}$$

Hence $\quad W^{ext} = -q\,V_{AB}$

Voltage Drop (cont.)

From the last slide: $W^{ext} = -q\,V_{AB}$

But we also have $W^{ext} = \mathrm{PE}\left(\underline{B}\right) - \mathrm{PE}\left(\underline{A}\right)$

(Remember: no change in kinetic energy.)

so

$$-qV_{AB} = \mathrm{PE}\left(\underline{B}\right) - \mathrm{PE}\left(\underline{A}\right)$$

or

$$V_{AB} = \frac{1}{q}\left[\mathrm{PE}\left(\underline{A}\right) - \mathrm{PE}\left(\underline{B}\right)\right]$$

Physical Interpretation of Voltage

Result:

$$V_{AB} = \mathrm{PE}\left(\underline{A}\right) - \mathrm{PE}\left(\underline{B}\right) \quad \left(q = 1\right)$$

Conclusion:

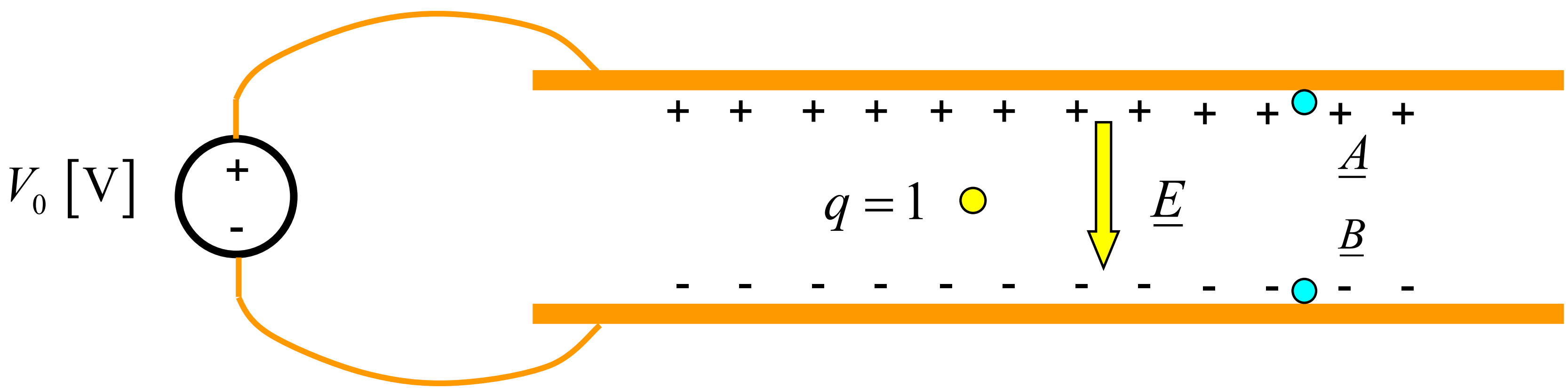
The voltage drop is equal to the change in potential energy of a unit test charge.

The voltage at a point can be thought of as the potential energy of a unit test charge at that point.

Physical Interpretation of Voltage

Example:

Point $\underline{A}$ is at a higher voltage, and hence a higher potential energy, than point $\underline{B}$.



