ECE 3318
Applied Electricity and Magnetism

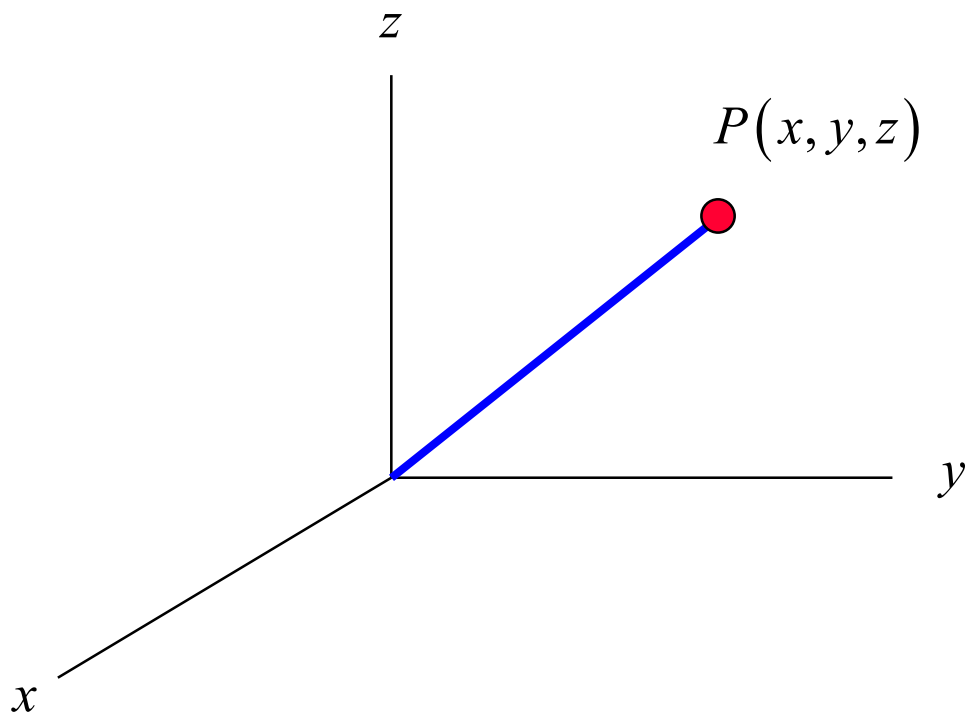
Spring 2023

Prof. David R. Jackson
Dept. of ECE

Notes 6
Coordinate Systems

Notes prepared by the EM Group
University of Houston

Review of Coordinate Systems



An understanding of coordinate systems is important for doing EM calculations.

Types of Integrals That Often Occur

Line integrals:

$$Q = \int_C \rho_\ell \, d\ell$$ (scalars in integral, scalar result)

$$V_{AB} = \int_C \underline{E} \cdot d\underline{r}$$ (vectors in integral, scalar result)

$$I = \int_C \underline{J}_s \cdot \hat{\underline{n}} \, dl$$ (vectors in integral, scalar result)

$$\underline{E} = \int_C \frac{\rho_\ell \hat{\underline{R}}}{4\pi\varepsilon_0 R^2} \, d\ell$$ (vector in integral, vector result)

Surface integrals:

$$Q = \int_S \rho_s \, dS$$ (scalars in integral, scalar result)

$$I = \int_S \underline{J} \cdot \hat{\underline{n}} \, dS$$ (vectors in integral, scalar result)

$$\underline{E} = \int_S \frac{\rho_s \hat{\underline{R}}}{4\pi\varepsilon_0 R^2} \, dS$$ (vector in integral, vector result)

Volume integrals:

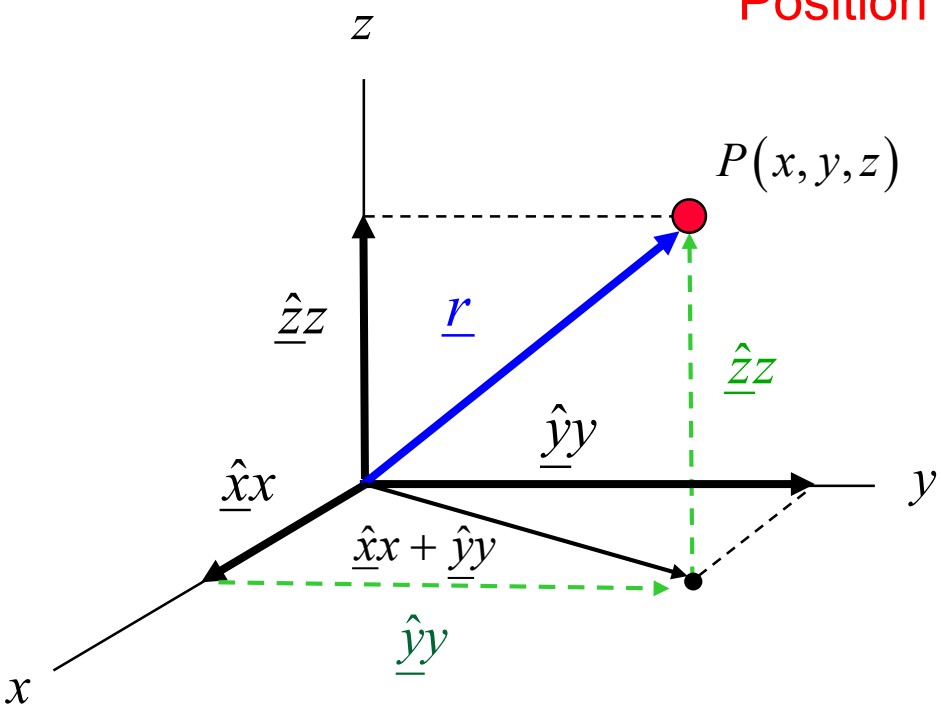
$$Q = \int_V \rho_v \, dV$$ (scalars in integral, scalar result)

$$\underline{E} = \int_V \frac{\rho_v \hat{\underline{R}}}{4\pi\varepsilon_0 R^2} \, dV$$ (vector in integral, vector result)

We wish to be able to perform all of these calculations in various coordinate systems.

Rectangular Coordinates

Position vector: $\underline{r} = \underline{\hat{x}}x + \underline{\hat{y}}y + \underline{\hat{z}}z$



Short hand notation:

$$\underline{r} = (x, y, z)$$

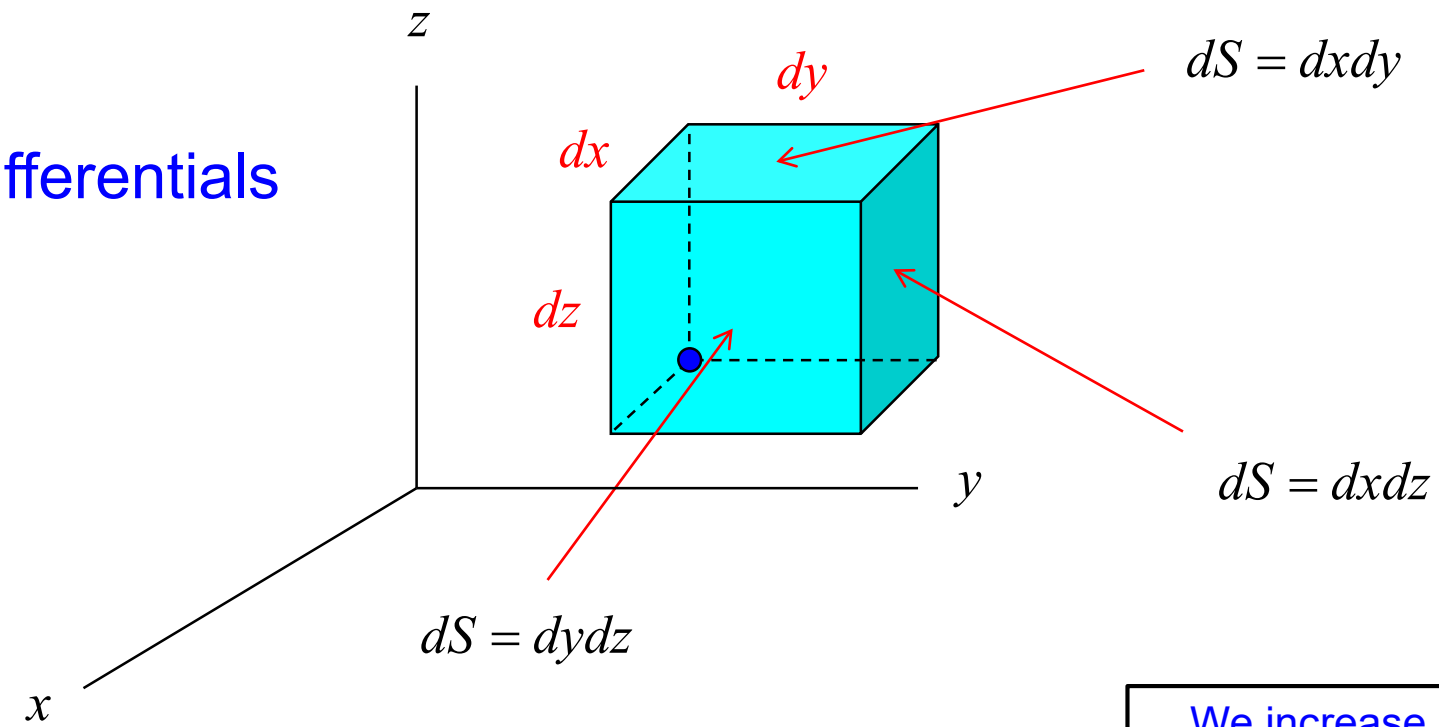
Note:
We have the "tip to tail" rule when adding vectors.

Note:
A unit vector direction is defined by <u>increasing</u> one coordinate variable while keeping the other two fixed.

Note: Different notations are used for vectors in the books.

Rectangular Coordinates

Differentials



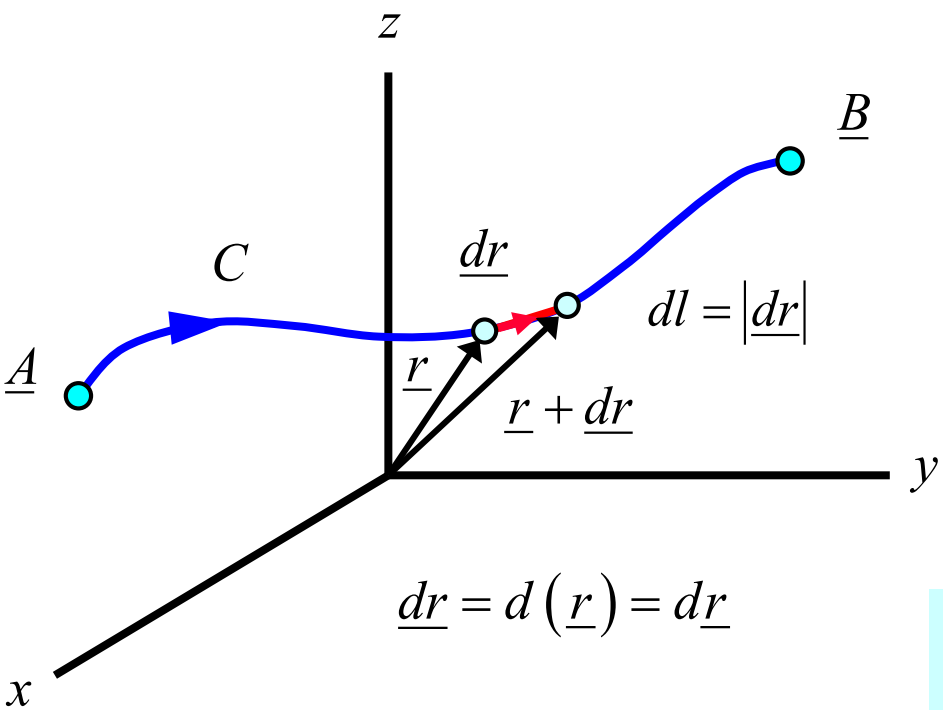
$$dV = dx\,dy\,dz$$

We <u>increase</u> x, y, or z starting from an initial point (blue dot).

Note:
dS may be in <u>three</u> different forms.

Rectangular (cont.)

