ECE 3318
Applied Electricity and Magnetism

Spring 2023

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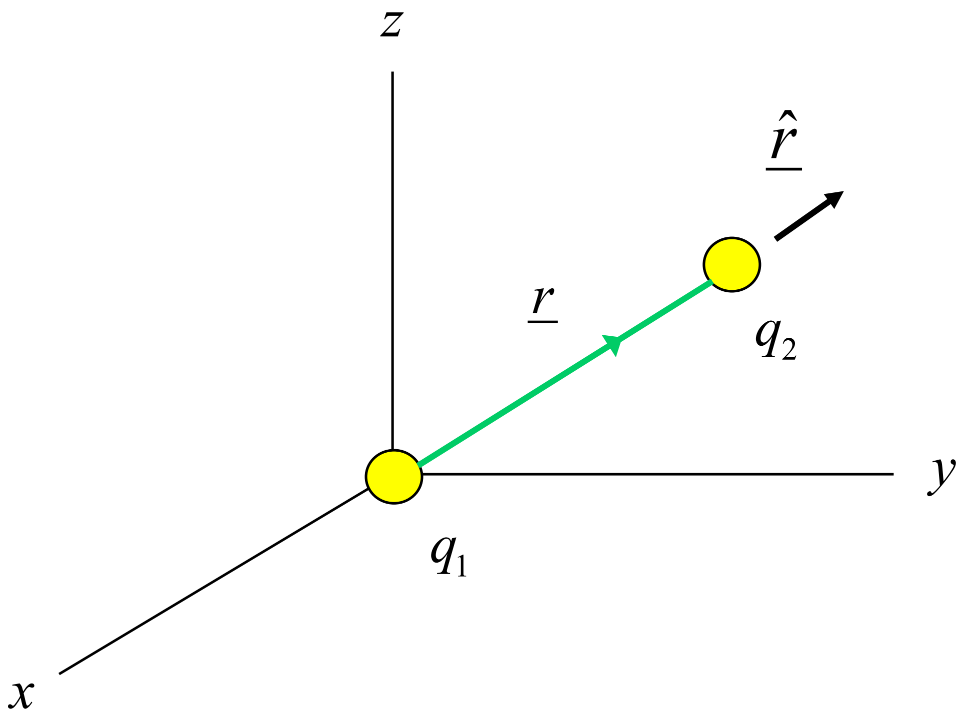
Dept. of ECE

Notes 7
Coulomb's Law I

Notes prepared by the EM Group
University of Houston

Coulomb's Law



Experimental law:

$$\underline{F}_2 = \frac{q_1 q_2}{4\pi\varepsilon_0 r^2}\,\hat{\underline{r}} \quad [\mathrm{N}]$$

$$\varepsilon_0 \doteq 8.854187818 \times 10^{-12} \quad [\mathrm{F/m}]$$

(permittivity of free space)

Charles-Augustin de Coulomb

Here is how we can calculate ε_0 accurately:

c = speed of light $\equiv 2.99792458 \times 10^8$ [m/s] (defined)

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

(from ECE 3317)

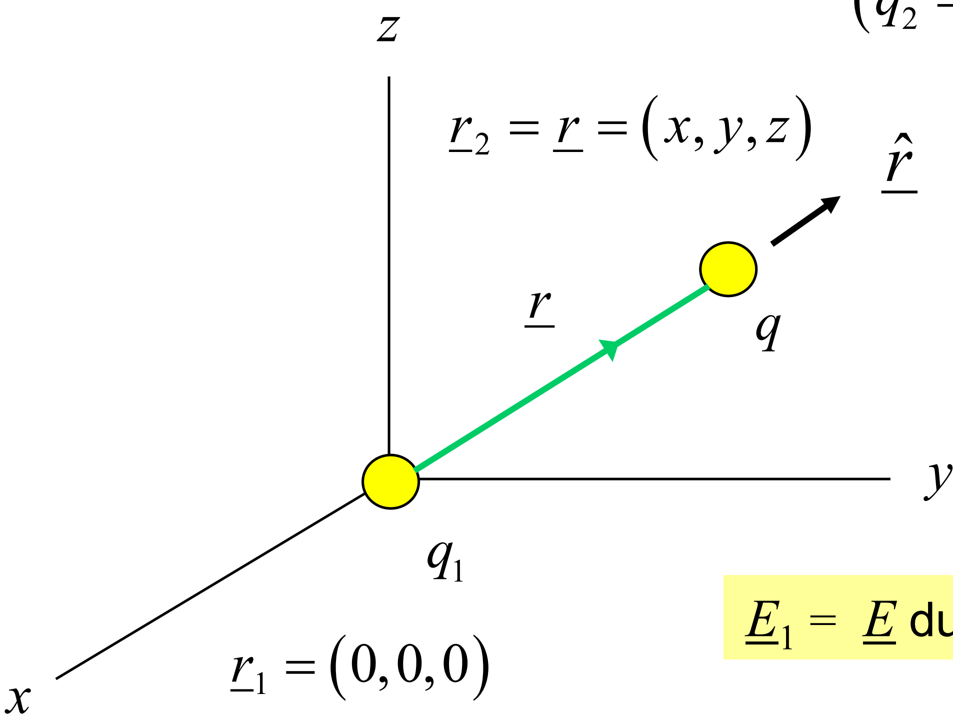
$$\mu_0 \doteq 4\pi \times 10^{-7} \quad [\mathrm{H/m}]$$

$$\Rightarrow \quad \varepsilon_0 = \frac{1}{\mu_0 c^2}$$

Coulomb's Law (cont.)

A "test" charge q is placed at $\underline{r}$ to measure the electric field there from charge q_1.

$$\left(q_2 = q\right)$$



$$\underline{F}_q = \frac{q_1 q_2}{4\pi\varepsilon_0 r^2}\,\hat{\underline{r}} = q\left(\frac{q_1}{4\pi\varepsilon_0 r^2}\,\hat{\underline{r}}\right)$$

But $\underline{F}_q = q\,\underline{E}_1(\underline{r})$

$\underline{E}_1 = \underline{E}$ due to q_1

$\underline{r}$ = location of charge q

Hence:

$$\underline{E}_1(\underline{r}) = \frac{q_1}{4\pi\varepsilon_0 r^2}\,\hat{\underline{r}}$$

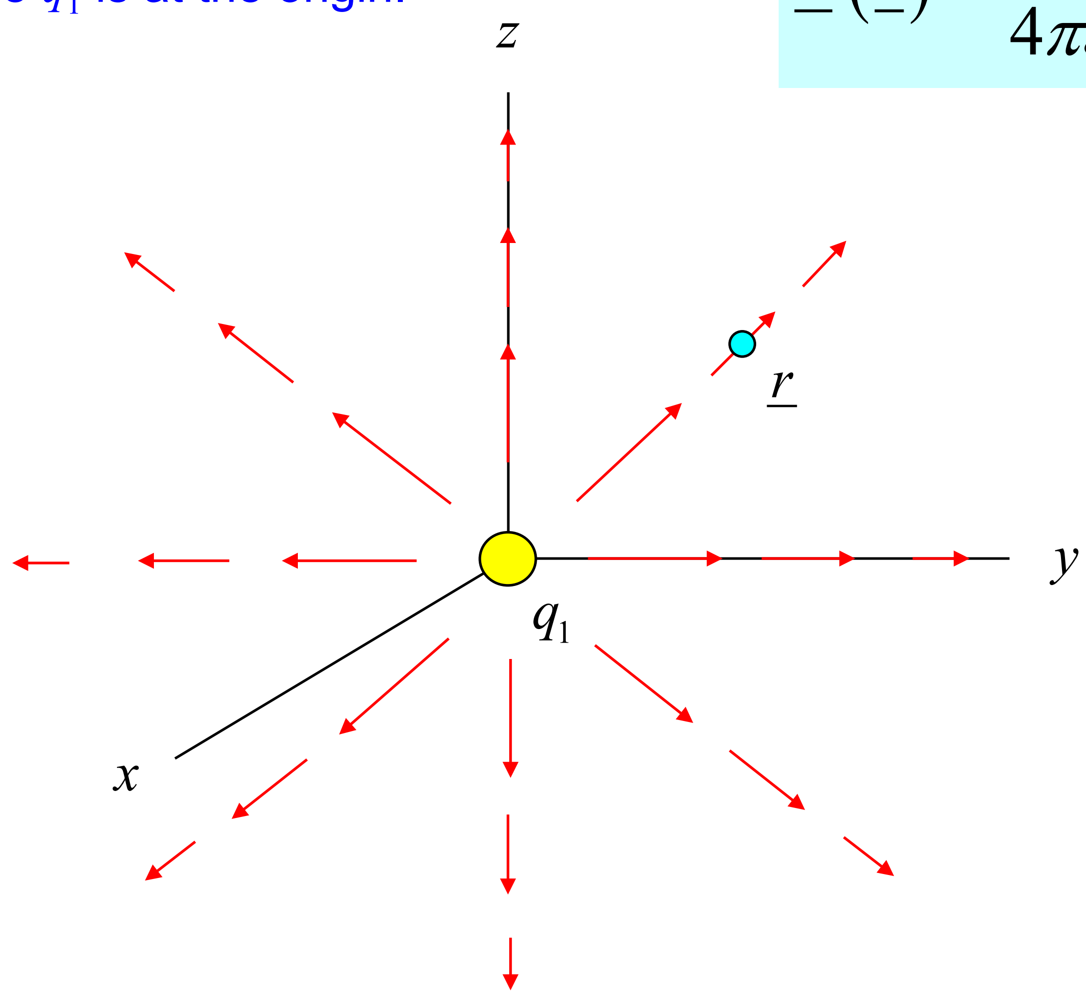
Note:
There is no self-force on charge 2 due to its own electric field.