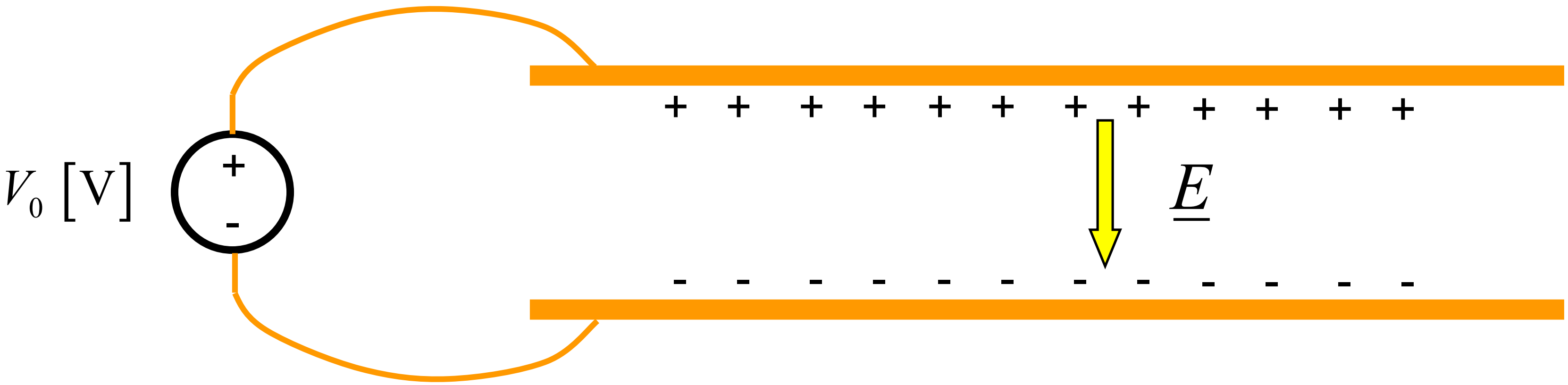
Parallel-plate capacitor

$$V_{AB} = V_0 > 0 \;\; \Rightarrow \;\; V_{AB} = \mathrm{PE}\left(\underline{A}\right) - \mathrm{PE}\left(\underline{B}\right) > 0$$

The charge is at a higher potential energy when near the top plate!

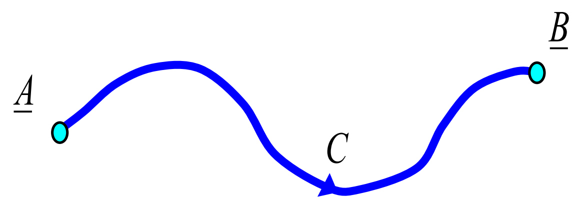
Comments

- The electric field vector points from positive charges to negative charges.
- Positive charges are at a higher voltage, and negative charges are at a lower voltage.
- The electric field points from higher voltage to lower voltage.



Review of Doing Line Integrals

$$V_{AB} = \int_{\underline{A}}^{\underline{B}} \underline{E} \cdot d\underline{r}$$



In rectangular coordinates,

$$\underline{E} = \underline{\hat{x}}\, E_x + \underline{\hat{y}}\, E_y + \underline{\hat{z}}\, E_z$$

$$\underline{r} = \underline{\hat{x}}\, x + \underline{\hat{y}}\, y + \underline{\hat{z}}\, z \quad \Rightarrow \quad d\underline{r} = \underline{\hat{x}}\, dx + \underline{\hat{y}}\, dy + \underline{\hat{z}}\, dz$$

Hence

$$V_{AB} = \int_{\underline{A}}^{\underline{B}} \left(E_x\, dx + E_y\, dy + E_z\, dz \right)$$