Path Integral (we need $\underline{dr}$)



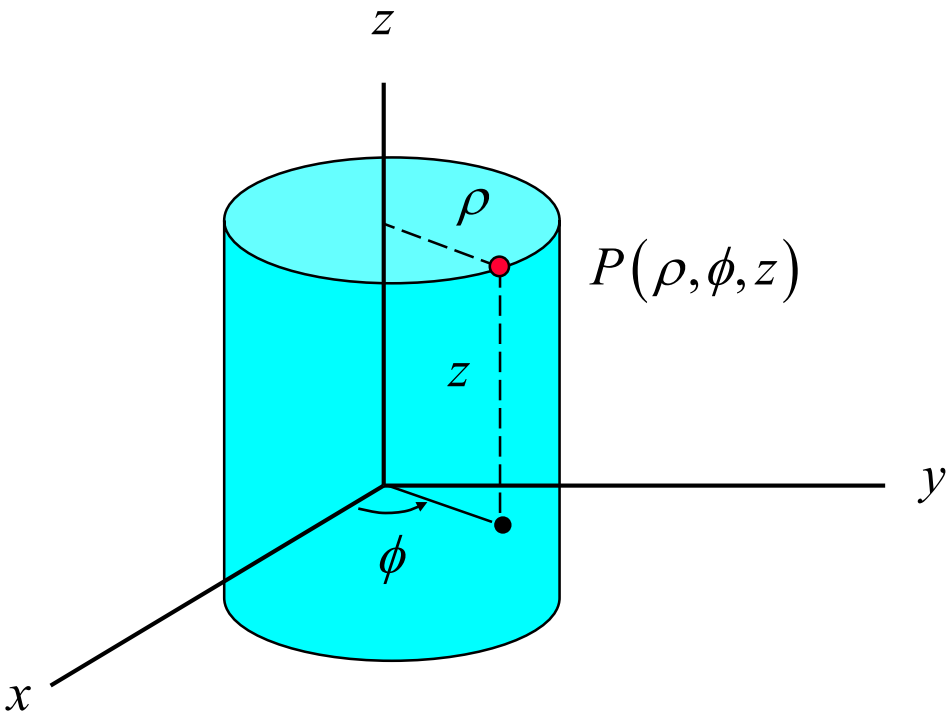
$$\underline{dr} = d\left(\underline{r}\right) = d\underline{r}$$

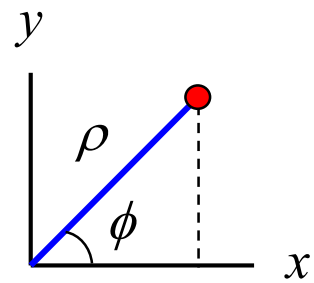
$$\underline{r} = \underline{\hat{x}}x + \underline{\hat{y}}y + \underline{\hat{z}}z$$

$$d\underline{r} = \underline{\hat{x}}\,dx + \underline{\hat{y}}\,dy + \underline{\hat{z}}\,dz$$

Note on notation: The symbol $\underline{dl}$ is often used instead of $\underline{dr}$.

Cylindrical Coordinates

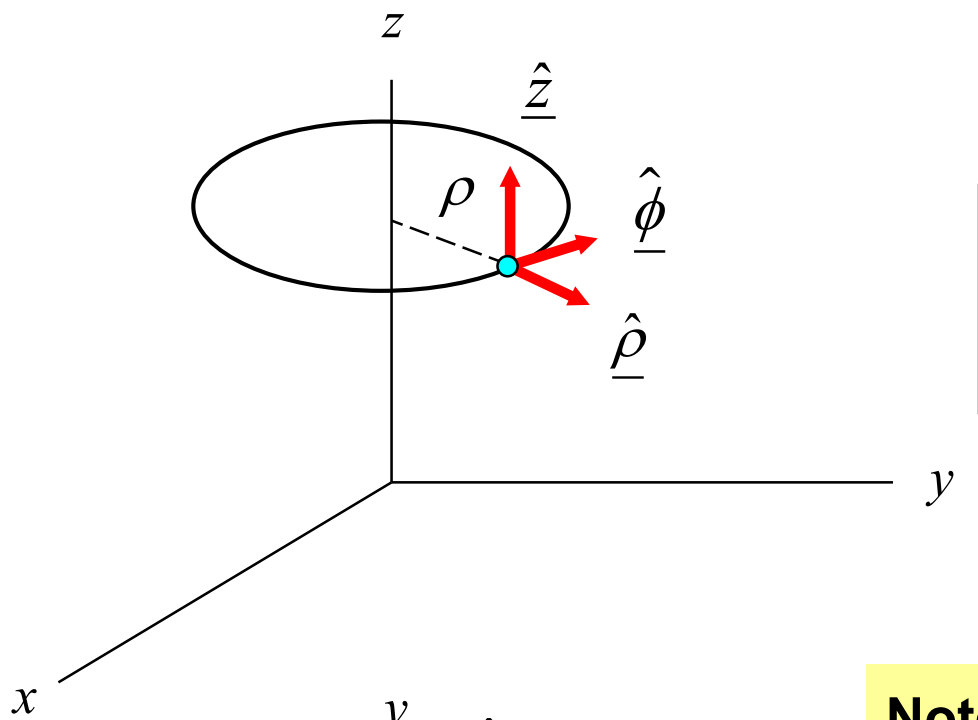




$$x = \rho\cos\phi$$
$$y = \rho\sin\phi$$
$$z = z$$

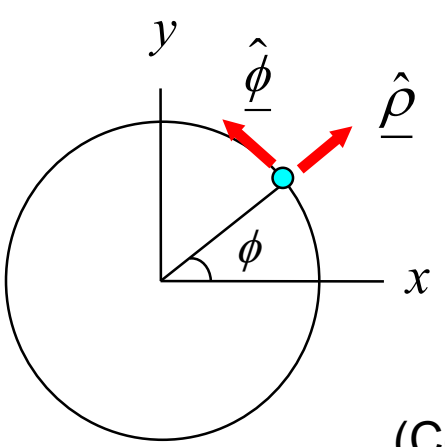
$$\rho = \sqrt{x^2 + y^2}$$
$$\phi = \tan^{-1}(y/x)$$
$$z = z$$

Cylindrical (cont.)



Unit Vectors

Note:
A unit vector direction is defined by <u>increasing</u> one coordinate variable while keeping the other two fixed.



Note: $\hat{\rho}$ and $\hat{\phi}$ depend on (x, y)

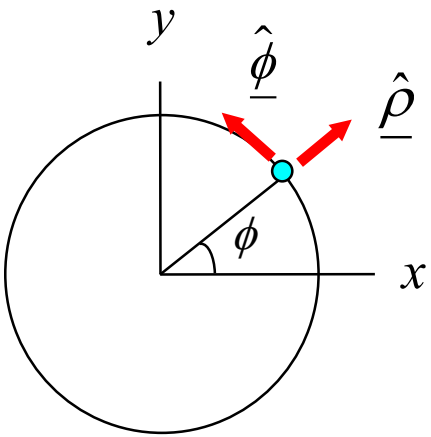
This is why we often prefer to express them in terms of

$\hat{x}$ and $\hat{y}$

(Constant vectors can be pulled outside of an integral.)

Cylindrical (cont.)

Expressions for unit vectors (illustrated for $\hat{\underline{\rho}}$)



Assume $\hat{\underline{\rho}} = A_1 \hat{\underline{x}} + A_2 \hat{\underline{y}}$

Solve for A_1:

$$\hat{\underline{\rho}} \cdot \hat{\underline{x}} = A_1 \, \hat{\underline{x}} \cdot \hat{\underline{x}} + A_2 \, \cancel{\hat{\underline{y}} \cdot \hat{\underline{x}}}$$

$$A_1 = \hat{\underline{\rho}} \cdot \hat{\underline{x}}$$

$$= \left|\hat{\underline{\rho}}\right| \left|\hat{\underline{x}}\right| \cos\phi$$

so $A_1 = \cos\phi$

Similarly,

$$A_2 = \hat{\underline{\rho}} \cdot \hat{\underline{y}}$$

$$= \cos\left(\frac{\pi}{2} - \phi\right)$$

$$= \sin\phi$$

Hence, we have $\hat{\underline{\rho}} = \hat{\underline{x}} \cos\phi + \hat{\underline{y}} \sin\phi$

Cylindrical (cont.)

Summary of Results

$$\underline{\hat{\rho}} = \underline{\hat{x}}\cos\phi + \underline{\hat{y}}\sin\phi$$

$$\underline{\hat{\phi}} = \underline{\hat{x}}\left(-\sin\phi\right) + \underline{\hat{y}}\cos\phi$$

$$\underline{\hat{z}} = \underline{\hat{z}}$$

$$\underline{\hat{x}} = \underline{\hat{\rho}}\cos\phi + \underline{\hat{\phi}}\left(-\sin\phi\right)$$

$$\underline{\hat{y}} = \underline{\hat{\rho}}\sin\phi + \underline{\hat{\phi}}\cos\phi$$

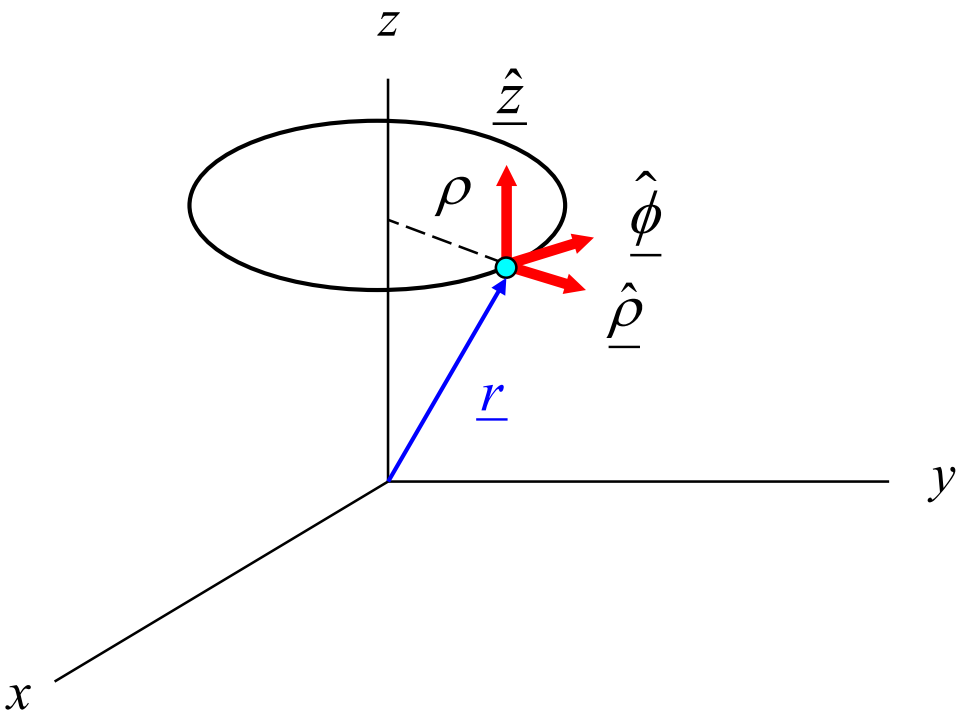
$$\underline{\hat{z}} = \underline{\hat{z}}$$

Note:

$$\cos\phi = \frac{x}{\sqrt{x^2 + y^2}}$$

$$\sin\phi = \frac{y}{\sqrt{x^2 + y^2}}$$

Cylindrical (cont.)



Example:

Express the $\underline{r}$ vector in cylindrical coordinates.

$$\underline{r} = \hat{\underline{x}}x + \hat{\underline{y}}y + \hat{\underline{z}}z$$

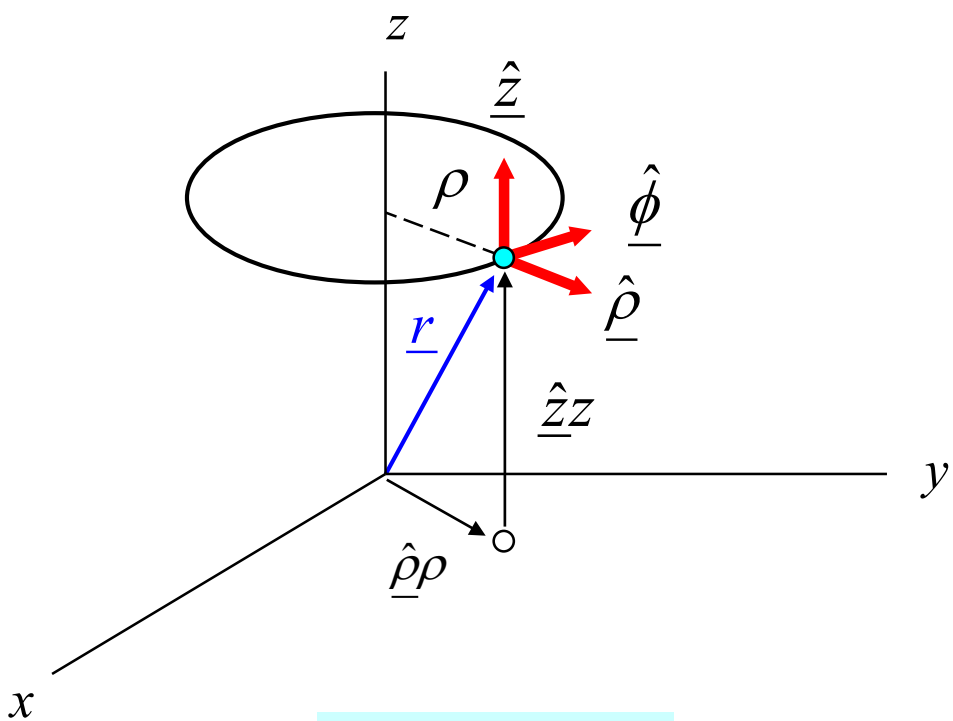
Substituting from the previous tables of unit vector transformations and coordinate transformations, we have:

$$\underline{r} = \left(\hat{\underline{\rho}}\cos\phi + \hat{\underline{\phi}}(-\sin\phi)\right)(\rho\cos\phi) + \left(\hat{\underline{\rho}}\sin\phi + \hat{\underline{\phi}}\cos\phi\right)(\rho\sin\phi) + \hat{\underline{z}}z$$

$$= \left(\hat{\underline{\rho}}\rho\right)\left(\cos^2\phi + \sin^2\phi\right) + \hat{\underline{z}}z$$

$$= \hat{\underline{\rho}}\rho + \hat{\underline{z}}z$$

Cylindrical (cont.)