Coulomb's Law (cont.)

Point charge summary

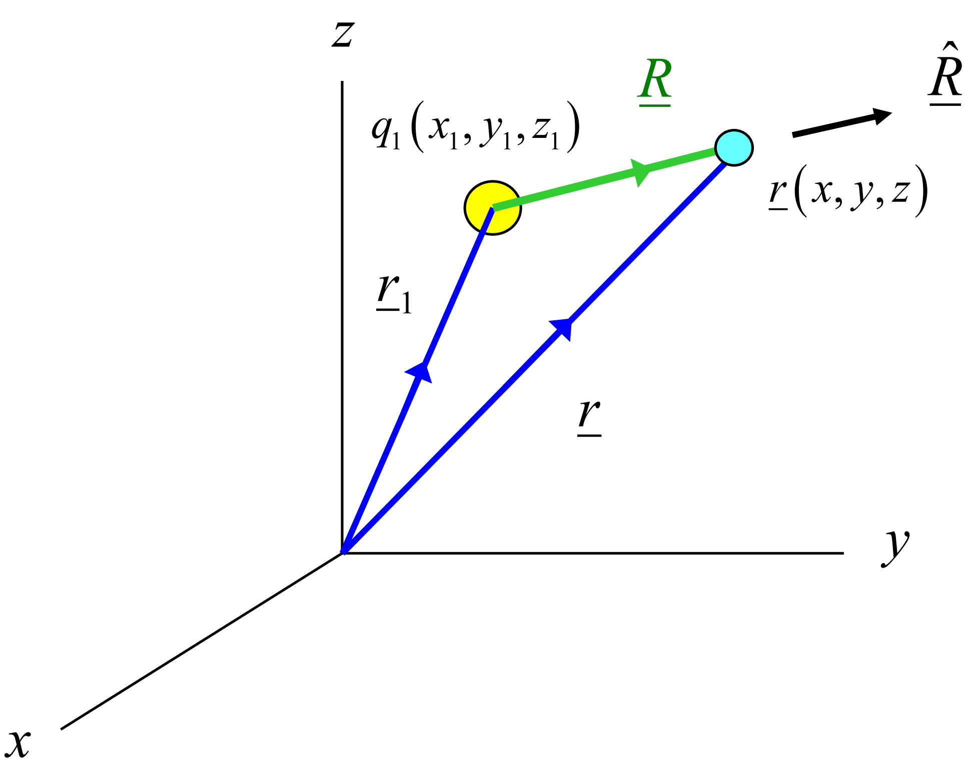
A point change q_1 is at the origin.

$$\underline{E}\left(\underline{r}\right) = \frac{q_1}{4\pi\varepsilon_0 r^2}\,\underline{\hat{r}}$$



Coulomb's Law (cont.)

Generalization (q_1 not at the origin):



$$\underline{r}_1 = (x_1, y_1, z_1)$$

$$\underline{r} = (x, y, z)$$

$$\underline{R} = \underline{r} - \underline{r}_1$$

$$\underline{R} = \hat{\underline{x}}(x - x_1) + \hat{\underline{y}}(y - y_1) + \hat{\underline{z}}(z - z_1)$$

$$R = |\underline{R}| = |\underline{r} - \underline{r}_1| = \left[(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2\right]^{1/2}$$

$$\underline{E}_1(\underline{r}) = \frac{q_1}{4\pi\varepsilon_0 R^2}\hat{\underline{R}}$$

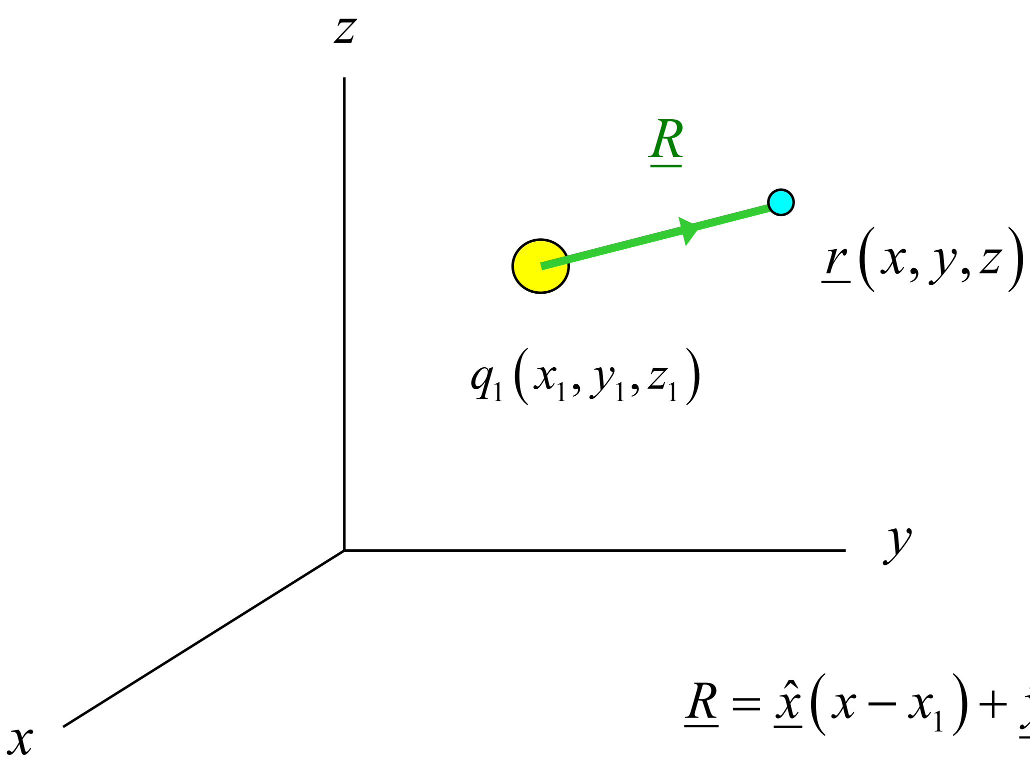
$$\hat{\underline{R}} = \frac{\underline{R}}{|\underline{R}|}$$

Coulomb's Law (cont.)

Point charge summary

A point change q_1 is arbitrarily located.

$$\underline{E}\left(\underline{r}\right)=\frac{q_1}{4\pi\varepsilon_0 R^2}\,\underline{\hat{R}}$$



$$\underline{\hat{R}}=\frac{\underline{R}}{R}$$

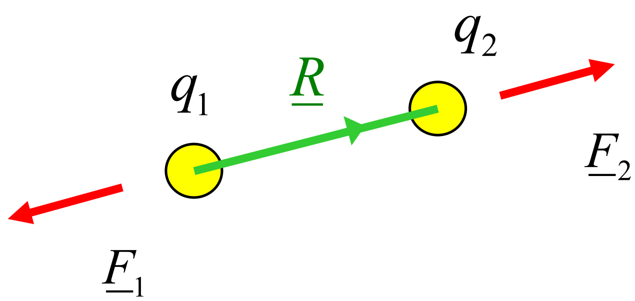
$$\underline{R}=\underline{\hat{x}}\left(x-x_1\right)+\underline{\hat{y}}\left(y-y_1\right)+\underline{\hat{z}}\left(z-z_1\right)$$

$$R=\left|\underline{R}\right|=\sqrt{\left(x-x_1\right)^2+\left(y-y_1\right)^2+\left(z-z_1\right)^2}$$

Example

$q_1 = 0.7$ [mC] located at $(3,5,7)$ [m]

$q_2 = 4.9$ [μC] located at $(1,2,1)$ [m]



Find $\underline{F}_1$, $\underline{F}_2$

$\underline{F}_1$ = force on charge q_1

$\underline{F}_2$ = force on charge q_2

Note:
The position vector for charge 2 is called $\underline{r}_2$ here instead of $\underline{r}$.

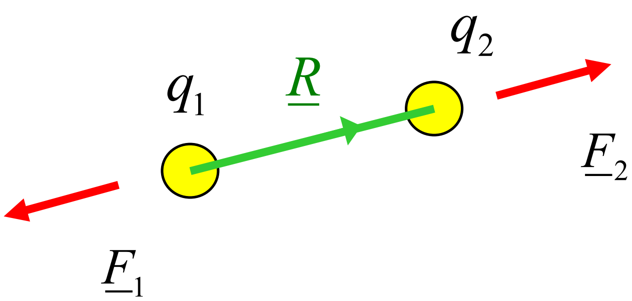
For $\underline{F}_2$: $\quad \underline{F}_2 = q_2\, \underline{E}_1\left(\underline{r}_2\right)$

$\underline{E}_1\left(\underline{r}_2\right)$ = electric field due to charge q_1, evaluated at point $\underline{r}_2$

Example (cont.)