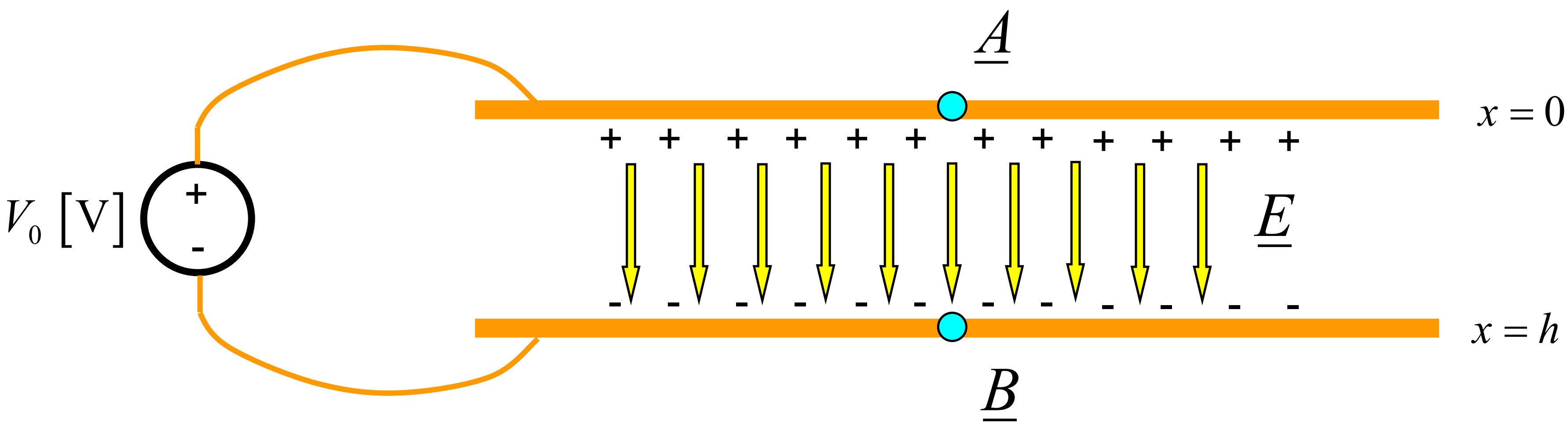
Note:
The limits are <u>vector</u> points.

so

$$V_{AB} = \int_{x_A}^{x_B} E_x\left(x, y, z\right) dx + \int_{y_A}^{y_B} E_y\left(x, y, z\right) dy + \int_{z_A}^{z_B} E_z\left(x, y, z\right) dz$$

Each integrand must be <u>parameterized</u> in terms of the respective integration variable. This requires knowledge of the path C.

Example



Find: $\underline{E}$

Assume: $\underline{E}(x, y, z) = \underline{\hat{x}}\, E_0$

"Ideal parallel-plate capacitor assumption"

$$V_{AB} = \int_{\underline{A}}^{\underline{B}} \underline{E}(x, y, z) \cdot d\underline{r} = V_0 \text{ [V]}$$

Example (cont.)

$$V_{AB} = \int_{\underline{A}}^{\underline{B}} \underline{E}(x,y,z) \cdot d\underline{r} = V_0 \text{ [V]}$$

Evaluate in rectangular coordinates:

$$\underline{E} = \underline{\hat{x}}\, E_x + \underline{\hat{y}}\, E_y + \underline{\hat{z}}\, E_z$$

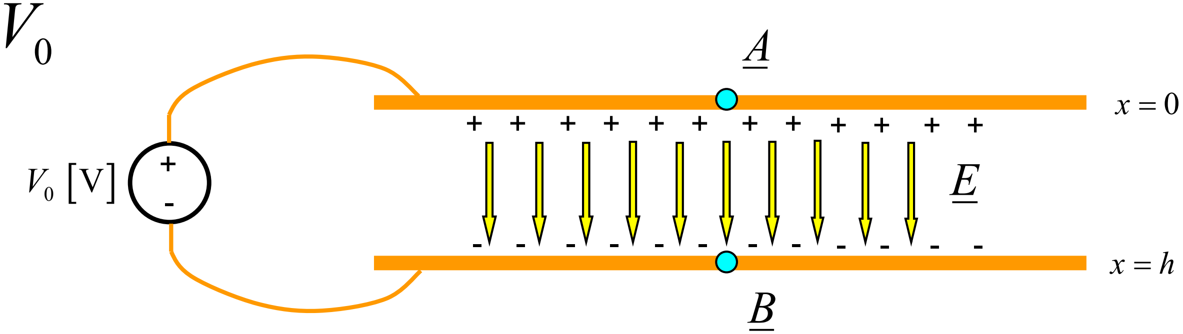
$$d\underline{r} = \underline{\hat{x}}\, dx + \underline{\hat{y}}\, dy + \underline{\hat{z}}\, dz$$

$$\underline{E} \cdot d\underline{r} = E_x\, dx + \cancel{E_y}\, dy + \cancel{E_z}\, dz$$

$$V_{AB} = \int_{x_A}^{x_B} E_x(x,y,z)\, dx = V_0 \text{ [V]}$$

Example (cont.)