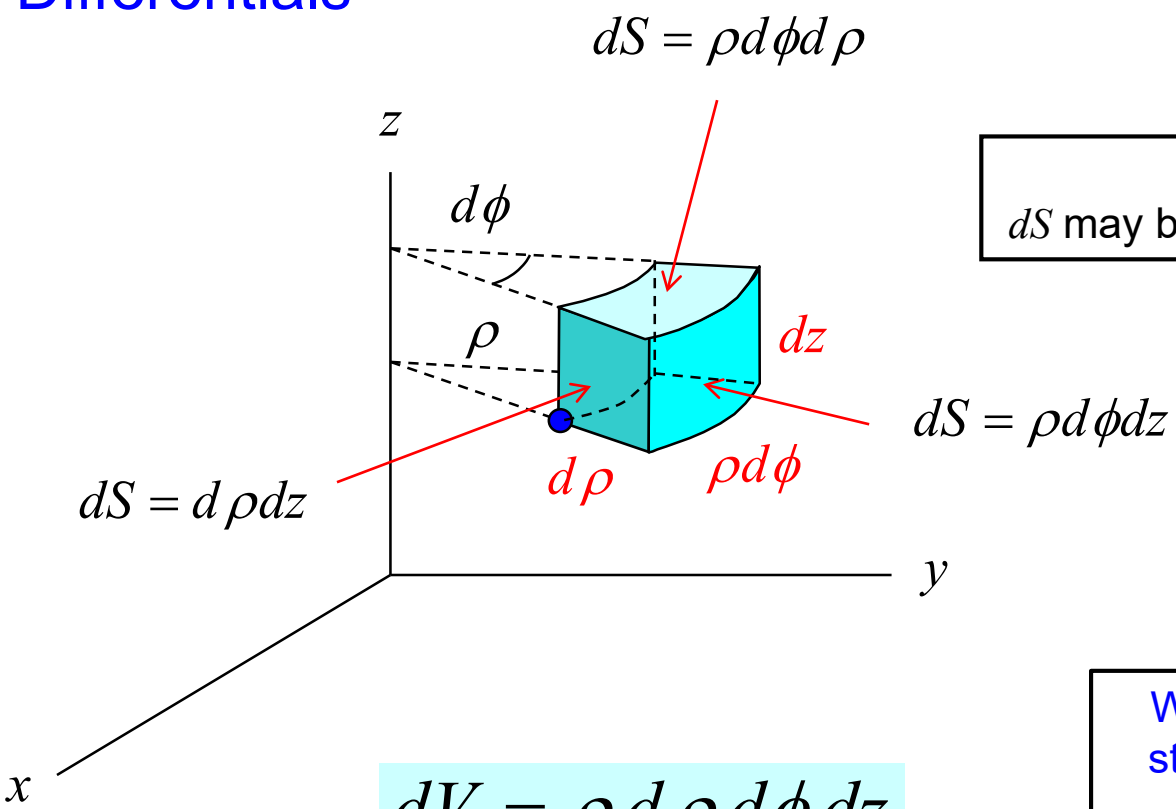


$$\underline{r} = \hat{\underline{\rho}}\rho + \hat{\underline{z}}z$$

Note: $\underline{r} \neq \hat{\underline{\rho}}\rho + \hat{\underline{\phi}}\phi + \hat{\underline{z}}z$

Cylindrical (cont.)

Differentials



Note:
dS may be in <u>three</u> different forms.

$$dV = \rho\, d\rho\, d\phi\, dz$$

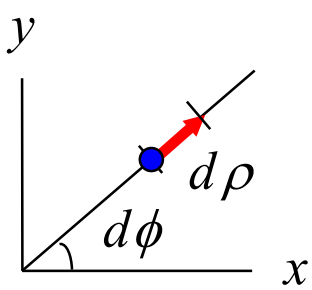
We <u>increase</u> ρ, ϕ, or z starting from an initial point (blue dot).

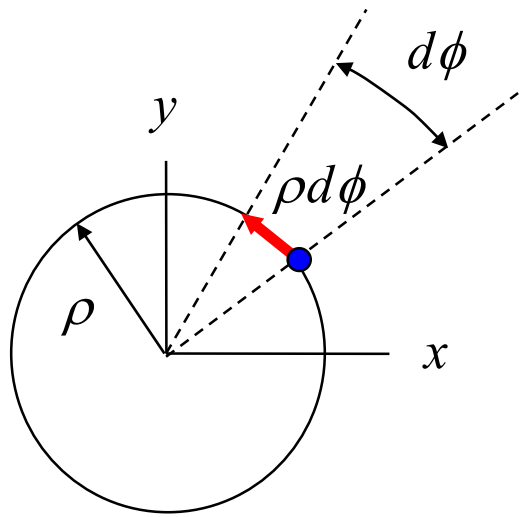
Note: The angle ϕ must be in radians here.

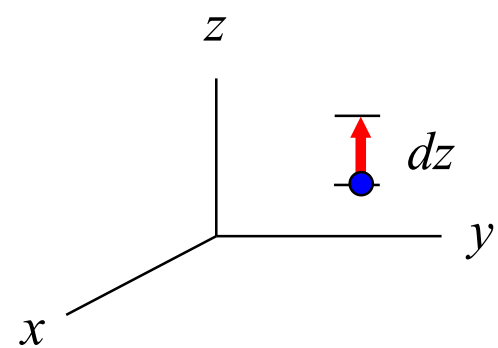
Cylindrical (cont.)

Path Integrals

First, consider differential changes along any of the three coordinate directions.







$$d\underline{r} = \underline{\hat{\rho}}\, d\rho \qquad d\underline{r} = \underline{\hat{\phi}}\left(\rho\, d\phi\right) \qquad d\underline{r} = \underline{\hat{z}}\, dz$$

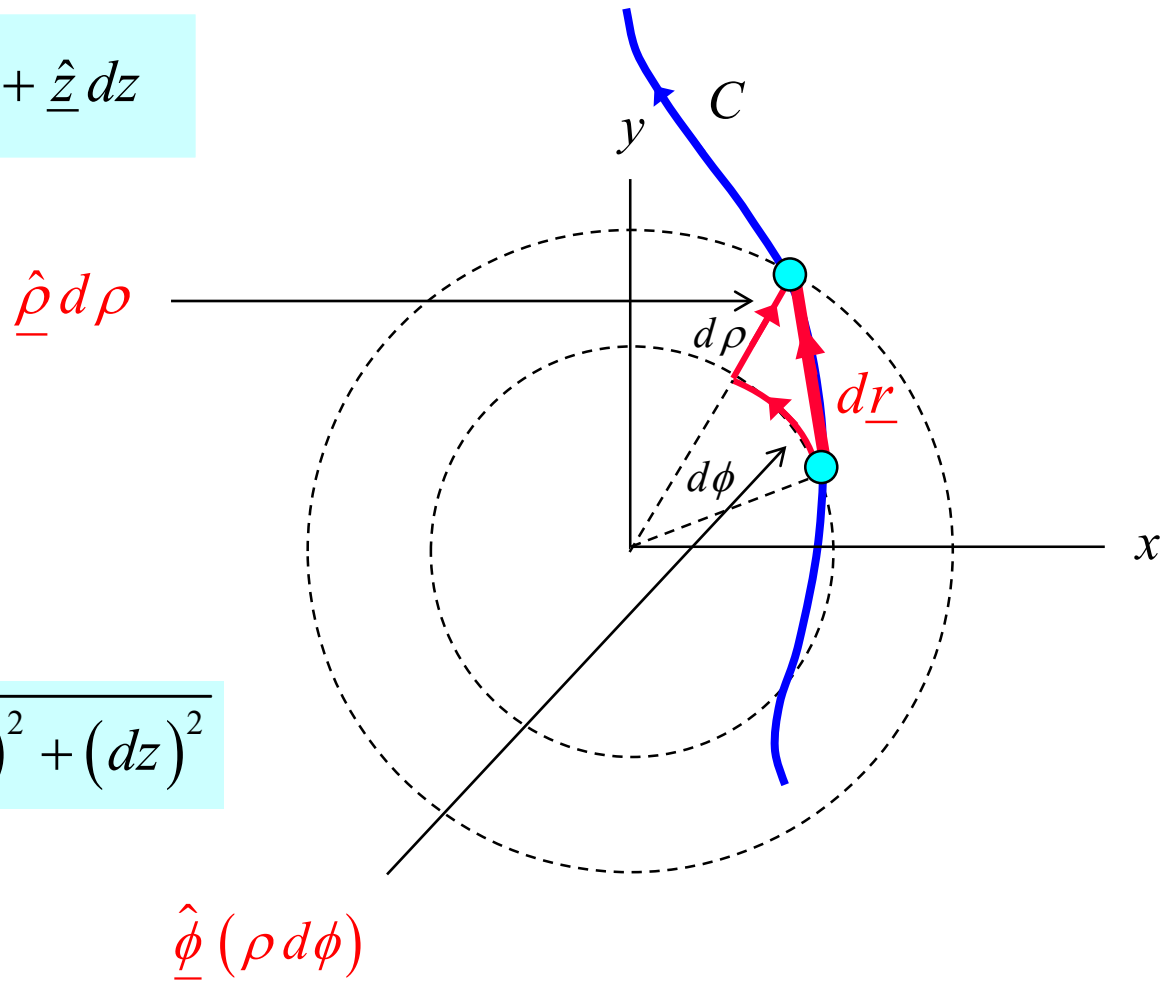
Note: The angle ϕ must be in radians here.

Cylindrical (cont.)

Note: A change in z is not shown, but is possible.

In general:

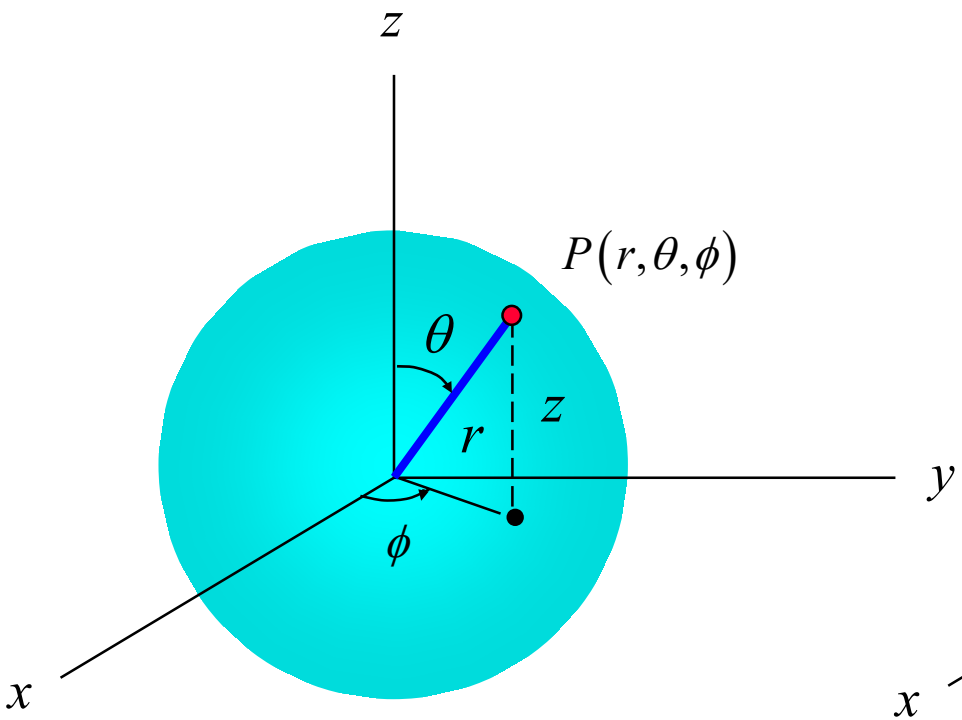
$$d\underline{r} = \underline{\hat{\rho}}\, d\rho + \underline{\hat{\phi}}\left(\rho\, d\phi\right) + \underline{\hat{z}}\, dz$$

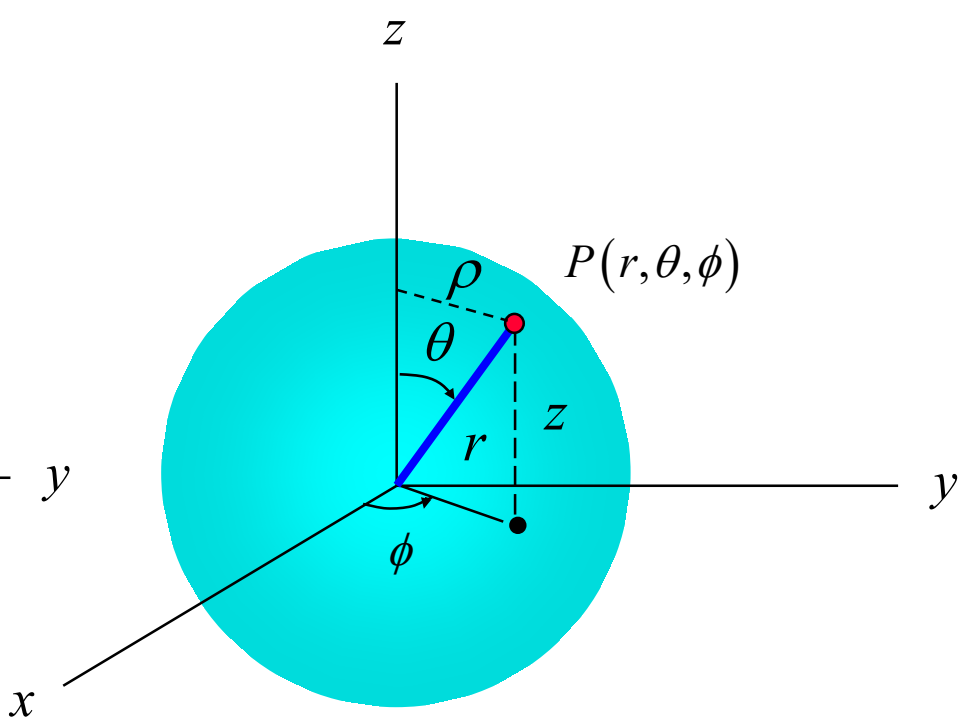


If we ever need to find the <u>length</u> along a contour:

$$d\ell = \left|d\underline{r}\right| = \sqrt{\left(d\rho\right)^2 + \left(\rho\, d\phi\right)^2 + \left(dz\right)^2}$$