$q_1 = 0.7$ [mC] located at $(3,5,7)$ [m]

$q_2 = 4.9$ [μC] located at $(1,2,1)$ [m]



$$\underline{E}_1(\underline{r}_2) = \frac{q_1}{4\pi\varepsilon_0 R^2}\underline{\hat{R}}$$

$$\underline{F}_2 = q_2\,\underline{E}_1(\underline{r}_2)$$

$$\underline{R} = \underline{r}_2 - \underline{r}_1 = \underline{\hat{x}}(1-3) + \underline{\hat{y}}(2-5) + \underline{\hat{z}}(1-7)$$
$$= \underline{\hat{x}}(-2) + \underline{\hat{y}}(-3) + \underline{\hat{z}}(-6)\ \text{[m]}$$

$$R = |\underline{R}| = \sqrt{(-2)^2 + (-3)^2 + (-6)^2} = 7\ \text{[m]}$$

$$\underline{\hat{R}} = \frac{\underline{R}}{|\underline{R}|} = \underline{\hat{x}}\left(\frac{-2}{7}\right) + \underline{\hat{y}}\left(\frac{-3}{7}\right) + \underline{\hat{z}}\left(\frac{-6}{7}\right)$$

Example (cont.)

$$\underline{E}_1(\underline{r}_2) = \frac{0.7\times10^{-3}}{4\pi(8.854\times10^{-12})(7)^2}\left[\underline{\hat{x}}\left(\frac{-2}{7}\right)+\underline{\hat{y}}\left(\frac{-3}{7}\right)+\underline{\hat{z}}\left(\frac{-6}{7}\right)\right]$$

$$= 1.834\times10^4\left[\underline{\hat{x}}(-2)+\underline{\hat{y}}(-3)+\underline{\hat{z}}(-6)\right] \quad \text{[V/m]}$$

$$\underline{F}_2 = q_2\,\underline{E}_1(\underline{r}_2) = 4.9\times10^{-6}\,\underline{E}_1(\underline{r}_2)$$

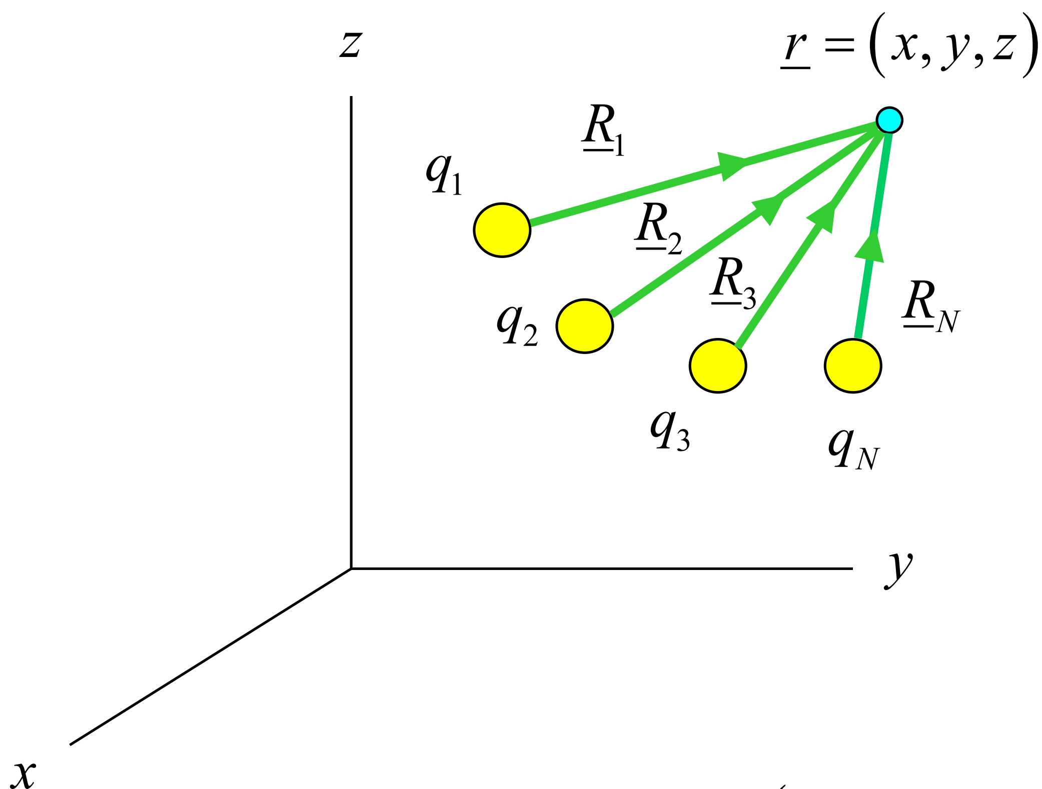
$$= 0.08988\left[\underline{\hat{x}}(-2)+\underline{\hat{y}}(-3)+\underline{\hat{z}}(-6)\right] \quad \text{[N]}$$

Also, we have $\underline{F}_1 = -\underline{F}_2$ (Newton's Law)

$$\underline{F}_2 = \underline{\hat{x}}(-0.180)+\underline{\hat{y}}(-0.270)+\underline{\hat{z}}(-0.539) \quad \text{[N]}$$

$$\underline{F}_1 = \underline{\hat{x}}(+0.180)+\underline{\hat{y}}(+0.270)+\underline{\hat{z}}(+0.539) \quad \text{[N]}$$

General Case: Multiple Charges



$q_1: \ \underline{r}_1 = (x_1, y_1, z_1)$

$q_2: \ \underline{r}_2 = (x_2, y_2, z_2)$

$\vdots$

$q_N: \ \underline{r}_N = (x_N, y_N, z_N)$

$\underline{R}_1 = \underline{r} - \underline{r}_1$

$\underline{R}_2 = \underline{r} - \underline{r}_2$

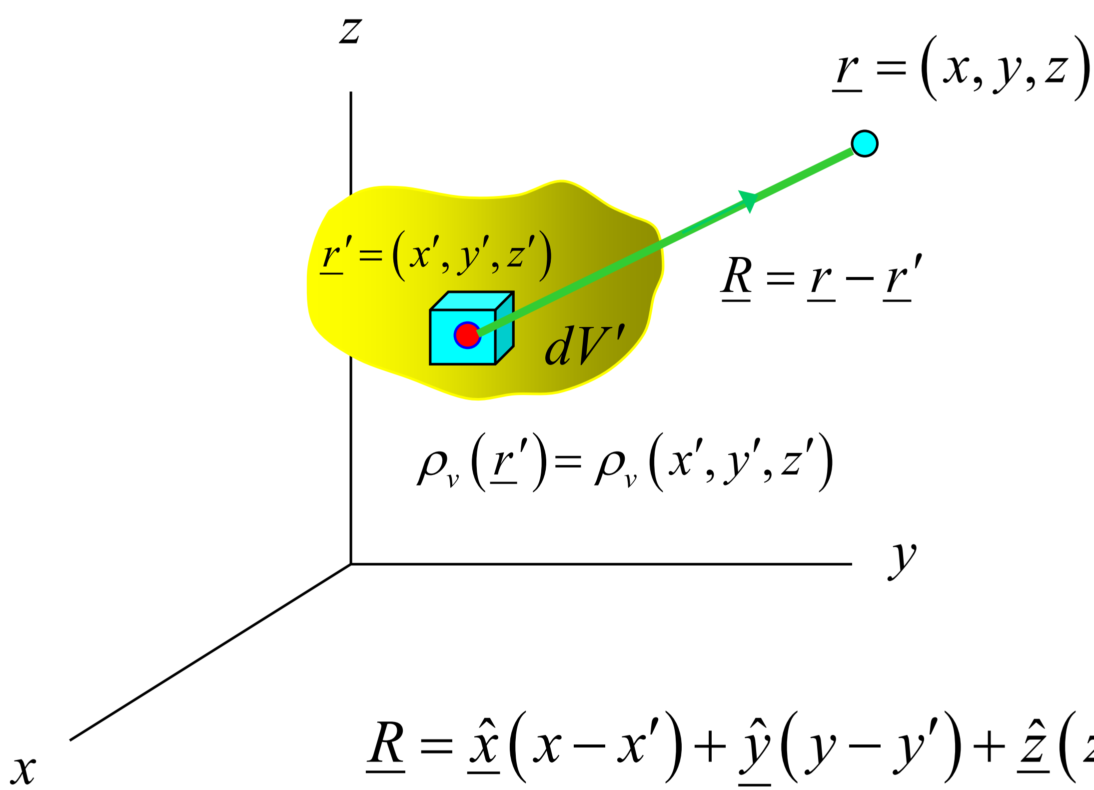
$\vdots$

$\underline{R}_N = \underline{r} - \underline{r}_N$

$\underline{E} = \underline{E}_1 + \underline{E}_2 + \ldots \underline{E}_N$ (superposition)

$$\underline{E}(\underline{r}) = \frac{q_1}{4\pi\varepsilon_0 R_1^{\,2}}\underline{\hat{R}}_1 + \frac{q_2}{4\pi\varepsilon_0 R_2^{\,2}}\underline{\hat{R}}_2 + \ldots + \frac{q_N}{4\pi\varepsilon_0 R_N^{\,2}}\underline{\hat{R}}_N$$

Field from Volume Charge



Note :