$$V_{AB} = \int_{x_A}^{x_B} E_x\left(x,y,z\right)dx = V_0$$



$$\underline{E}\left(x,y,z\right) = \underline{\hat{x}}\, E_0$$

or

$$\int_0^h E_0\, dx = V_0$$

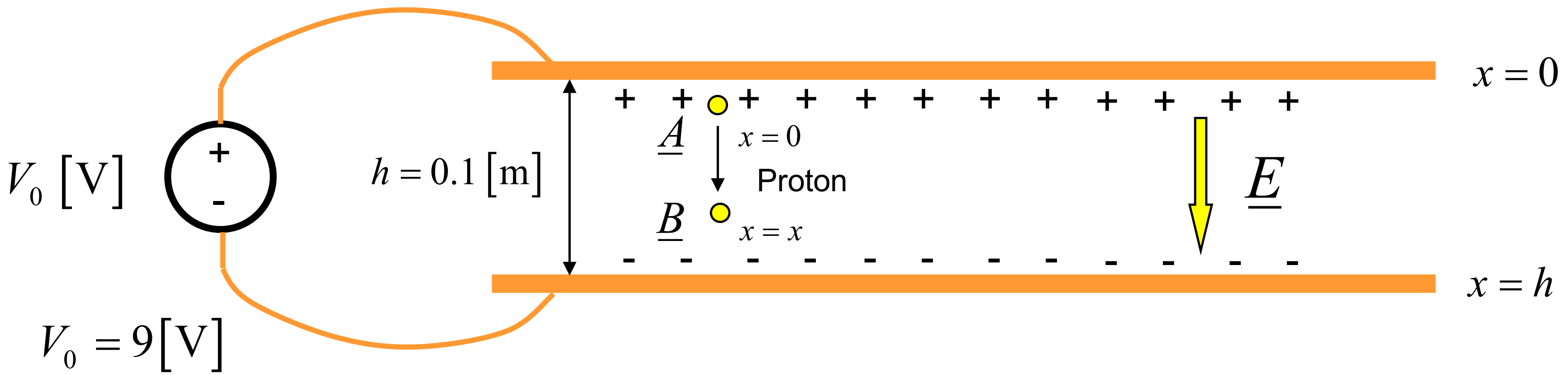
or

$$E_0\, h = V_0 \quad \Rightarrow \quad E_0 = V_0 / h$$

Recall that $\underline{E}\left(x,y,z\right) = \underline{\hat{x}}\, E_0$

Hence, we have $$\boxed{\underline{E}\left(x,y,z\right) = \underline{\hat{x}}\left(\frac{V_0}{h}\right) \ \text{[V/m]}}$$

Example



A proton is released at point $\underline{A}$ on the top plate with zero velocity.
Find the velocity $v(x)$ of the proton at distance x from the top plate (at point $\underline{B}$).

Conservation of energy: $\mathrm{KE}(x) - \cancel{\mathrm{KE}(0)} = \mathrm{PE}(0) - \mathrm{PE}(x)$

$$\frac{1}{2} m v^2(x) = \mathrm{PE}(0) - \mathrm{PE}(x) = q\left[V(0) - V(x)\right]$$

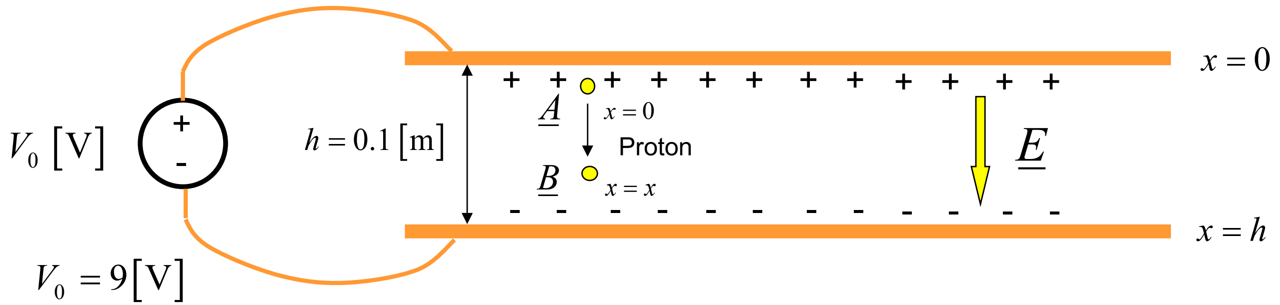
Example (cont.)