Spherical Coordinates

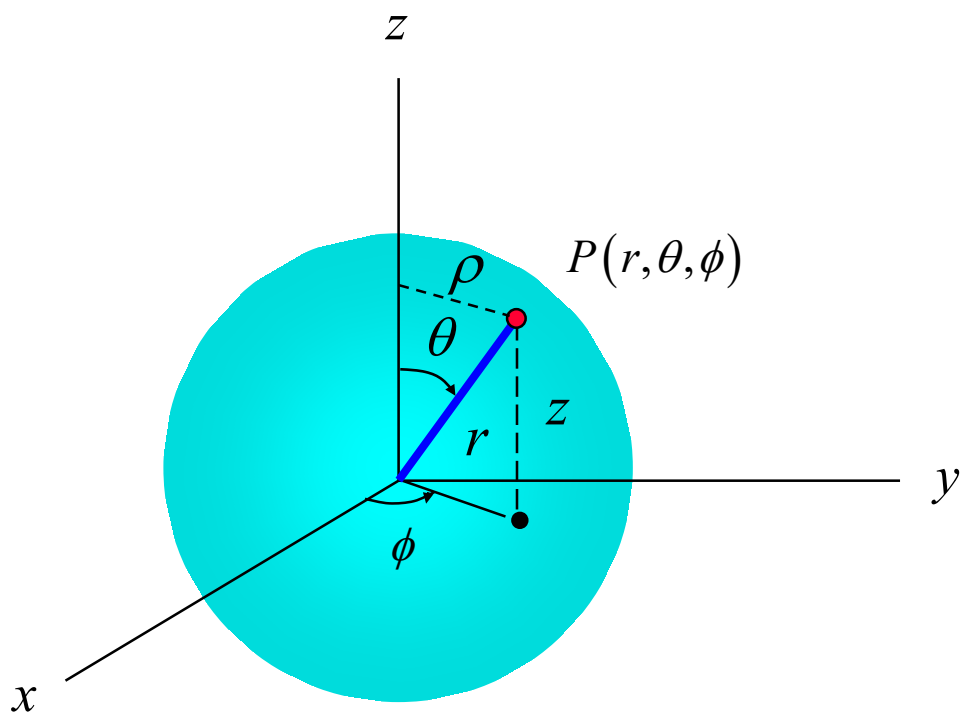




Note: $0 \le \theta \le \pi$

Note: $\rho = r \sin\theta$

Spherical (cont.)



$$x = r\sin\theta\,\cos\phi$$
$$y = r\sin\theta\,\sin\phi$$
$$z = r\cos\theta$$

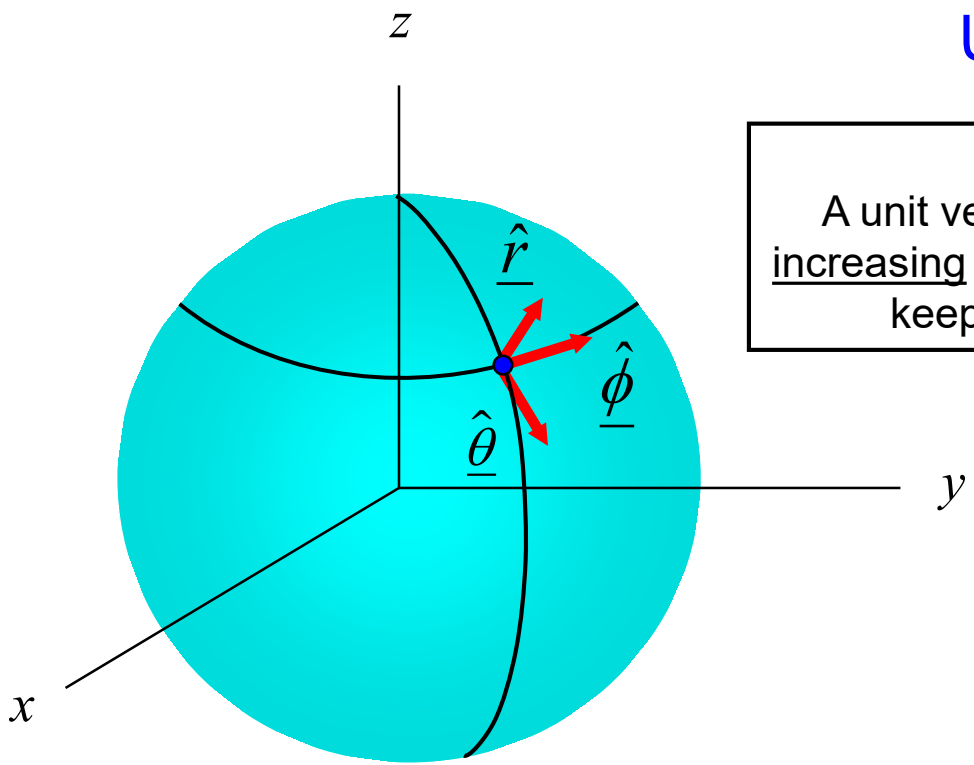
$$r = \sqrt{x^2 + y^2 + z^2}$$
$$\theta = \cos^{-1}\left(z/r\right)$$
$$\phi = \tan^{-1}\left(y/x\right)$$

Note: $\rho = r\sin\theta$

Recall: $x = \rho\cos\phi$

$y = \rho\sin\phi$

Spherical (cont.)



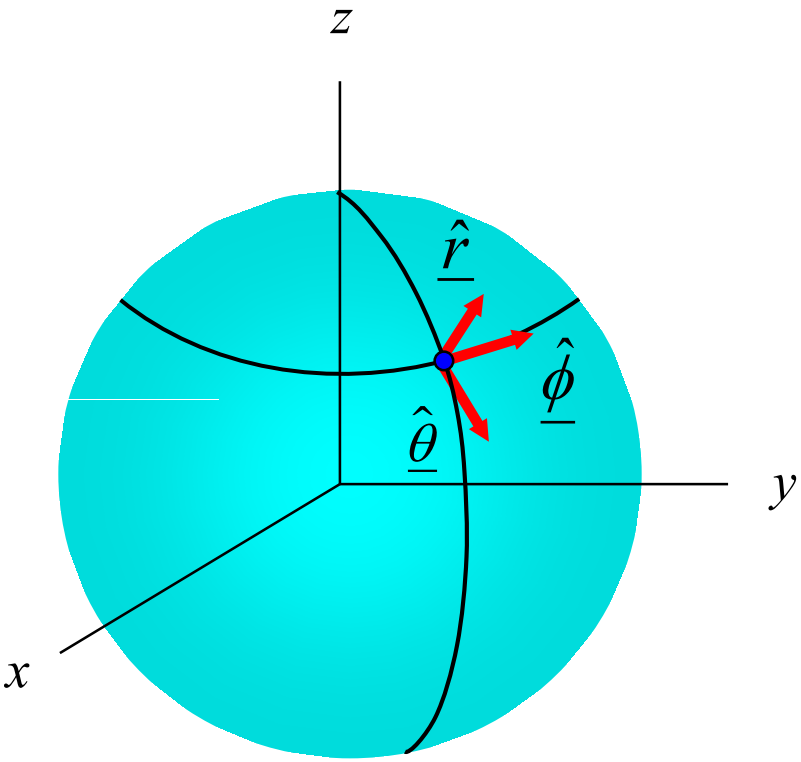
Unit Vectors

Note:
A unit vector direction is defined by <u>increasing</u> one coordinate variable while keeping the other two fixed.

Note: $\hat{\underline{r}}, \hat{\underline{\theta}}, \hat{\underline{\phi}}$ depend on (x, y, z)

Spherical (cont.)

Transformation of Unit Vectors



$$\hat{\underline{r}} = \hat{\underline{x}}\sin\theta\cos\phi + \hat{\underline{y}}\sin\theta\sin\phi + \hat{\underline{z}}\cos\theta$$

$$\hat{\underline{\theta}} = \hat{\underline{x}}\cos\theta\cos\phi + \hat{\underline{y}}\cos\theta\sin\phi + \hat{\underline{z}}(-\sin\theta)$$

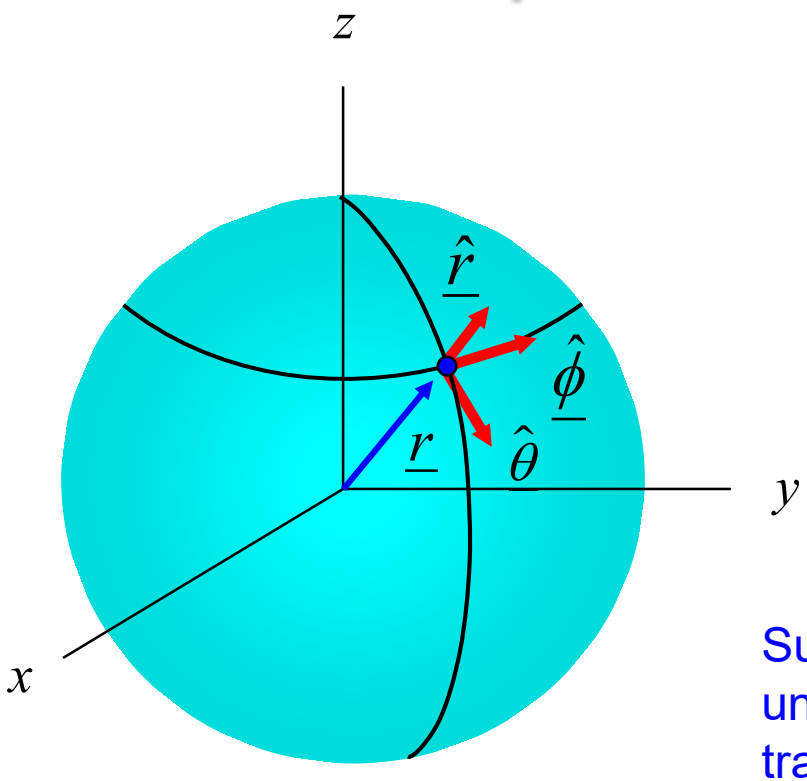
$$\hat{\underline{\phi}} = \hat{\underline{x}}(-\sin\phi) + \hat{\underline{y}}\cos\phi$$

$$\hat{\underline{x}} = \hat{\underline{r}}\sin\theta\cos\phi + \hat{\underline{\theta}}\cos\theta\cos\phi + \hat{\underline{\phi}}(-\sin\phi)$$

$$\hat{\underline{y}} = \hat{\underline{r}}\sin\theta\sin\phi + \hat{\underline{\theta}}\cos\theta\sin\phi + \hat{\underline{\phi}}\cos\phi$$

$$\hat{\underline{z}} = \hat{\underline{r}}\cos\theta + \hat{\underline{\theta}}(-\sin\theta)$$

Spherical (cont.)



Example:

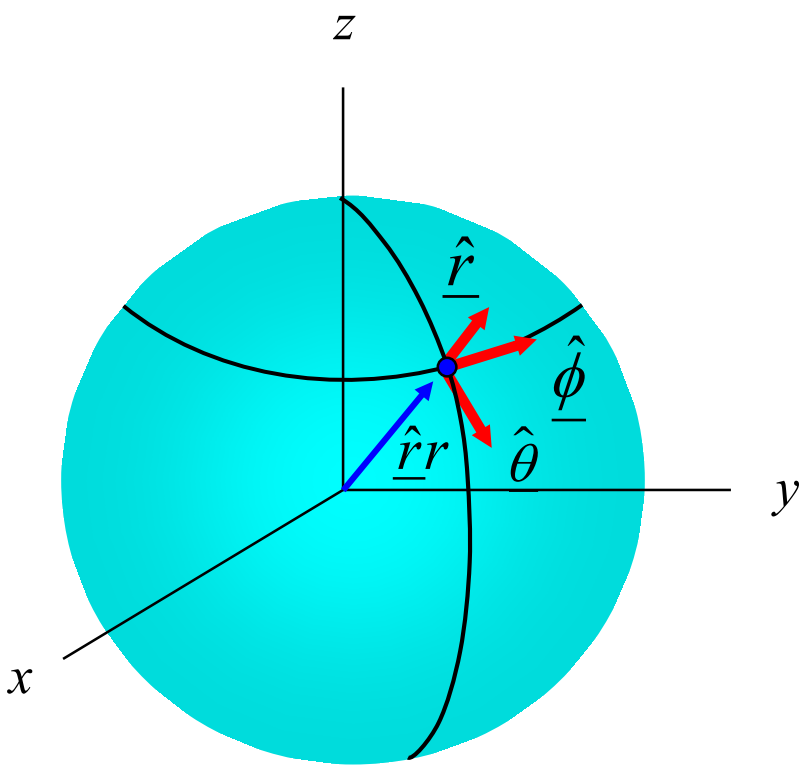
Express the $\underline{r}$ vector in spherical coordinates.

$$\underline{r} = \underline{\hat{x}}x + \underline{\hat{y}}y + \underline{\hat{z}}z$$

Substituting from the previous tables of unit vector transformations and coordinate transformations, we have:

$$\begin{aligned}\underline{r} = &\left(\underline{\hat{r}}\sin\theta\cos\phi + \underline{\hat{\theta}}\cos\theta\cos\phi + \underline{\hat{\phi}}\left(-\sin\phi\right)\right)\left(r\sin\theta\cos\phi\right)\\ &+\left(\underline{\hat{r}}\sin\theta\sin\phi + \underline{\hat{\theta}}\cos\theta\sin\phi + \underline{\hat{\phi}}\cos\phi\right)\left(r\sin\theta\sin\phi\right)\\ &+\left(\underline{\hat{r}}\cos\theta + \underline{\hat{\theta}}\left(-\sin\theta\right)\right)r\cos\theta\end{aligned}$$

Spherical (cont.)