The blue dot is associated with $\underline{r}$

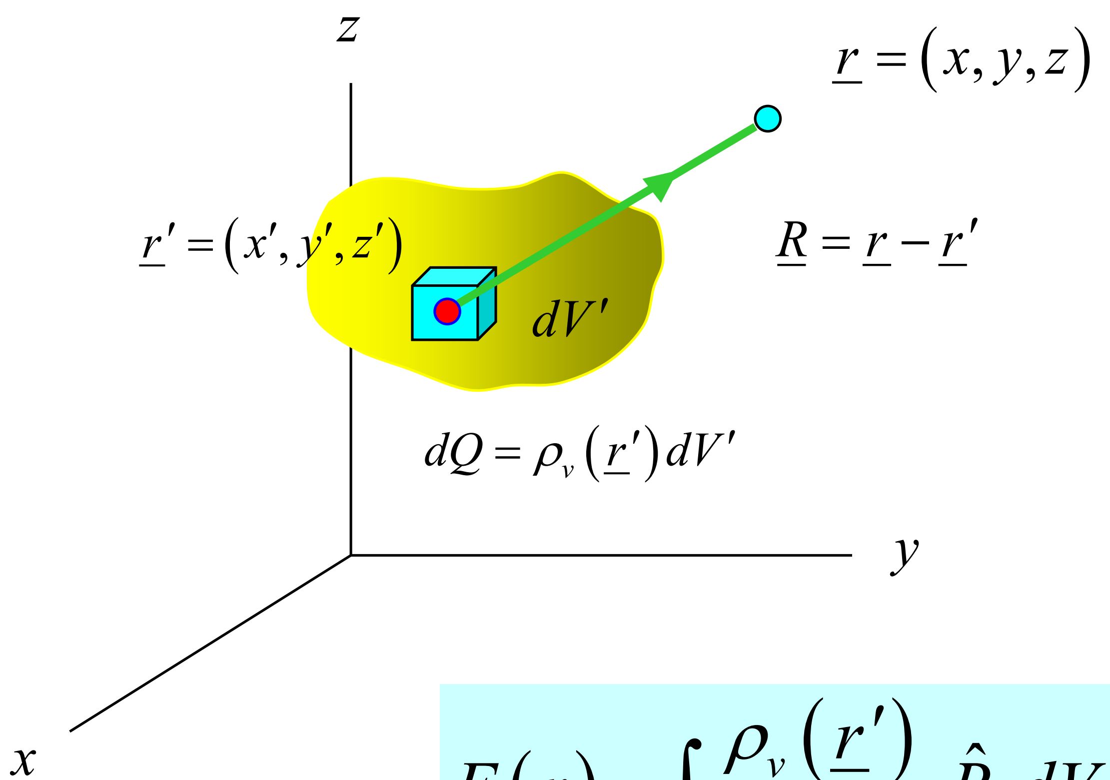
The red dot is associated with $\underline{r}'$

$$\underline{R} = \underline{\hat{x}}(x - x') + \underline{\hat{y}}(y - y') + \underline{\hat{z}}(z - z')$$

$$R = |\underline{R}| = \left[(x - x')^2 + (y - y')^2 + (z - z')^2\right]^{1/2}$$

$$\underline{\hat{R}} = \frac{\underline{R}}{|\underline{R}|}$$

Field from Volume Charge (cont.)

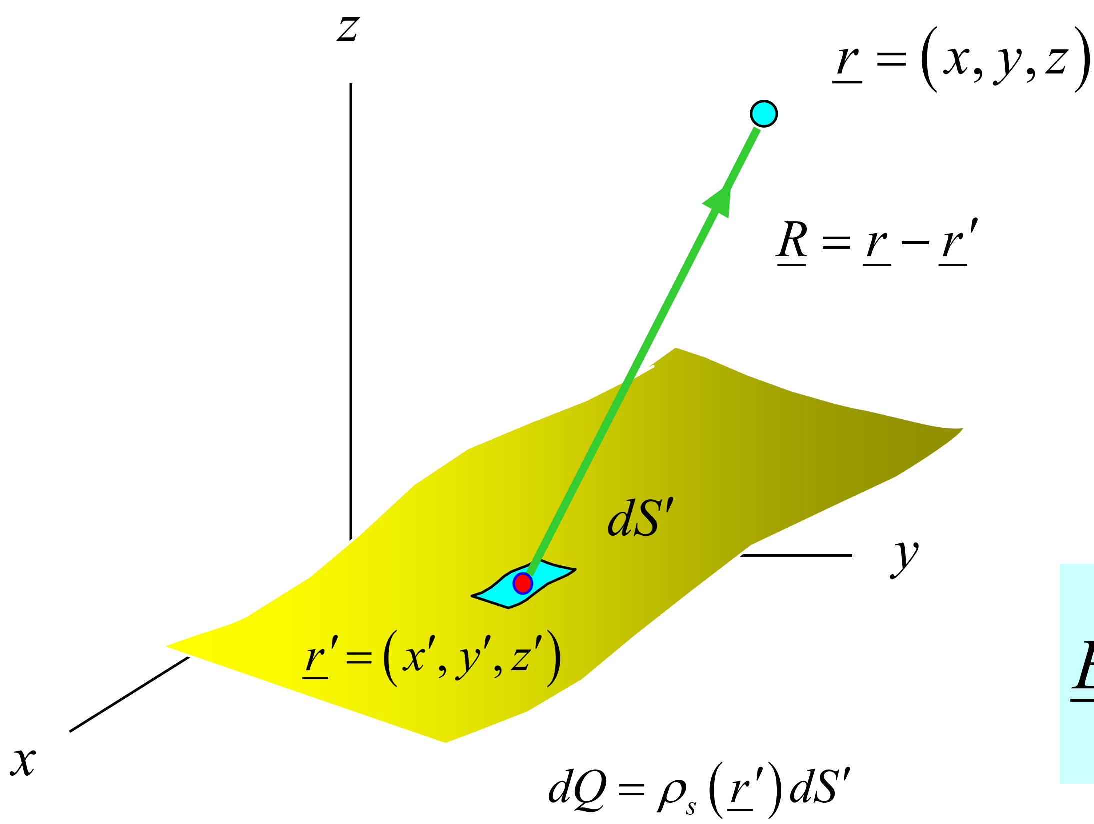


$$d\underline{E} = \frac{dQ}{4\pi\varepsilon_0 R^2}\,\underline{\hat{R}} = \frac{\rho_v(x', y', z')\, dV'}{4\pi\varepsilon_0 R^2}\,\underline{\hat{R}}$$

$$\underline{E}(\underline{r}) = \int_V \frac{\rho_v(\underline{r}')}{4\pi\varepsilon_0 R^2}\,\underline{\hat{R}}\; dV'$$

$$\underline{R} = \underline{r} - \underline{r}' \qquad R = |\underline{R}| = |\underline{r} - \underline{r}'|$$

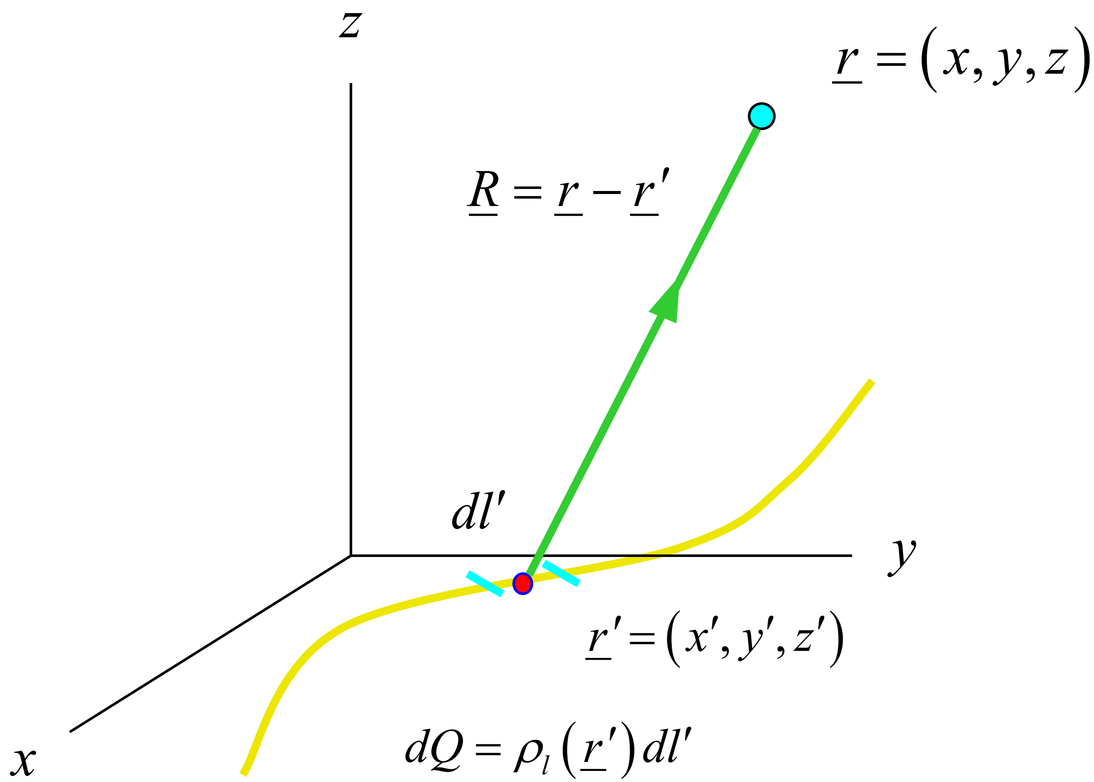
Field from Surface Charge



$$\underline{E}(\underline{r}) = \int_S \frac{\rho_s(\underline{r}')}{4\pi\varepsilon_0 R^2}\,\hat{\underline{R}}\;dS'$$

$$\underline{R} = \underline{r} - \underline{r}' \qquad R = \left|\underline{R}\right| = \left|\underline{r} - \underline{r}'\right|$$