From last slide: $$\frac{1}{2}m v^2(x) = q\left[V(0) - V(x)\right]$$

$$V(0) - V(x) = \int_{\underline{A}}^{\underline{B}} \underline{E} \cdot d\underline{r} = \int_0^x E_x(x,y,z)\,dx = \int_0^x \left(\frac{V_0}{h}\right) dx = \left(\frac{V_0}{h}\right) x$$

Hence: $$\frac{1}{2}m v^2(x) = q\left(\frac{V_0}{h}\right) x$$

so $$v(x) = \sqrt{\left(\frac{2qV_0}{h\,m}\right) x}$$



Example (cont.)

$$v(x) = \sqrt{\left(\frac{2qV_0}{h\,m}\right)x}$$

$V_0 = 9\ \ [\mathrm{V}] \qquad h = 0.1\ [\mathrm{m}]$

$q = 1.602 \times 10^{-19}\ [\mathrm{C}]$

$m = 1.673 \times 10^{-27}\ [\mathrm{kg}]$

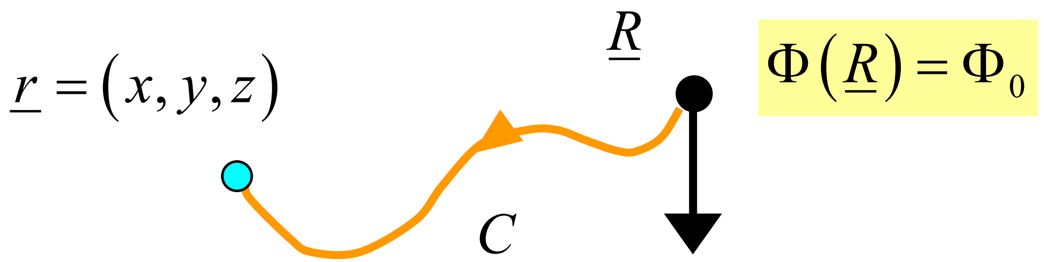
(See Appendix B of the Hayt & Buck book or Appendix D of the Shen & Kong book for these values.)

Hence:

$$v(x) = 5.627 \times 10^{6}\sqrt{x}\quad [\mathrm{m/s}]$$

Reference Point

- A reference point $\underline{R}$ is a point where the voltage is assigned.
- This makes the voltage unique at all points in space.
- The voltage at a given point is often called the "potential" Φ at the point.

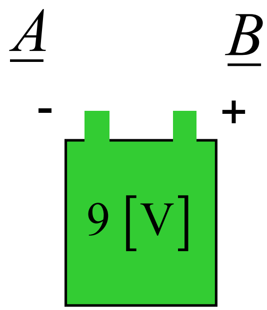


Integrating the electric field along C will determine the potential at point $\underline{r}$.

Note: Φ_0 is often chosen to be zero, but this is not necessary.

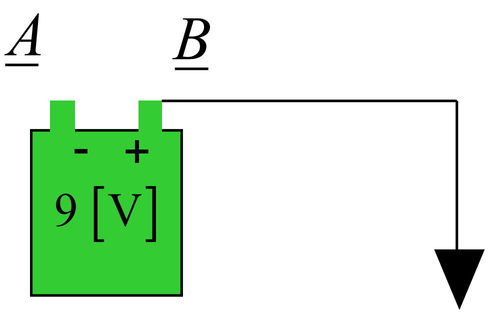
Example

A 9 volt battery:



$$V_{BA} = 9\ [\mathrm{V}]$$

Find the potential on the left terminal (cathode) assuming the reference point $\underline{R}$ is on the right terminal (anode) and $\Phi = 0$ at the reference point.