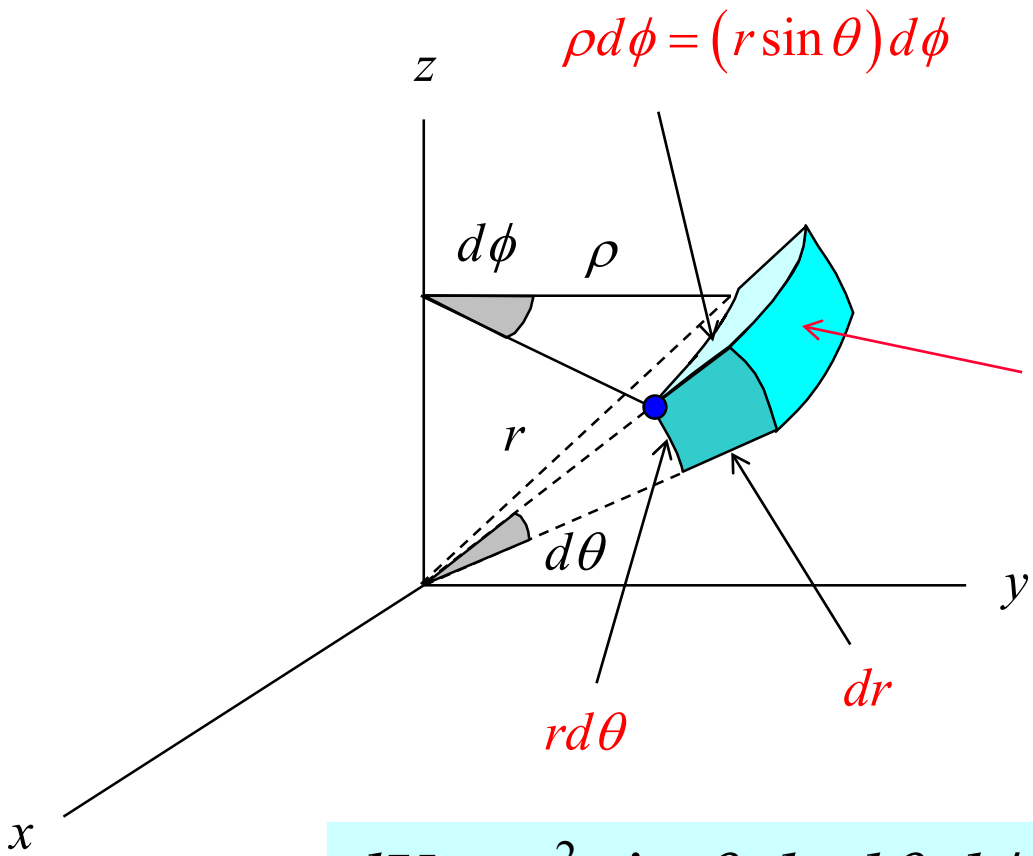
After simplifying:

$$\underline{r} = \underline{\hat{r}}r$$

Note : $\underline{r} \neq \underline{\hat{r}}r + \underline{\hat{\theta}}\theta + \underline{\hat{\phi}}\phi$

Spherical (cont.)

Differentials



$$dS = \left(r^2 \sin\theta\right) d\theta d\phi$$

We <u>increase</u> r, θ, or ϕ starting from an initial point (blue dot).

$$dV = r^2 \sin\theta \, dr \, d\theta \, d\phi$$

Note:
dS may be in <u>three</u> different forms (only one is shown). The other two are:

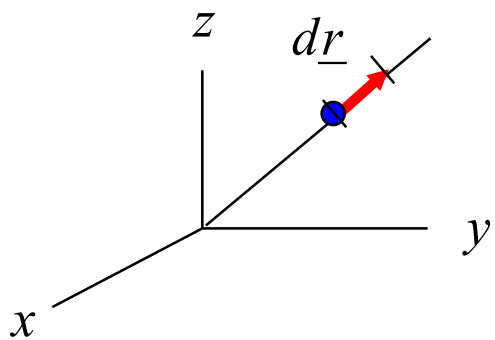
$$dS = r\sin\theta \, dr d\phi$$

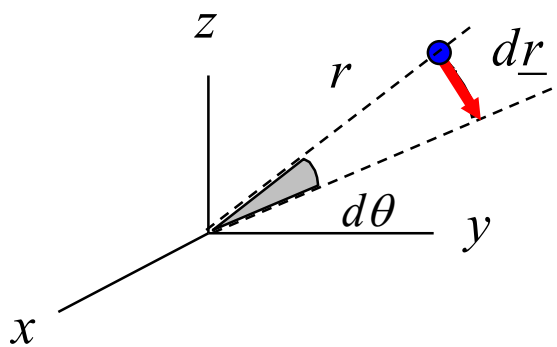
$$dS = r \, dr d\theta$$

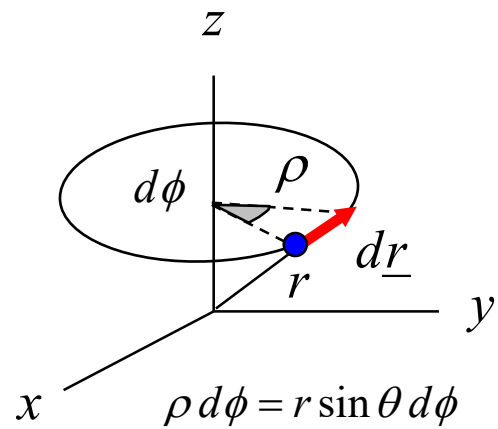
Note: The angles θ and ϕ must be in radians here.

Spherical (cont.)

Path Integrals







$$d\underline{r} = \hat{\underline{r}}\, dr$$

$$d\underline{r} = \hat{\underline{\theta}}\left(r\, d\theta\right)$$

$$d\underline{r} = \hat{\underline{\phi}}\left(r \sin\theta\, d\phi\right)$$

$$d\underline{r} = \hat{\underline{r}}\, dr + \hat{\underline{\theta}}\, r d\theta + \hat{\underline{\phi}}\, r \sin\theta\, d\phi$$

Note: The angles θ and ϕ must be in radians here.

Summary of Differentials

$$d\underline{r} = \underline{\hat{x}}\,dx + \underline{\hat{y}}\,dy + \underline{\hat{z}}\,dz$$

$$d\underline{r} = \underline{\hat{\rho}}\,d\rho + \underline{\hat{\phi}}\left(\rho\,d\phi\right) + \underline{\hat{z}}\,dz$$

$$d\underline{r} = \underline{\hat{r}}\,dr + \underline{\hat{\theta}}\,r\,d\theta + \underline{\hat{\phi}}\,r\sin\theta\,d\phi$$

$$dV = dx\,dy\,dz$$

$$dV = \rho\,d\rho\,d\phi\,dz$$

$$dV = r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$dS = \rho d\rho d\phi$$

(on horizontal surface)

$$dS = \rho d\phi dz$$

(on surface of cylinder)

$$dS = \left(r^2\sin\theta\right)d\theta d\phi$$

(on surface of sphere)

Note on $d\underline{r}$ Vector

Note that the formulas for the $d\underline{r}$ vector never change, no matter which direction we go along a path (<u>we never add a minus sign</u>!).

Example: Integrating along a horizontal radial path in cylindrical coordinates.

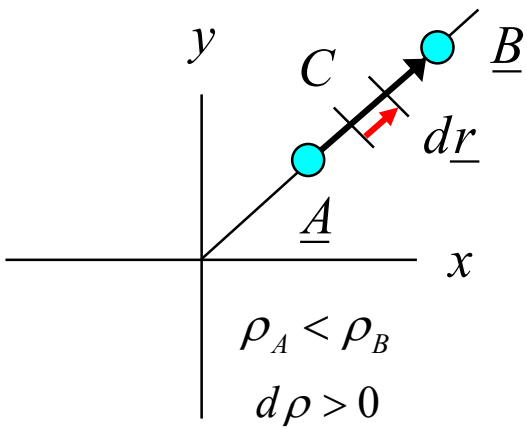
$$d\underline{r} = \underline{\hat{\rho}}\, d\rho + \underline{\hat{\phi}}\left(\rho\, \cancel{d\phi}\right) + \underline{\hat{z}}\, \cancel{dz}$$

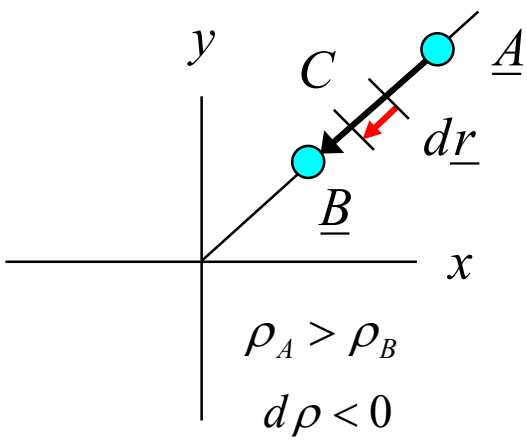
$$V_{AB} = \int_{\underline{A}}^{\underline{B}} \underline{E} \cdot d\underline{r}$$

$$d\underline{r} = \underline{\hat{\rho}}\, d\rho$$

$$\underline{E} = \underline{\hat{\rho}} E_\rho + \underline{\hat{\phi}} E_\phi + \underline{\hat{z}} E_z$$

$$\Rightarrow V_{AB} = \int_{\rho_A}^{\rho_B} E_\rho\, d\rho$$





Note:
This form does not change, <u>regardless</u> of which limit is larger.

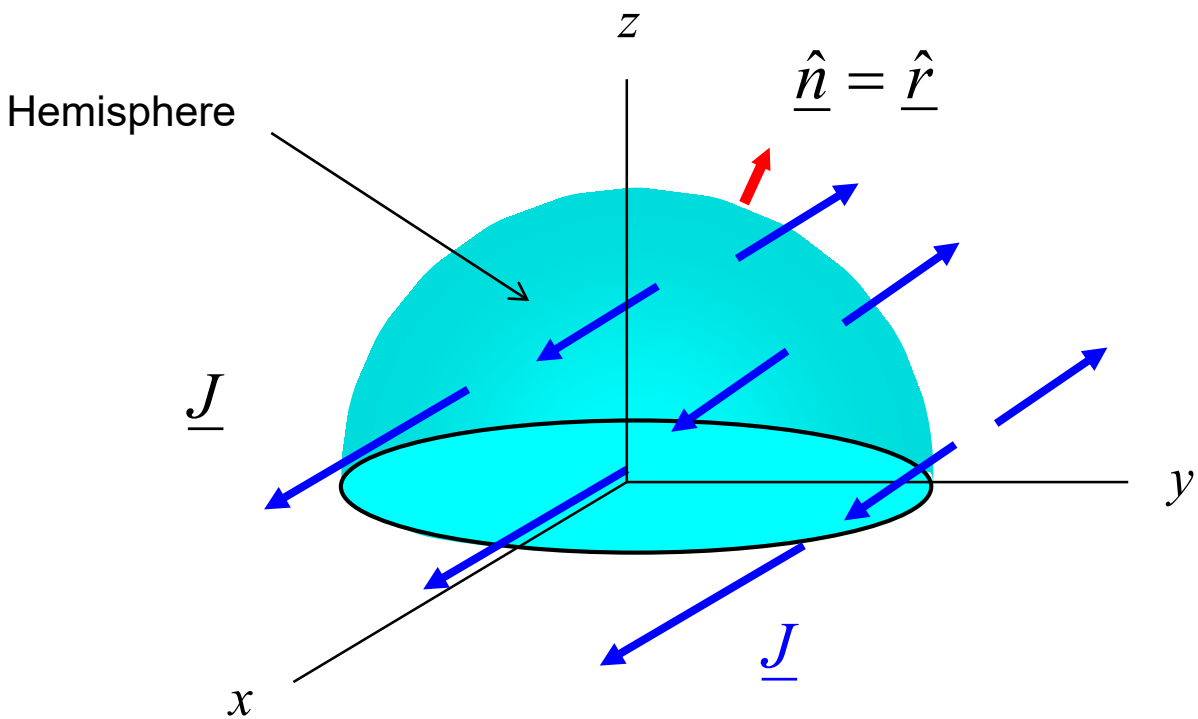
The limits take care of the direction of $d\underline{r}$.