Field from Line Charge



$$\underline{E} = \int_C \frac{\rho_l(\underline{r}')}{4\pi\varepsilon_0 R^2}\,\hat{\underline{R}}\;dl'$$

$$\underline{R} = \underline{r} - \underline{r}'$$

$$R = \left|\underline{R}\right| = \left|\underline{r} - \underline{r}'\right|$$

Note : $dl' = \left|d\underline{r}'\right|$

Summary of Coulomb's Law

$$\underline{E}(\underline{r}) = \frac{q_1}{4\pi\varepsilon_0 R_1^{\,2}}\hat{\underline{R}}_1 + \frac{q_2}{4\pi\varepsilon_0 R_2^{\,2}}\hat{\underline{R}}_2 + \ldots + \frac{q_N}{4\pi\varepsilon_0 R_N^{\,2}}\hat{\underline{R}}_N$$

$$\underline{E} = \int_C \frac{\rho_l(\underline{r}')}{4\pi\varepsilon_0 R^2}\hat{\underline{R}}\; dl'$$

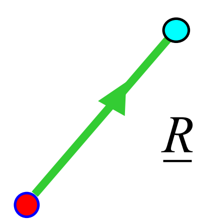
$$\underline{E}(\underline{r}) = \int_S \frac{\rho_s(\underline{r}')}{4\pi\varepsilon_0 R^2}\hat{\underline{R}}\; dS'$$

$$\underline{E}(\underline{r}) = \int_V \frac{\rho_v(\underline{r}')}{4\pi\varepsilon_0 R^2}\hat{\underline{R}}\; dV'$$

$$\underline{R} = \underline{\hat{x}}(x - x') + \underline{\hat{y}}(y - y') + \underline{\hat{z}}(z - z')$$

$$R = |\underline{R}| = \left[(x - x')^2 + (y - y')^2 + (z - z')^2\right]^{1/2}$$

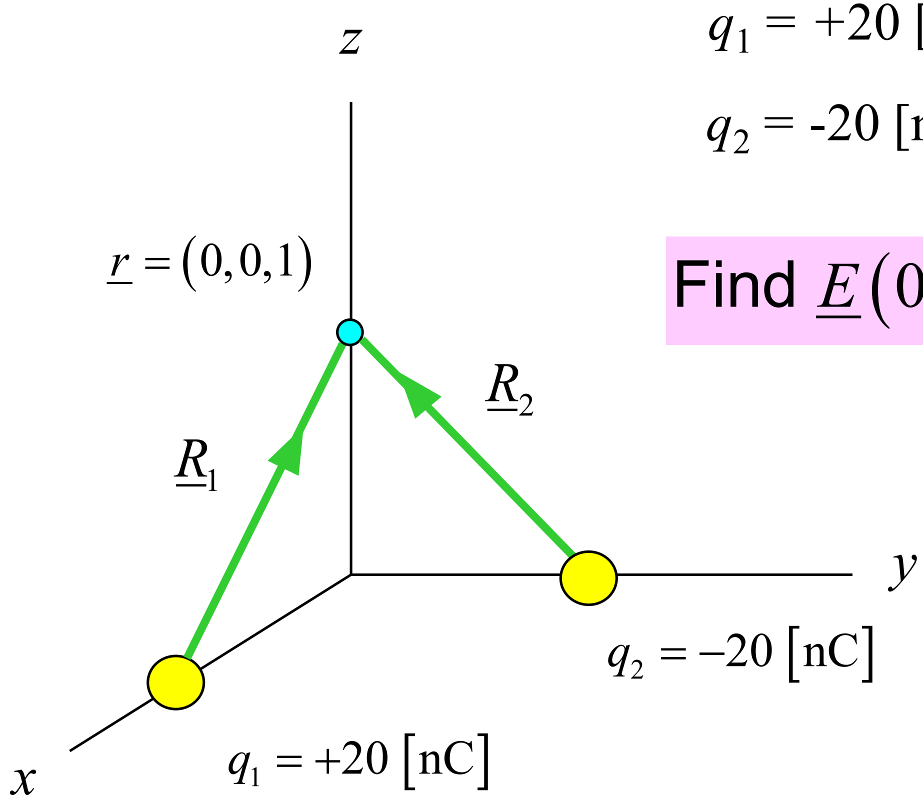
$$\hat{\underline{R}} = \frac{\underline{R}}{|\underline{R}|}$$



Example

$q_1 = +20$ [nC] located at $(1,0,0)$ [m]

$q_2 = -20$ [nC] located at $(0,1,0)$ [m]



Find $\underline{E}(0,0,1)$

Find the displacement vectors:

$$\underline{R}_1 = (0,0,1) - (1,0,0)$$

$$\underline{R}_2 = (0,0,1) - (0,1,0)$$

$$\underline{E} = \frac{q_1}{4\pi\varepsilon_0 R_1^2}\hat{\underline{R}}_1 + \frac{q_2}{4\pi\varepsilon_0 R_2^2}\hat{\underline{R}}_2$$

$$\underline{R}_1 = (-1,0,1) \qquad \underline{R}_2 = (0,-1,1)$$

$$R_1 = \sqrt{2} \qquad R_2 = \sqrt{2}$$

$$\hat{\underline{R}}_1 = \frac{1}{\sqrt{2}}(-1,0,1) \qquad \hat{\underline{R}}_2 = \frac{1}{\sqrt{2}}(0,-1,1)$$

Example (cont.)

$$\underline{E} = \frac{q_1}{4\pi\varepsilon_0 R_1^{\,2}}\underline{\hat{R}}_1 + \frac{q_2}{4\pi\varepsilon_0 R_2^{\,2}}\underline{\hat{R}}_2$$

$$= \frac{20\times10^{-9}}{4\pi\left(8.854\times10^{-12}\right)\left(\sqrt{2}\right)^2}\left[\frac{1}{\sqrt{2}}(-1,0,1)\right] + \frac{-20\times10^{-9}}{4\pi\left(8.854\times10^{-12}\right)\left(\sqrt{2}\right)^2}\left[\frac{1}{\sqrt{2}}(0,-1,1)\right]$$

$$= 63.55\left[(-1,0,1)-(0,-1,1)\right]$$

$$= 63.55\left[(-1,+1,0)\right]$$

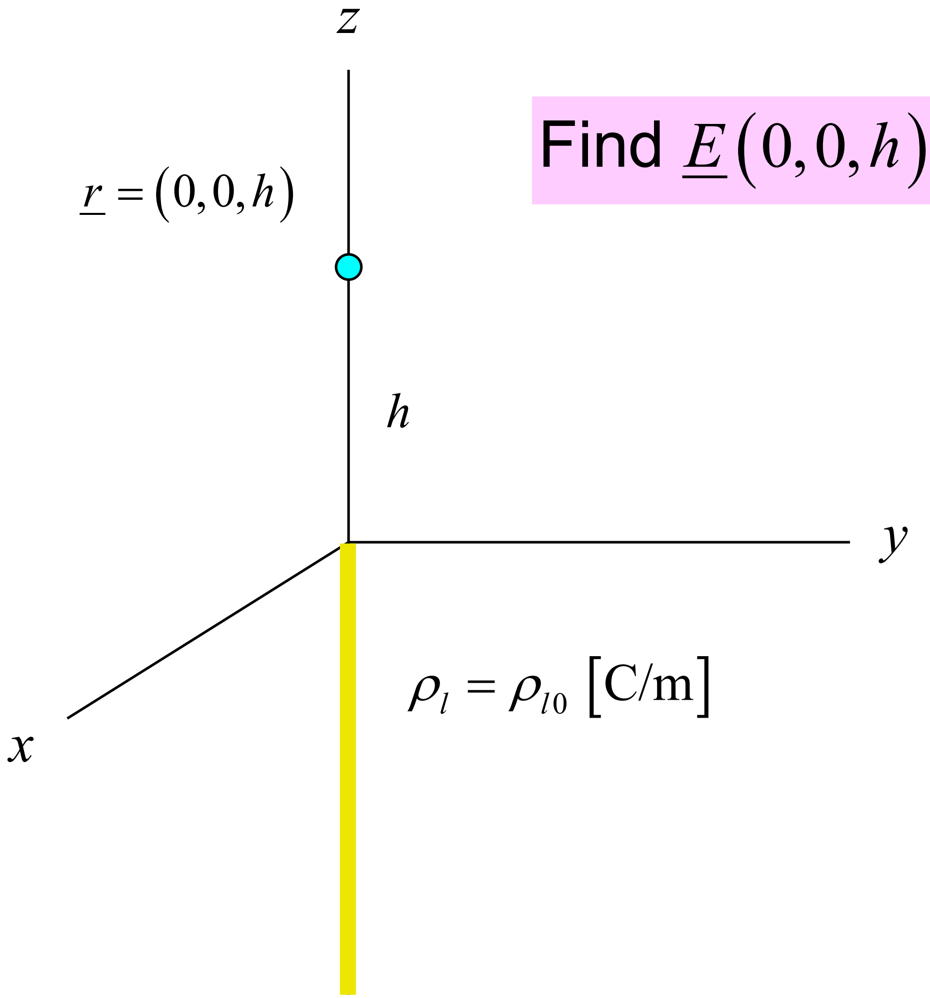
Hence we have

$$\underline{E} = 63.55\left(-\underline{\hat{x}} + \underline{\hat{y}}\right) \quad [\text{V/m}]$$