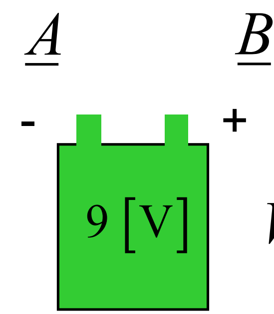
$$\Phi(\underline{R}) \equiv 0$$

$$V_{BA} = \cancel{\Phi(\underline{B})} - \Phi(\underline{A}) = 9\ [\mathrm{V}]$$

$$\boxed{\Phi(\underline{A}) = -9\ [\mathrm{V}]}$$

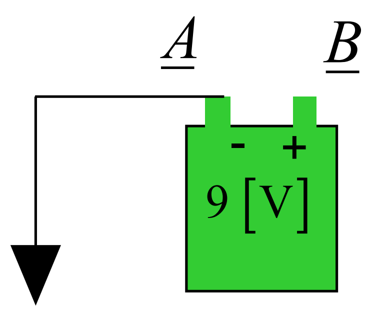
Example

A 9 volt battery:



$$V_{BA} = 9\ [\mathrm{V}]$$

Find the potential on the right terminal (anode) assuming the reference point $\underline{R}$ is on the left terminal (cathode) and $\Phi = 0$ at the reference point.



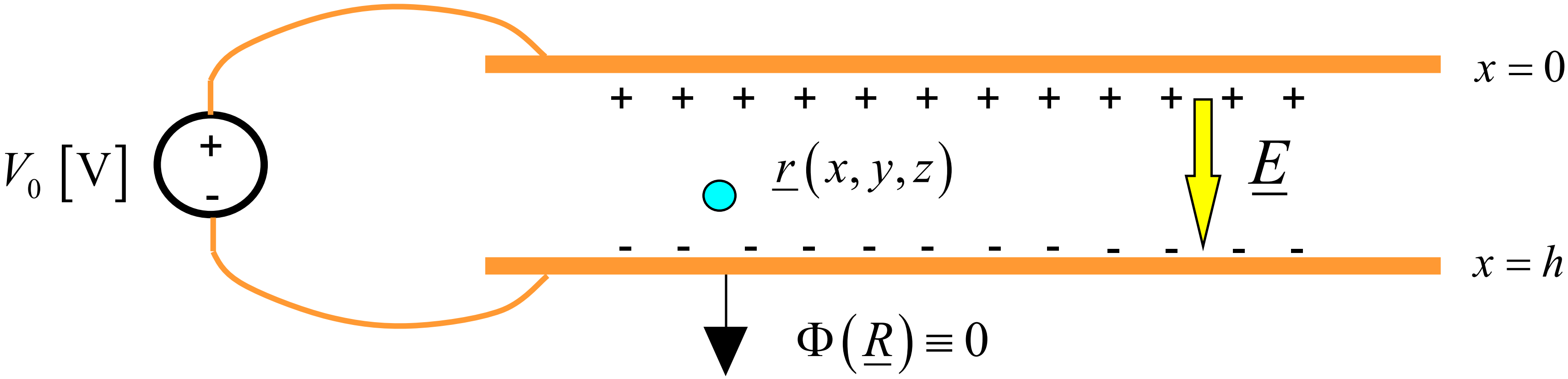
$$\Phi\left(\underline{R}\right) \equiv 0$$

$$V_{BA} = \Phi\left(\underline{B}\right) - \cancel{\Phi\left(\underline{A}\right)} = 9\ [\mathrm{V}]$$