Example

Given:

$$\underline{J} = \underline{\hat{x}}(x) \quad [\mathrm{A/m^2}]$$

Find the current I crossing a hemisphere ($z > 0$) of radius a, in the <u>outward</u> direction.



$$I = \int_S \underline{J} \cdot \underline{\hat{n}}\, dS$$

Example (cont.)

$$I = \int_S \underline{J} \cdot \underline{\hat{n}}\, dS$$

$$= \int_S \underline{J} \cdot \underline{\hat{r}}\, dS$$

$$= \int_S \left(\underline{\hat{x}} J_x\right) \cdot \underline{\hat{r}}\, dS$$

$$= \int_S J_x \left(\sin\theta \cos\phi\right) dS$$

$$= \int_S (x)\left(\sin\theta \cos\phi\right) dS$$

$$= \int_S \left(a \sin\theta \cos\phi\right)\left(\sin\theta \cos\phi\right) dS$$

$$= a \int_S \left(\sin^2\theta \cos^2\phi\right) dS$$

$$\underline{\hat{r}} = \underline{\hat{x}} \sin\theta \cos\phi + \underline{\hat{y}} \sin\theta \sin\phi + \underline{\hat{z}} \cos\theta$$

$$\underline{\hat{\theta}} = \underline{\hat{x}} \cos\theta \cos\phi + \underline{\hat{y}} \cos\theta \sin\phi + \underline{\hat{z}}\left(-\sin\theta\right)$$

$$\underline{\hat{\phi}} = \underline{\hat{x}}\left(-\sin\phi\right) + \underline{\hat{y}} \cos\phi$$

$$\underline{\hat{x}} = \underline{\hat{r}} \sin\theta \cos\phi + \underline{\hat{\theta}} \cos\theta \cos\phi + \underline{\hat{\phi}}\left(-\sin\phi\right)$$

$$\underline{\hat{y}} = \underline{\hat{r}} \sin\theta \sin\phi + \underline{\hat{\theta}} \cos\theta \sin\phi + \underline{\hat{\phi}} \cos\phi$$

$$\underline{\hat{z}} = \underline{\hat{r}} \cos\theta + \underline{\hat{\theta}}\left(-\sin\theta\right)$$

$$x = r \sin\theta \cos\phi$$

$$y = r \sin\theta \sin\phi$$

$$z = r \cos\theta$$

Example (cont.)

$$I = a\int_S \left(\sin^2\theta\cos^2\phi\right)dS$$

$$dS = r^2\sin\theta\, d\theta d\phi$$

$$= a\int_0^{2\pi}\int_0^{\pi/2}\left(\sin^2\theta\cos^2\phi\right)\underline{a^2\sin\theta\, d\theta d\phi}$$

$$= a^3\int_0^{2\pi}\int_0^{\pi/2}\left(\sin^3\theta\cos^2\phi\right)d\theta d\phi$$

$$a^3\int_0^{2\pi}\cos^2\phi\, d\phi\int_0^{\pi/2}\sin^3\theta\, d\theta$$

Note :

$$\int_0^{2\pi}\cos^2\phi\, d\phi = (2\pi)\frac{1}{2} = \pi$$

$$= a^3(\pi)\int_0^{\pi/2}\sin^3\theta\, d\theta$$

Note :

$$\int_0^{\pi/2}\sin^3\theta\, d\theta = \frac{2}{3}$$

$$= a^3(\pi)\left(\frac{2}{3}\right)$$

$$I = \left(\frac{2\pi}{3}\right)a^3 \quad [\mathrm{A}]$$

Appendix

Here we work out some more examples.

Example

Given: $\left.\begin{array}{l} P_1\left(4, 60^\circ, 1\right) \\ P_2\left(3, 180^\circ, -1\right) \end{array}\right\}$ Cylindrical coordinates (ρ, ϕ, z)

(with distances in meters)

Find d = distance between points

$$d = \sqrt{\left(x_1 - x_2\right)^2 + \left(y_1 - y_2\right)^2 + \left(z_1 - z_2\right)^2}$$

This formula only works in rectangular coordinates!

$$x = \rho \cos\phi$$
$$y = \rho \sin\phi$$
$$z = z$$

$$x_1 = 4\cos\left(60^\circ\right) = 2$$
$$y_1 = 4\sin\left(60^\circ\right) = 3.4641$$
$$z_1 = 1$$

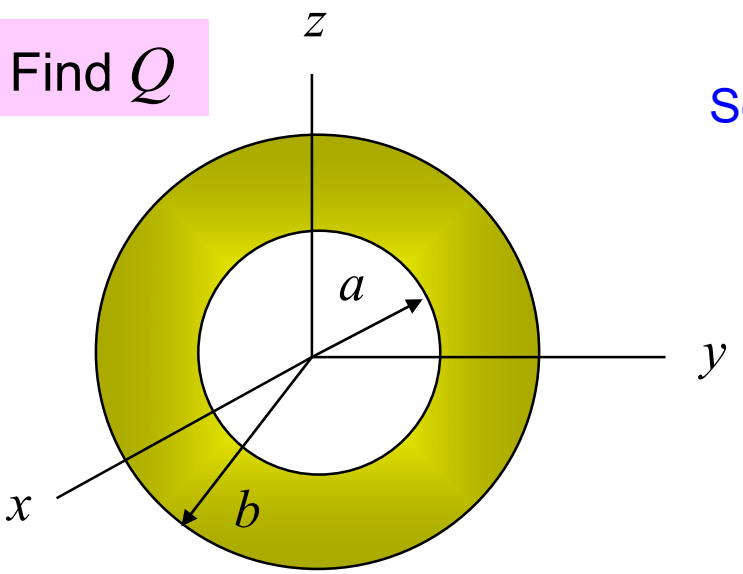
$$x_2 = 3\cos\left(180^\circ\right) = -3$$
$$y_2 = 3\sin\left(180^\circ\right) = 0$$
$$z_2 = -1$$

$$d = 6.403 \text{ [m]}$$

Example

Given: $\rho_v = -3\times10^{-8}\left(\cos^2\phi / r^4\right)\ \left[\mathrm{C/m^3}\right],\quad 2 < r < 5\ [\mathrm{m}]$

Find Q



$a = 2[\mathrm{m}],\ \ b = 5[\mathrm{m}]$

"A sphere with a hole in it"

Solution:

Note:
The integrand is separable <u>and</u> the limits are fixed.

$$Q = \int_V \rho_v\, dV$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_a^{b} \rho_v(r,\phi)\, r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= -3\times10^{-8}\int_0^{2\pi}\int_0^{\pi}\int_a^{b}\left(\cos^2\phi\frac{1}{r^2}\sin\theta\right) dr\, d\theta\, d\phi$$

$$= -3\times10^{-8}\int_0^{2\pi}\cos^2\phi\, d\phi\ \int_a^{b}\frac{1}{r^2}dr\ \int_0^{\pi}\sin\theta\, d\theta$$

Example (cont.)

$$\int_0^{2\pi} \cos^2\phi\, d\phi = \int_0^{2\pi} \frac{1+\cos 2\phi}{2}\, d\phi$$

$$= \left[\frac{1}{2}\phi + \frac{\sin 2\phi}{4}\right]_0^{2\pi} = \pi$$

Note: The average value of $\cos^2\phi$ is $1/2$.

$$\int_0^{2\pi} \cos^2\phi\, d\phi = 2\pi\left(\frac{1}{2}\right) = \pi$$

$$\int_a^b \frac{1}{r^2}\, dr = \left[-\frac{1}{r}\right]_a^b$$

$$= \frac{1}{a} - \frac{1}{b} = 3/10$$

$$\int_0^{\pi} \sin\theta\, d\theta = \left[-\cos\theta\right]_0^{\pi} = 2$$

$$Q = -5.655\times 10^{-8}\ \text{[C]}$$

Example

Derive $\underline{\hat{z}} = \underline{\hat{r}}\cos\theta + \underline{\hat{\theta}}\left(-\sin\theta\right)$

Let $\underline{\hat{z}} = \underline{\hat{r}}\,A_1 + \underline{\hat{\theta}}\,A_2 + \underline{\hat{\phi}}\,A_3$

Dot multiply both sides with $\underline{\hat{r}},\ \underline{\hat{\theta}},\ \underline{\hat{\phi}}$

Then

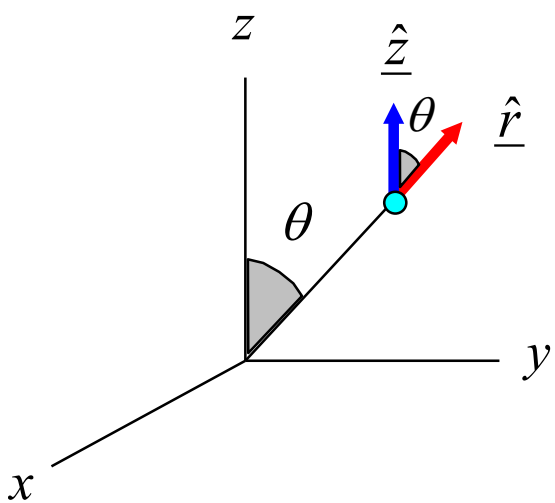
$$A_1 = \underline{\hat{z}} \cdot \underline{\hat{r}}$$

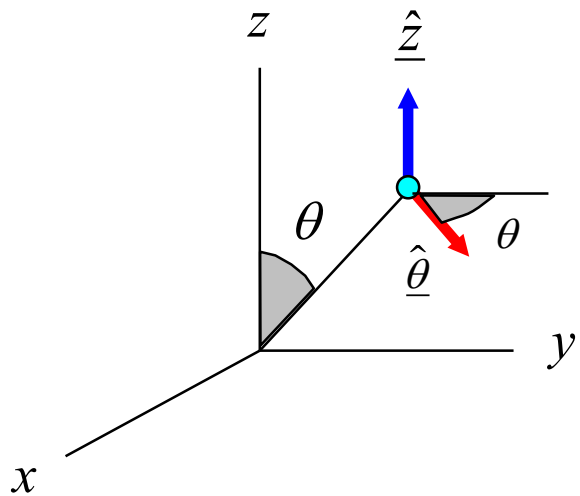
$$A_2 = \underline{\hat{z}} \cdot \underline{\hat{\theta}}$$

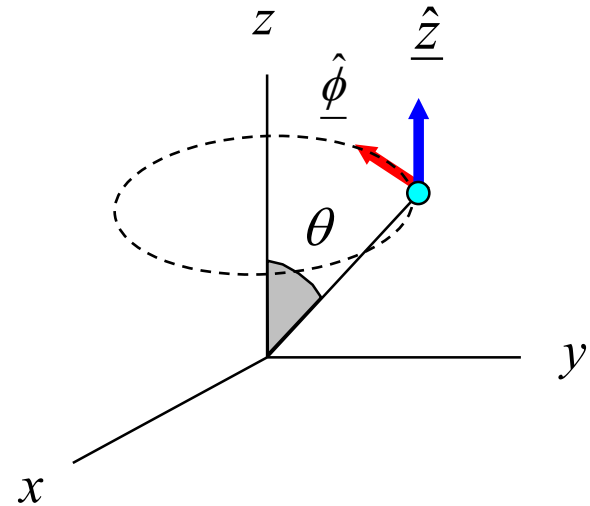
$$A_3 = \underline{\hat{z}} \cdot \underline{\hat{\phi}}$$

Example (cont.)

$$\hat{\underline{z}} = \hat{\underline{r}}\, A_1 + \hat{\underline{\theta}}\, A_2 + \hat{\underline{\phi}}\, A_3$$







$$A_1 = \hat{\underline{z}} \cdot \hat{\underline{r}}$$
$$\hat{\underline{z}} \cdot \hat{\underline{r}} = \cos\theta$$
$$A_1 = \cos\theta$$

$$A_2 = \hat{\underline{z}} \cdot \hat{\underline{\theta}}$$
$$\hat{\underline{z}} \cdot \hat{\underline{\theta}} = \cos\left(\frac{\pi}{2} + \theta\right)$$
$$A_2 = -\sin\theta$$

$$A_3 = \hat{\underline{z}} \cdot \hat{\underline{\phi}}$$
$$\hat{\underline{z}} \cdot \hat{\underline{\phi}} = 0$$
$$A_3 = 0$$

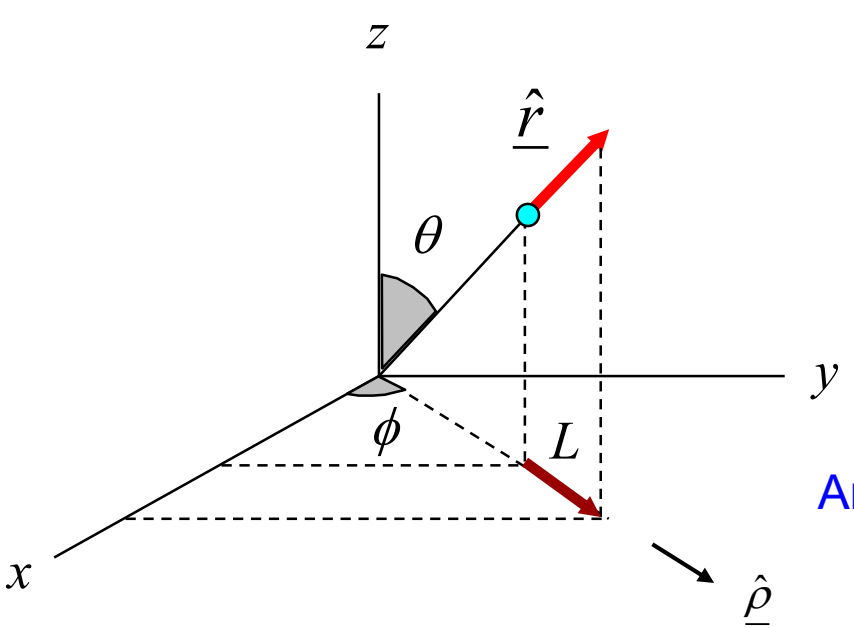
Result: $\hat{\underline{z}} = \hat{\underline{r}} \cos\theta + \hat{\underline{\theta}}\left(-\sin\theta\right)$