Example

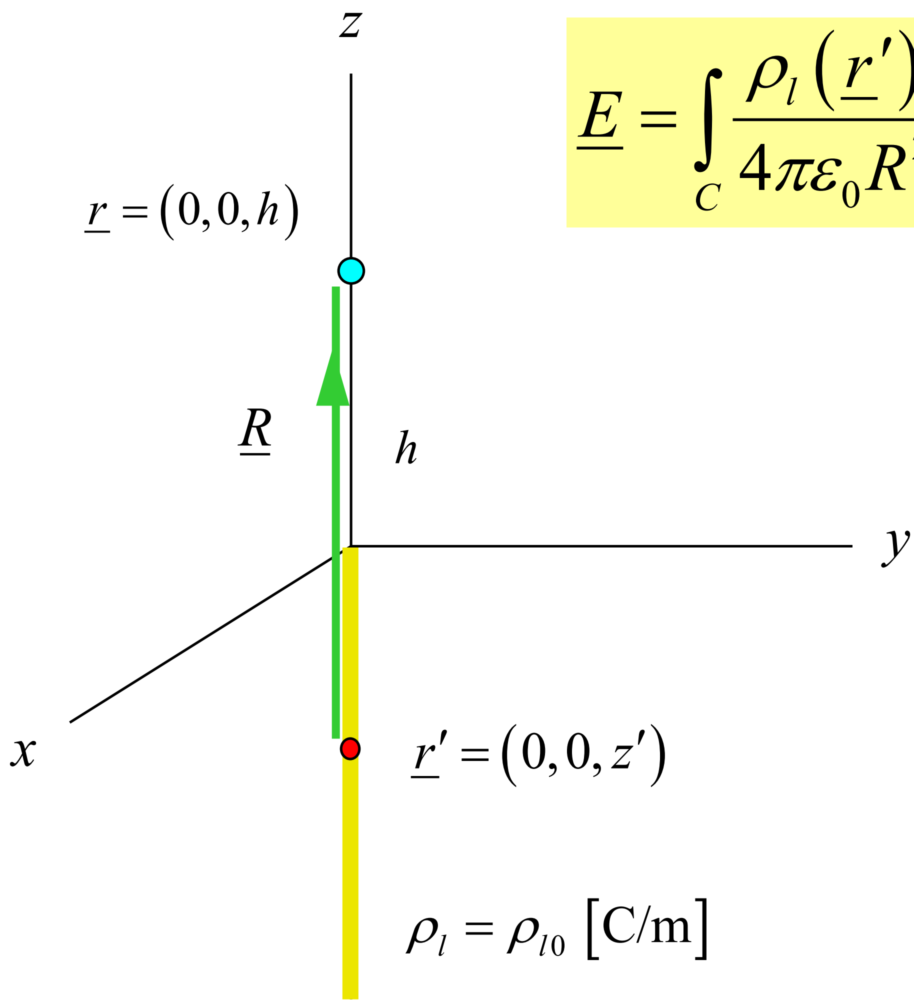
Semi-infinite uniform line charge

Find $\underline{E}(0,0,h)$



Example (cont.)

$$\underline{E} = \int_C \frac{\rho_l(\underline{r}')}{4\pi\varepsilon_0 R^2}\,\underline{\hat{R}}\; dl'$$



$$\underline{R} = \underline{\hat{x}}(x - x') + \underline{\hat{y}}(y - y') + \underline{\hat{z}}(z - z')$$
$$= \underline{\hat{z}}(h - z')$$

$$R = \sqrt{(h - z')^2} = |h - z'| = h - z'$$

$$\underline{\hat{R}} = \underline{\hat{z}}$$

$$dl' = dz'$$

Example (cont.)

$$\underline{E} = \frac{\rho_{l0}}{4\pi\varepsilon_0}\int_{-\infty}^{0}\frac{\hat{\underline{z}}}{\left(h-z'\right)^2}\,dz'$$

$$= \frac{\hat{\underline{z}}\,\rho_{l0}}{4\pi\varepsilon_0}\int_{-\infty}^{0}\frac{1}{\left(h-z'\right)^2}\,dz'$$

$$= \frac{\hat{\underline{z}}\,\rho_{l0}}{4\pi\varepsilon_0}\left[\frac{1}{\left(h-z'\right)}\right]_{-\infty}^{0}$$

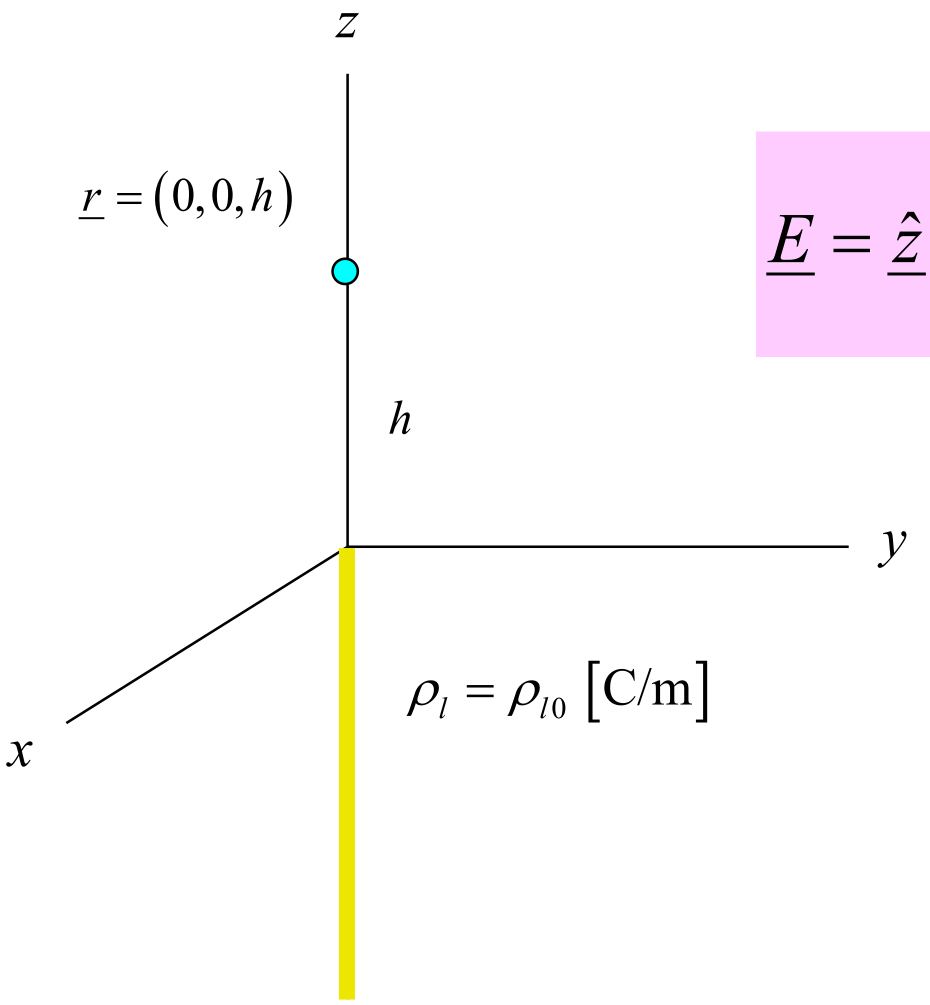
$$= \frac{\hat{\underline{z}}\,\rho_{l0}}{4\pi\varepsilon_0}\left[\frac{1}{h}\right]$$

Note:
The upper limit must be greater than the lower limit to keep dl' positive.

(This is different from voltage drop calculations, where the upper limit can be smaller than the lower limit, after parameterizing the line integral.)

Example (cont.)