$$\boxed{\Phi\left(\underline{B}\right) = 9\ [\mathrm{V}]}$$

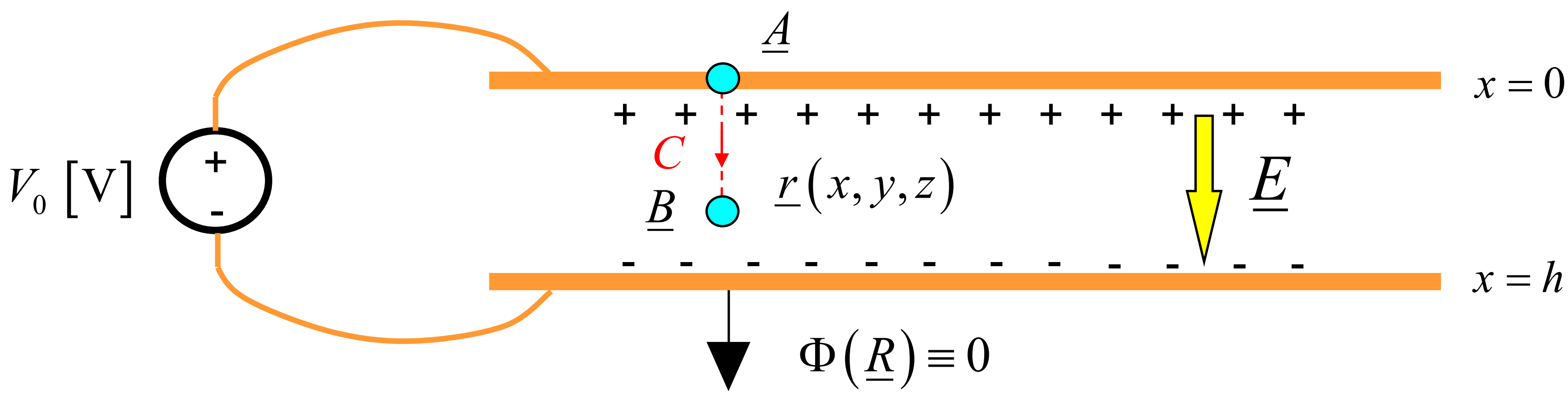
This is the more usual choice for the reference point (on the negative terminal).

Example



Find the potential function $\Phi(x)$, $0 < x < h$, assuming that the reference point $\underline{R}$ is on the bottom plate, and the voltage at $\underline{R}$ is zero.

Example (cont.)



Find the potential function $\Phi(x)$, $0 < x < h$, assuming that the reference point $\underline{R}$ is on the bottom plate, and the voltage at $\underline{R}$ is zero.

$$V_{AB} = \Phi(0) - \Phi(x) = \int_0^x E_x(x, y, z)\,dx = \int_0^x \left(\frac{V_0}{h}\right) dx = \left(\frac{V_0}{h}\right) x$$

Also, $\Phi(0) - \cancel{\Phi(h)} = V_0 \quad \Rightarrow \quad \Phi(0) = V_0$

Hence $$\boxed{\Phi(x) = V_0 - \left(\frac{V_0}{h}\right) x \quad [\mathrm{V}]}$$

Voltage Drop in Dynamics

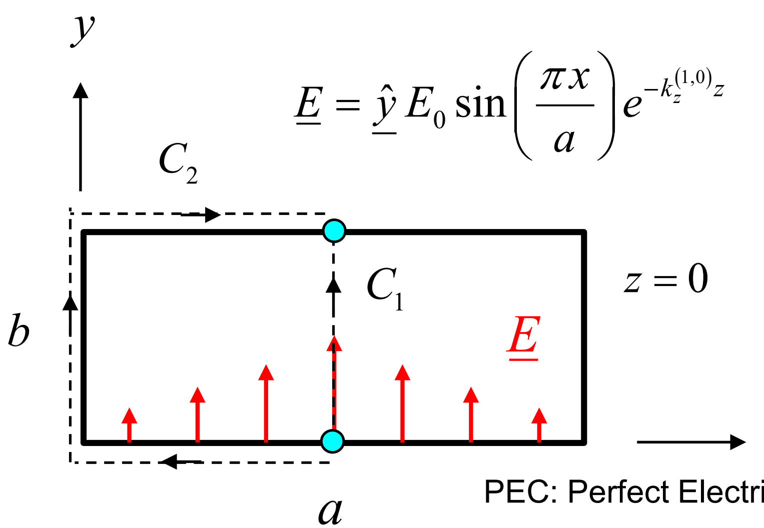
In statics, the voltage drop is unique, and does not depend on the shape of the path. In dynamics, this is not generally true.