Example

Derive $\underline{\hat{r}} = \underline{\hat{x}} \sin\theta \cos\phi + \underline{\hat{y}} \sin\theta \sin\phi + \underline{\hat{z}} \cos\theta$

Let $\underline{\hat{r}} = \underline{\hat{x}}\, A_1 + \underline{\hat{y}}\, A_2 + \underline{\hat{z}}\, A_3$

Dot multiply both sides with $\underline{\hat{x}},\ \underline{\hat{y}},\ \underline{\hat{z}}$



$A_1 = \underline{\hat{r}} \cdot \underline{\hat{x}}$ (x component of $\underline{\hat{r}}$)

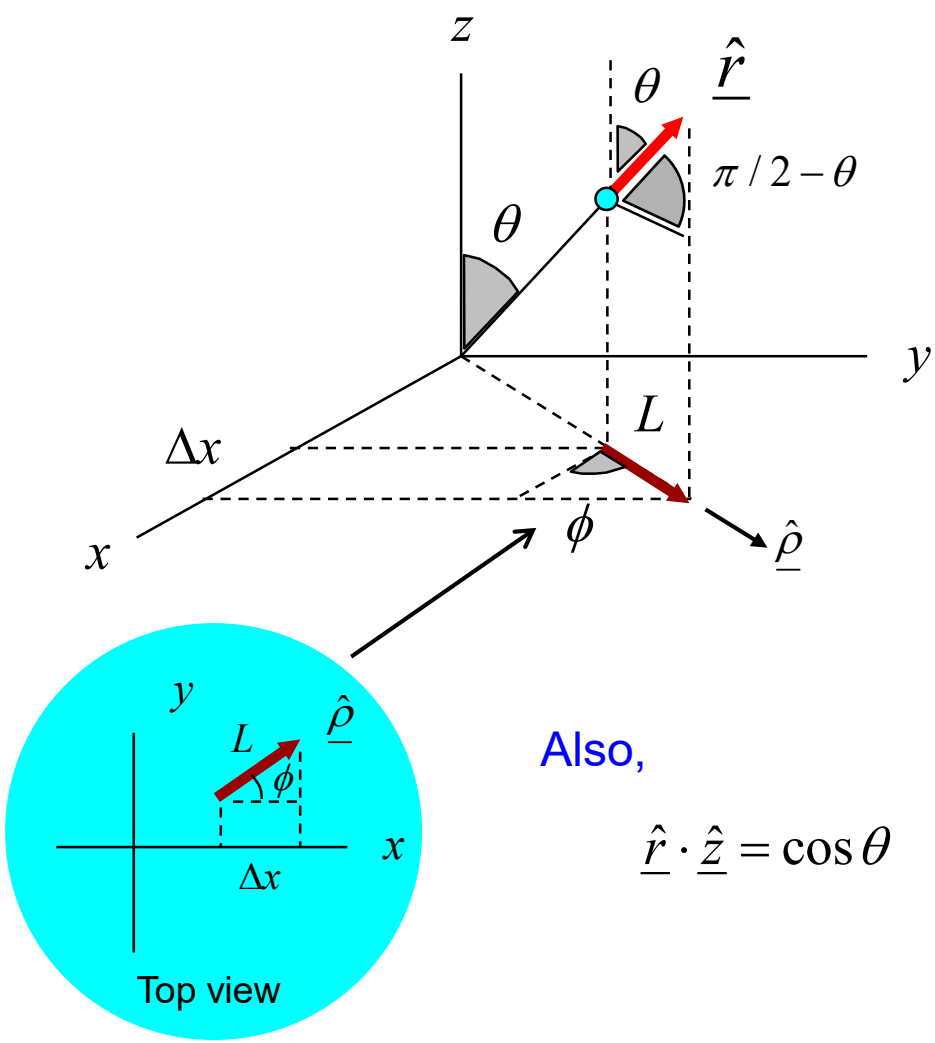
$A_2 = \underline{\hat{r}} \cdot \underline{\hat{y}}$ (y component of $\underline{\hat{r}}$)

$A_3 = \underline{\hat{r}} \cdot \underline{\hat{z}}$ (z component of $\underline{\hat{r}}$)

An illustration of finding the x component of $\underline{\hat{r}}$

(We use a two-step process.)

Example (cont.)



$$A_1 = \underline{\hat{r}} \cdot \underline{\hat{x}} = L\cos\phi$$

$$L = \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$

Hence

$$\underline{\hat{r}} \cdot \underline{\hat{x}} = L\cos\phi = \sin\theta\cos\phi$$

Similarly,

$$\underline{\hat{r}} \cdot \underline{\hat{y}} = L\sin\phi = \sin\theta\sin\phi$$

Also,

$$\underline{\hat{r}} \cdot \underline{\hat{z}} = \cos\theta$$

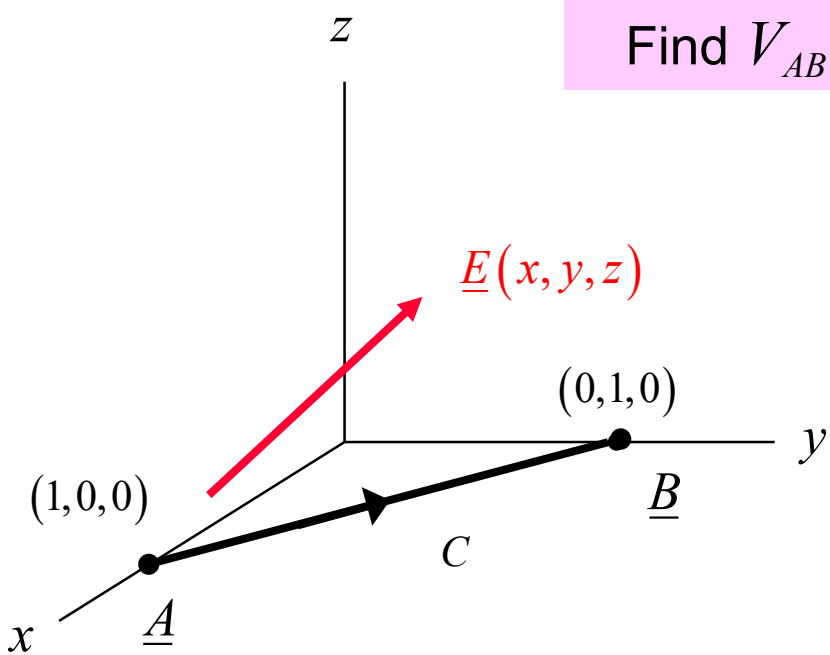
Result: $\underline{\hat{r}} = \underline{\hat{x}}\sin\theta\cos\phi + \underline{\hat{y}}\sin\theta\sin\phi + \underline{\hat{z}}\cos\theta$

Example (Part 1)

$$\underline{E} = \underline{\hat{x}}\left(3xy\right) + \underline{\hat{y}}\left(xy^2\right) + \underline{\hat{z}}\left(2z+1\right)$$

(This is <u>not</u> an electrostatic field.)

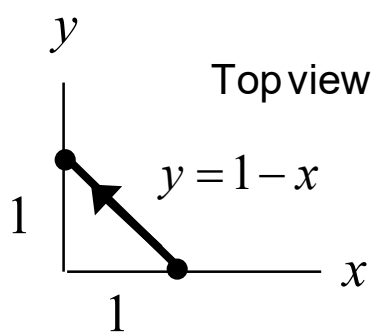
Find V_{AB} using path C shown below.



$$V_{AB} = \int_{\underline{A}}^{\underline{B}} \underline{E} \cdot \underline{dr} = \int_{\underline{A}}^{\underline{B}} \left(E_x\, dx + E_y\, dy + \cancel{E_z\, dz}\right)$$

$$= \int_{\underline{A}}^{\underline{B}} \left(\left(3xy\right)dx + \left(xy^2\right)dy\right)$$

Use : $y = 1 - x, \quad dy = -dx$



$$V_{AB} = \int_{1}^{0} \left[\left(3x\right)\left(1-x\right) - x\left(1-x\right)^2\right] dx$$

Note the limits!

(Parametrized in terms of x.)

Example (cont.)

Completing the calculus:

$$V_{AB} = \int_1^0 \left[(3x)(1-x) - x(1-x)^2 \right] dx$$

$$= \int_1^0 \left[3x - 3x^2 - x + 2x^2 - x^3 \right] dx$$

$$= \int_0^1 \left[-3x + 3x^2 + x - 2x^2 + x^3 \right] dx$$

$$= \int_0^1 \left(x^3 + x^2 - 2x \right) dx$$

$$= \left(\frac{1}{4}\right) + \left(\frac{1}{3}\right) - 2\left(\frac{1}{2}\right) = \frac{3+4-12}{12} = -\frac{5}{12}$$

$$V_{AB} = -5/12 \ [\mathrm{V}]$$

Example (cont.)

Alternative calculation (we parameterize differently):

Note:
In this example, there are 4 different ways we could have parameterized.

$$V_{AB} = \int_{\underline{A}}^{\underline{B}} \underline{E} \cdot \underline{dr} = \int_{\underline{A}}^{\underline{B}} \left(E_x \, dx + E_y \, dy + \cancel{E_z \, dz} \right)$$

$$= \int_1^0 (3xy)\, dx + \int_0^1 \left(xy^2 \right) dy$$

(Parametrized in terms of x.)

$$y = 1 - x$$

(Parametrized in terms of y.)

$$x = 1 - y$$

$$V_{AB} = \int_1^0 \left[(3x)(1-x) \right] dx + \int_0^1 \left[(1-y)\left(y^2\right) \right] dy$$

Note the limits!

$$= \int_1^0 \left(3x - 3x^2 \right) dx + \int_0^1 \left(y^2 - y^3 \right) dy$$

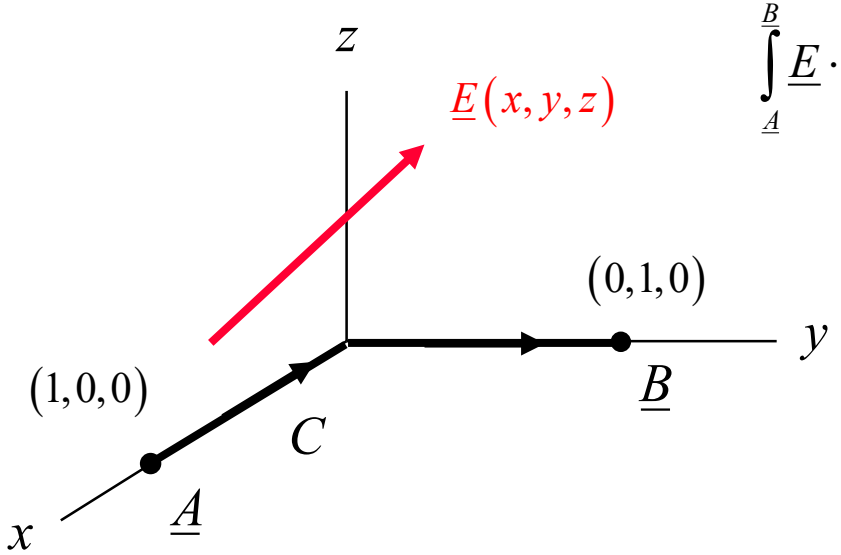
$$= \left(-\frac{3}{2} + 1 \right) + \left(\frac{1}{3} - \frac{1}{4} \right)$$

$$V_{AB} = -5/12 \ [\mathrm{V}]$$

Example (Part 2)

$$\underline{E} = \underline{\hat{x}}\left(3xy\right) + \underline{\hat{y}}\left(xy^2\right) + \underline{\hat{z}}\left(2z+1\right)$$ (same field as in Part 1)

Find V_{AB} using path C shown below.



$$\int_{\underline{A}}^{\underline{B}} \underline{E} \cdot \underline{dr} = \int_{\underline{A}}^{\underline{B}} \left(E_x\, dx + E_y\, dy + E_z\, dz\right)$$

$$= \int_{\underline{A}}^{\underline{B}} \left(\left(3xy\right) dx + \left(xy^2\right) dy\right)$$

$$= \int_{\underline{A}}^{0} \left(\left(3xy\right) dx + \left(xy^2\right) dy\right) + \int_{\underline{0}}^{\underline{B}} \left(\left(3xy\right) dx + \left(xy^2\right) dy\right)$$

$$= \int_{1}^{0} 0\, dx + \int_{0}^{1} 0\, dy$$

$$= 0$$

$$V_{AB} = 0 \ [\mathrm{V}]$$

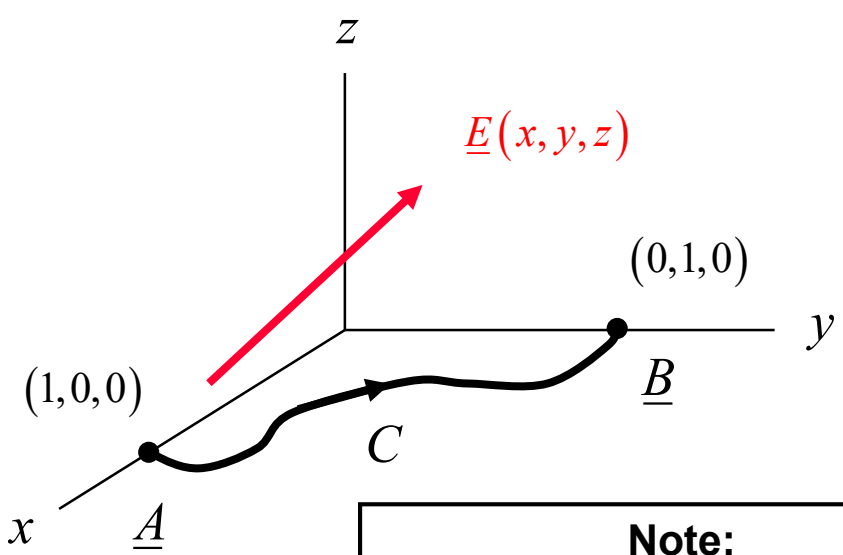
The voltage drop depends on the path!
For this electric field, the voltage at a point in space is not definable.