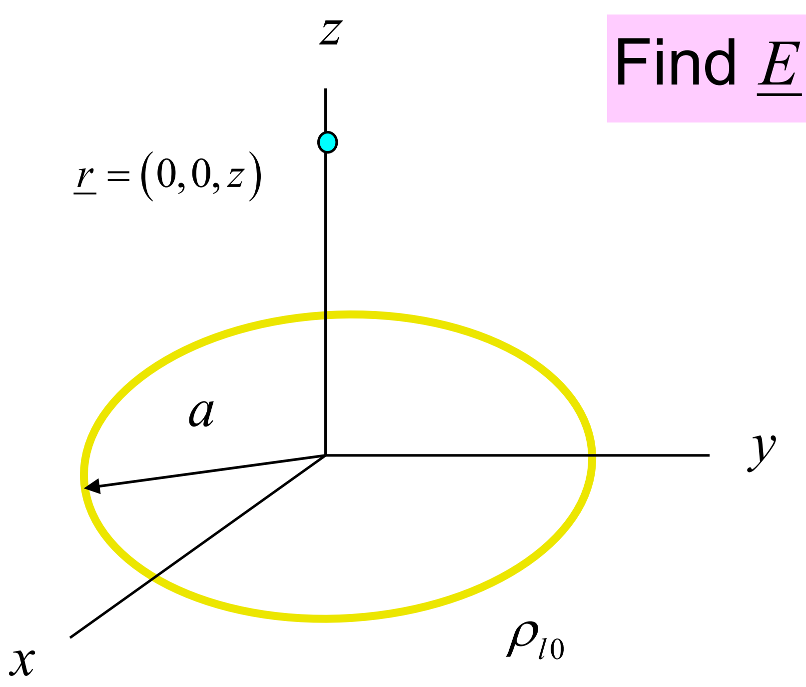
Summary



$$\underline{E} = \underline{\hat{z}}\left(\frac{\rho_{l0}}{4\pi\varepsilon_0 h}\right) \quad [\text{V/m}]$$

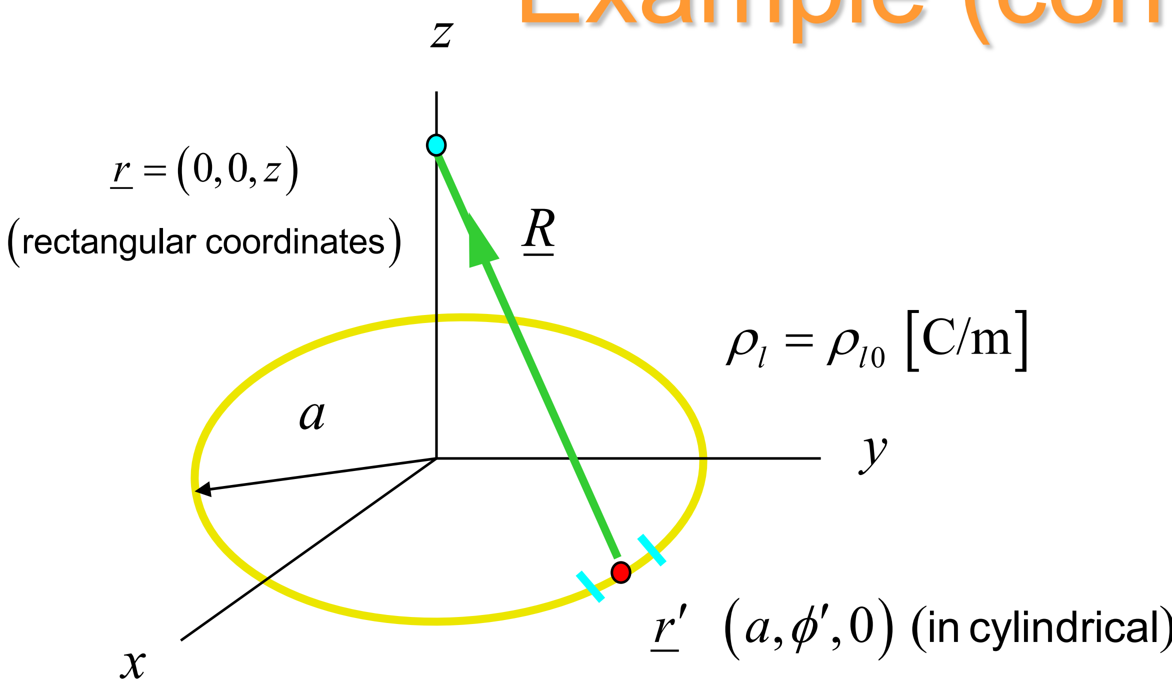
Example

Circular-loop uniform line charge

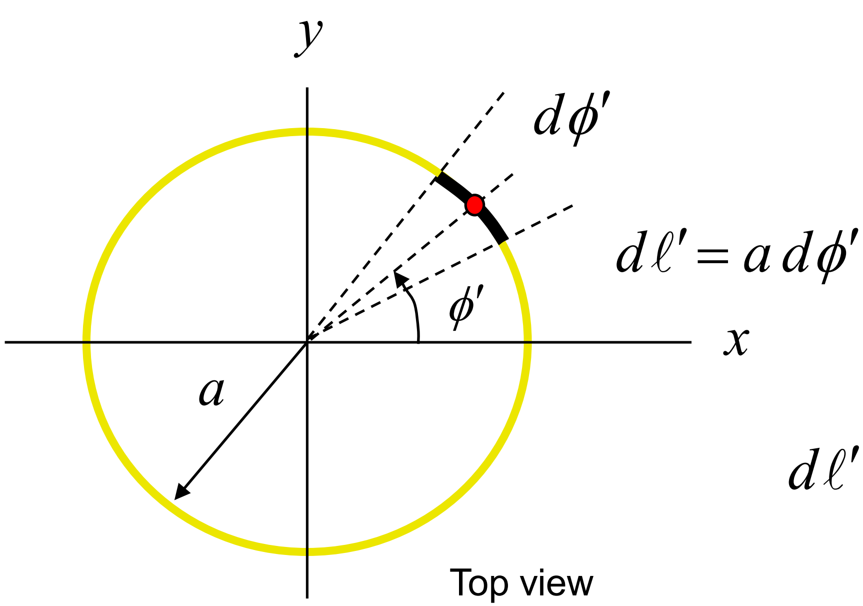


Find $\underline{E}(0,0,z)$

Example (cont.)



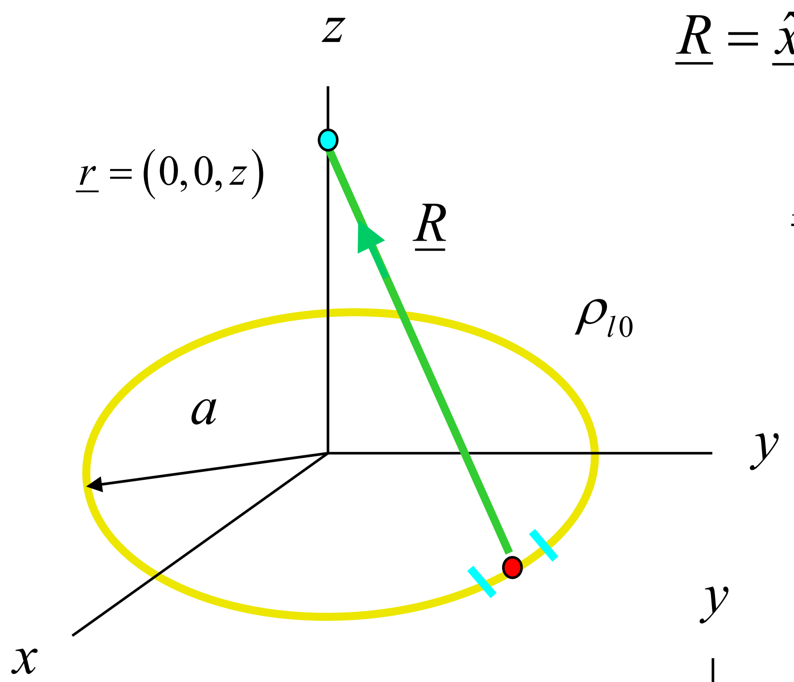
$$\underline{E} = \int_C \frac{\rho_l(\underline{r}')}{4\pi\varepsilon_0 R^2} \hat{\underline{R}} \, dl'$$



Note:
The upper limit must be greater than the lower limit to keep dl' positive.

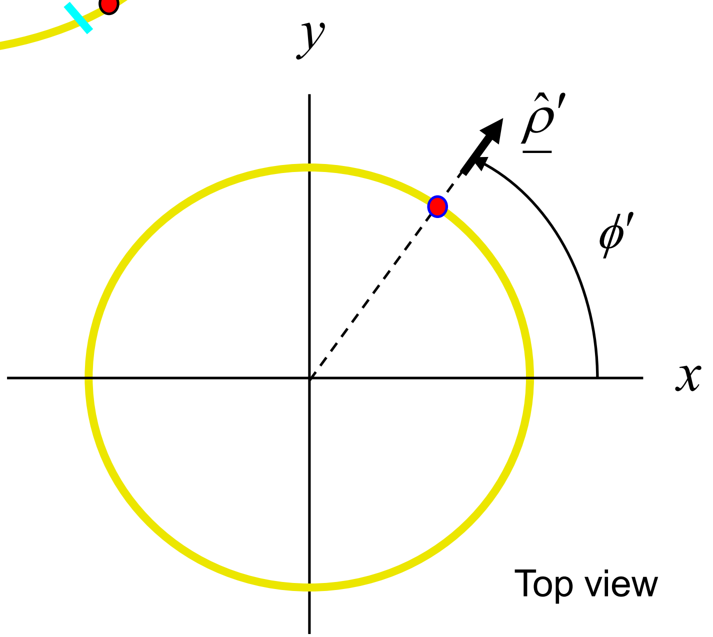
$d\ell' > 0$ so $\int_C (\;) \, d\ell' \rightarrow \int_0^{2\pi} (\;) \, a\, d\phi'$

Example (cont.)