Example: TE_{10} mode of rectangular waveguide

$$\underline{E} = \underline{\hat{y}}\, E_0 \sin\left(\frac{\pi x}{a}\right) e^{-k_z^{(1,0)} z}$$

(phasor $\underline{E}$ field)

(microwave frequency)



PEC: Perfect Electric Conductor

- There is a voltage drop along path C_1 ($V_{AB} = E_0$).
- There is no voltage drop along path C_2 (which stays inside the PEC metal).

So…the voltage drop is not uniquely defined!

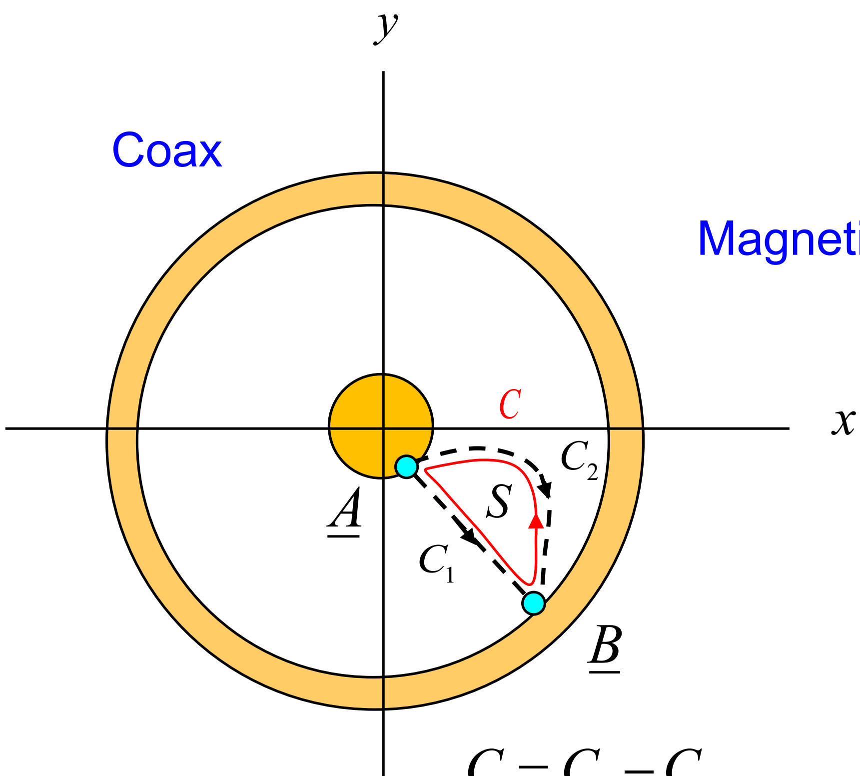
Voltage Drop in Dynamics (cont.)

One situation where voltage is uniquely defined, even at high frequency: **a TEM mode on a transmission line**.

$$\int_C \underline{E} \cdot d\underline{r} = -\frac{d\psi}{dt}$$

(Faraday's law)

Coax



Magnetic flux:

$$\psi = \int_S \underline{B} \cdot \underline{\hat{n}} \, dS = \int_S \underline{B} \cdot \underline{\hat{z}} \, dS = \int_S \cancel{B_z} \, dS$$

TEM mode

$$\int_C \underline{E} \cdot d\underline{r} = 0$$

$$C = C_1 - C_2$$

$$V_{AB} \equiv \int_{\underline{A}}^{\underline{B}} \underline{E} \cdot d\underline{r}$$

The counterclockwise closed path C goes from $\underline{A}$ to $\underline{B}$ along C_1, and then from $\underline{B}$ back to $\underline{A}$ along C_2.

$$\int_{C_1} \underline{E} \cdot d\underline{r} = \int_{C_2} \underline{E} \cdot d\underline{r}$$