Example

$$\underline{E} = \underline{\hat{x}}(3x) + \underline{\hat{y}}(y^2) + \underline{\hat{z}}(2z)$$

(This is a valid electrostatic field.)

Find V_{AB} using an *arbitrary* path C in the xy plane.



Note:
The path does not have to be parameterized! Hence, only the <u>endpoints</u> are important.

$$\int_{\underline{A}}^{\underline{B}} \underline{E} \cdot \underline{dr} = \int_{\underline{A}}^{\underline{B}} \left(E_x\, dx + E_y\, dy + \cancel{E_z\, dz} \right)$$

$$= \int_{\underline{A}}^{\underline{B}} \left[(3x)\, dx + (y^2)\, dy \right]$$

$$= \int_1^0 (3x)\, dx + \int_0^1 (y^2)\, dy$$

$$V_{AB} = -\int_0^1 (3x)\, dx + \int_0^1 (y^2)\, dy$$

$$-3\left(\frac{1}{2}\right) + \frac{1}{3}$$

$$= -7/6$$

$$V_{AB} = -7/6 \ [\mathrm{V}]$$

The integral is path independent!

Example (cont.)

Note: If we have an electric field of the form:

$$\underline{E} = \underline{\hat{x}}\left(f\left(x\right)\right) + \underline{\hat{y}}\left(g\left(y\right)\right) + \underline{\hat{z}}\left(h\left(z\right)\right)$$

$$\Downarrow$$

$$\nabla \times \underline{E} = \underline{0}$$ (discussed later)

$$\Downarrow$$

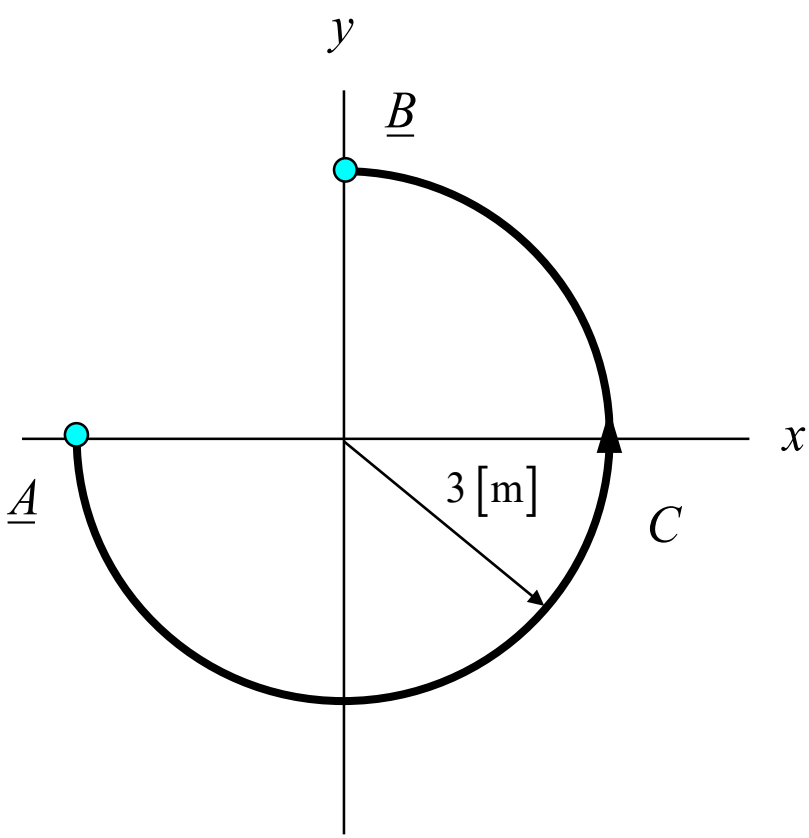
V_{AB} is path independent.

If the electric field vector is in this mathematical form, the path does not have to be parameterized.

Example

$$\underline{E} = \underline{\hat{x}}(x) + \underline{\hat{y}}(2y)$$

Find V_{AB} using path C shown below.



$$V_{AB} = \int_{\underline{A}}^{\underline{B}} \underline{E} \cdot d\underline{r}$$

For $\underline{dr}$:

$$\underline{dr} = \underline{\hat{\rho}}\, \cancel{d\rho} + \underline{\hat{\phi}}(\rho\, d\phi) + \cancel{\underline{\hat{z}} dz}$$

$$= \underline{\hat{\phi}}(3\, d\phi)$$

For $\underline{E}$:

$$x = \rho \cos\phi = 3\cos\phi$$

$$y = \rho \sin\phi = 3\sin\phi$$

$$\underline{\hat{x}} = \underline{\hat{\rho}} \cos\phi + \underline{\hat{\phi}}(-\sin\phi)$$

$$\underline{\hat{y}} = \underline{\hat{\rho}} \sin\phi + \underline{\hat{\phi}} \cos\phi$$

Example (cont.)

$$\underline{E} = \underline{\hat{x}}(x) + \underline{\hat{y}}(2y)$$

$$\underline{E} = \left[\underline{\hat{\rho}}\cos\phi - \underline{\hat{\phi}}\sin\phi\right](3\cos\phi) + \left[\underline{\hat{\rho}}\sin\phi + \underline{\hat{\phi}}\cos\phi\right](2(3\sin\phi))$$

$$d\underline{r} = \underline{\hat{\phi}}(3\,d\phi)$$

$$\underline{E}\cdot d\underline{r} = (-9+18)\sin\phi\cos\phi\,d\phi = 9\sin\phi\cos\phi\,d\phi$$

$$V_{AB} = \int_{-\pi}^{\pi/2} 9\sin\phi\cos\phi\,d\phi$$

$$= \int_{-\pi}^{\pi/2} \frac{9}{2}\sin(2\phi)\,d\phi$$

$$= \frac{9}{2}\left[-\frac{\cos(2\phi)}{2}\right]_{-\pi}^{\pi/2}$$

$$= \frac{9}{2}\left[\frac{1}{2} - \left(-\frac{1}{2}\right)\right] = \frac{9}{2}$$

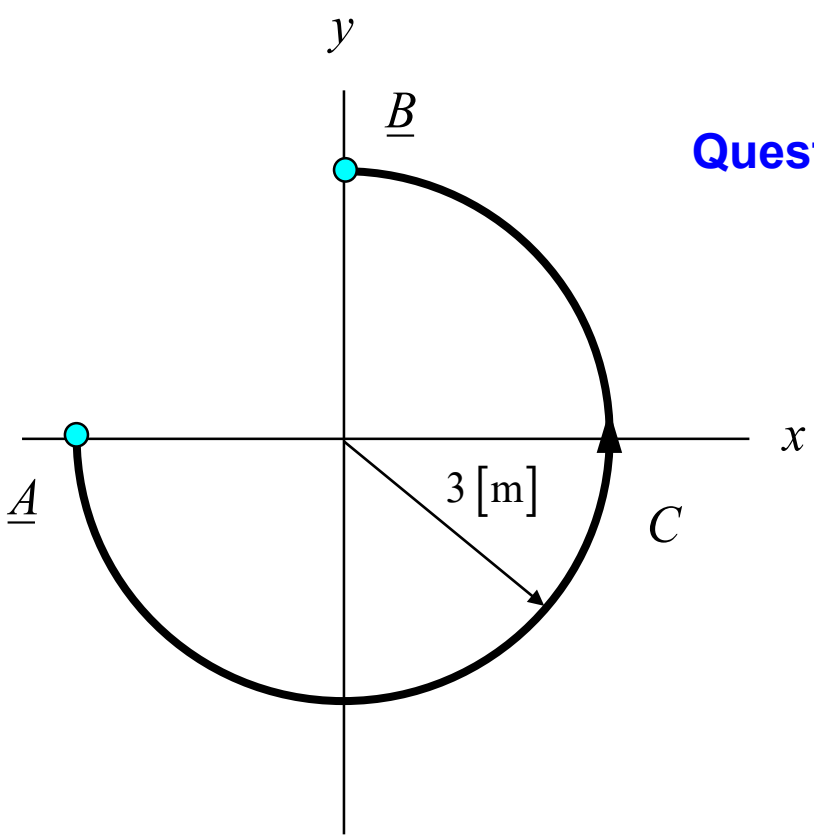
Note:
The angle ϕ must change continuously along the path. If we take the angle ϕ to be $\pi/2$ at point $\underline{B}$, then the angle ϕ must be $-\pi$ at point $\underline{A}$.

$$V_{AB} = 9/2 \ [\mathrm{V}]$$

Example (cont.)

Let's examine this same electric field once again:

$$\underline{E} = \underline{\hat{x}}(x) + \underline{\hat{y}}(2y)$$



$$V_{AB} = \int_{\underline{A}}^{\underline{B}} \underline{E} \cdot \underline{dr}$$

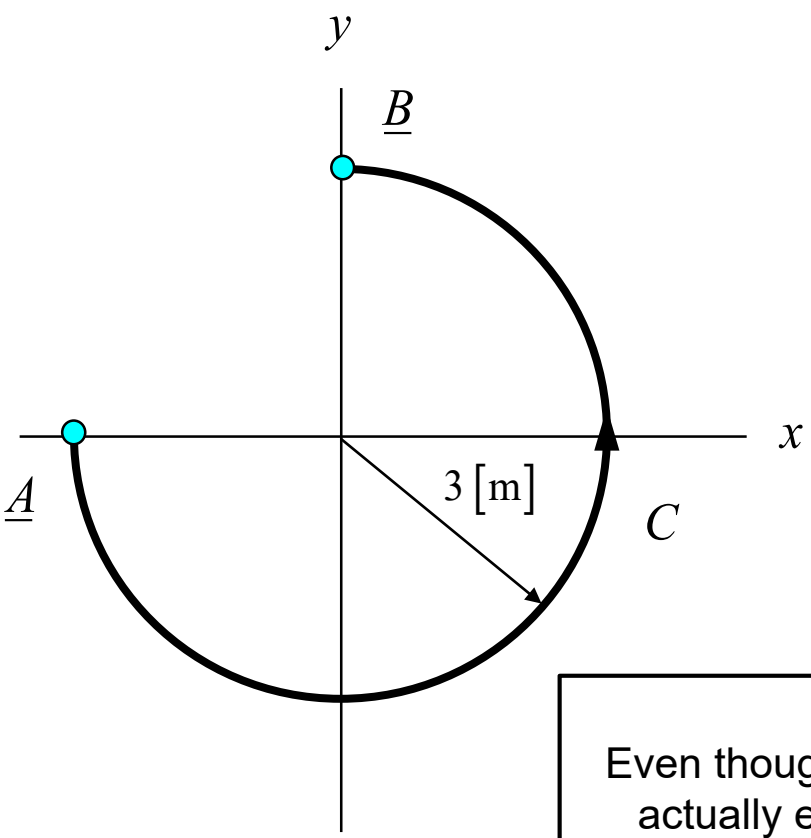
Question: Is this integral path independent?

Note:
The answer is yes because the curl of the electric field is zero.
(We will talk about curl later.)

Example (cont.)

$$\underline{E} = \underline{\hat{x}}(x) + \underline{\hat{y}}(2y)$$

$$V_{AB} = \int_{\underline{A}}^{\underline{B}} \underline{E} \cdot d\underline{r}$$



Question: Is this integral path independent?

Let's find out from the calculus:

$$\int_{\underline{A}}^{\underline{B}} \underline{E} \cdot \underline{dr} = \int_{\underline{A}}^{\underline{B}} \left(E_x\, dx + E_y\, dy + E_z\, dz \right)$$

$$= \int_{\underline{A}}^{\underline{B}} \left[(x)\, dx + (2y)\, dy \right]$$

$$= \int_{-3}^{0} (x)\, dx + \int_{0}^{3} (2y)\, dy$$

$$= -\left(\frac{9}{2} \right) + 9$$

$$= 9/2$$

Yes, it is path independent: no need to parameterize.

$$V_{AB} = 9/2 \ [\mathrm{V}]$$

Note:
Even though the path was given as a circle, the calculus was actually easier in rectangular coordinates in this example!