$$\underline{R} = \underline{\hat{x}}\left(x - x'\right) + \underline{\hat{y}}\left(y - y'\right) + \underline{\hat{z}}\left(z - z'\right)$$

$$\underline{R} = \underline{\hat{x}}\left(0 - a\cos\phi'\right) + \underline{\hat{y}}\left(0 - a\sin\phi'\right) + \underline{\hat{z}}\left(z - 0\right)$$
$$= -a\left(\underline{\hat{x}}\cos\phi' + \underline{\hat{y}}\sin\phi'\right) + \underline{\hat{z}}\, z$$



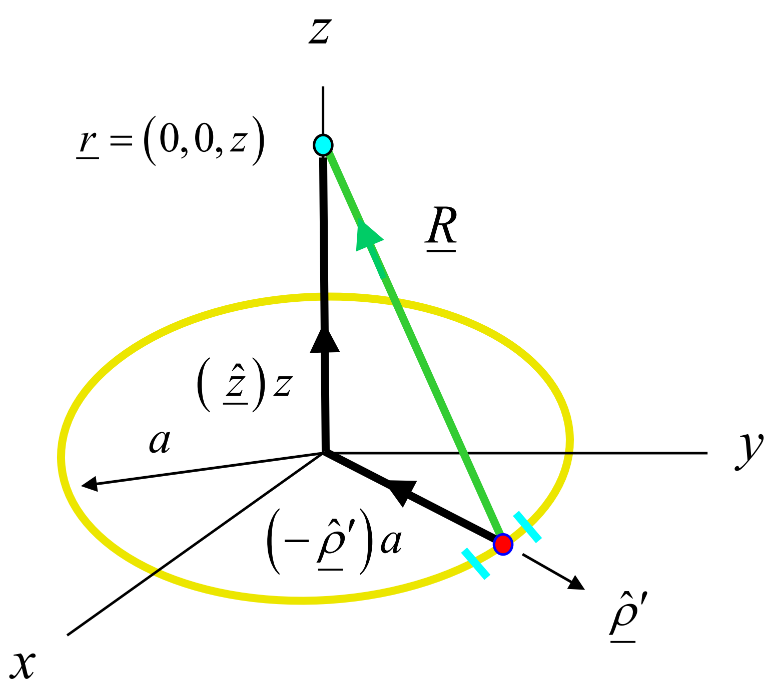
Note:

$$\underline{\hat{\rho}}' = \underline{\hat{x}}\cos\phi' + \underline{\hat{y}}\sin\phi'$$

Hence

$$\underline{R} = -a\,\underline{\hat{\rho}}' + z\,\underline{\hat{z}}$$

Example (cont.)

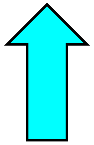


$$\underline{E} = \int_C \frac{\rho_l(\underline{r}')}{4\pi\varepsilon_0 R^2}\, \hat{\underline{R}}\; dl'$$

$$\underline{R} = \left(-\hat{\underline{\rho}}'\right) a + \left(\hat{\underline{z}}\right) z$$

$$R = \sqrt{a^2 + z^2}$$

$$\hat{\underline{R}} = \frac{-a\,\hat{\underline{\rho}}' + z\hat{\underline{z}}}{\sqrt{a^2 + z^2}}$$



We can also get these results geometrically, by simply looking at the picture.

Example (cont.)

Continuing with the calculation…

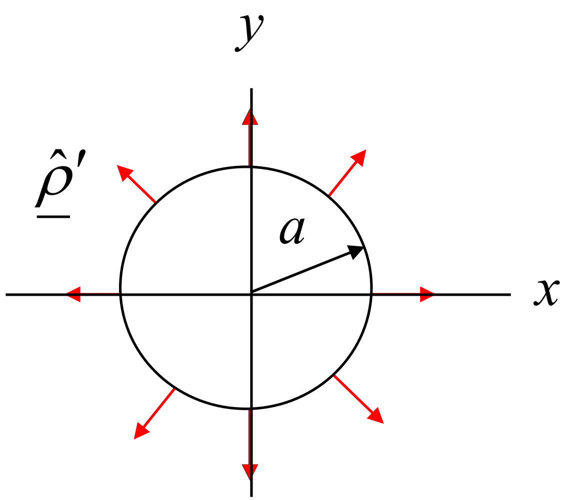
$$\underline{E} = \int_C \frac{\rho_l(\underline{r}')}{4\pi\varepsilon_0 R^2}\,\hat{\underline{R}}\,dl'$$

$$= \int_0^{2\pi} \left(\frac{\rho_{l0}}{4\pi\varepsilon_0}\right)\left(\frac{1}{\left(a^2+z^2\right)}\right)\left(\frac{-a\,\hat{\underline{\rho}}' + z\hat{\underline{z}}}{\left(a^2+z^2\right)^{1/2}}\right) a\,d\phi'$$

$$= \left(\frac{\rho_{l0}}{4\pi\varepsilon_0}\frac{a}{\left(a^2+z^2\right)^{3/2}}\right)\left[-a\int_0^{2\pi}\hat{\underline{\rho}}'\,d\phi' + \hat{\underline{z}}\,z\int_0^{2\pi}d\phi'\right]$$

Reminder: The upper limit must be greater than the lower limit to keep dl' positive.

Example (cont.)



$$\int_0^{2\pi} \underline{\hat{\rho}}' \, d\phi' = 0 \qquad \left(\underline{\hat{\rho}}' = \underline{\hat{x}} \cos\phi' + \underline{\hat{y}} \sin\phi' \right)$$

Also, $\displaystyle\int_0^{2\pi} d\phi' = 2\pi$

Hence

$$\underline{E} = \left(\frac{\rho_{l0}\, a}{4\pi\varepsilon_0 \left(a^2 + z^2\right)^{3/2}} \right) \left[\underline{\hat{z}}\, z \left(2\pi\right) \right]$$

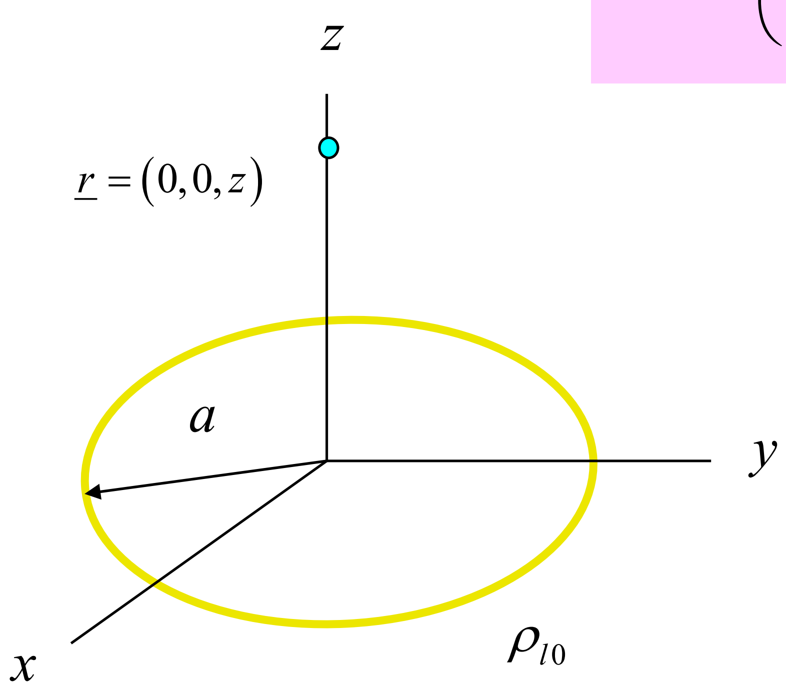
or

$$\underline{E} = \underline{\hat{z}} \left(\frac{\rho_{l0}\, a}{2\,\varepsilon_0} \right) \left(\frac{z}{\left(a^2 + z^2\right)^{3/2}} \right) \qquad [\mathrm{V/m}]$$

Example (cont.)

Summary

$$\underline{E} = \underline{\hat{z}} \left(\frac{\rho_{l0}\, a}{2\,\varepsilon_0} \right) \left(\frac{z}{\left(a^2 + z^2\right)^{3/2}} \right) \quad [\text{V/m}]$$



Example (cont.)

Limiting case: $z \to \infty$

$$\underline{E} = \underline{\hat{z}}\left(\frac{\rho_{l0}\,a}{2\varepsilon_0}\right)\left(\frac{z}{\left(z^2\right)^{3/2}\left(1+a^2/z^2\right)^{3/2}}\right) \to \underline{\hat{z}}\left(\frac{\rho_{l0}\,a}{2\varepsilon_0}\right)\left(\frac{z}{\left(z^2\right)^{3/2}}\right) = \underline{\hat{z}}\left(\frac{\rho_{l0}\,a}{2\varepsilon_0}\right)\left(\frac{z}{\left|z^3\right|}\right)$$

$$= \underline{\hat{z}}\left(\frac{\rho_{l0}\,a}{2\varepsilon_0}\right)\left(\pm\frac{1}{z^2}\right) \quad \left(+\text{ when } z>0, \quad -\text{ when } z<0\right)$$

$$= \underline{\hat{z}}\,\frac{\rho_{l0}\left(2\pi a\right)}{4\pi\varepsilon_0}\,\frac{1}{z^2}\left(\pm 1\right)$$

$$= \pm\underline{\hat{z}}\,\frac{Q}{4\pi\varepsilon_0\,z^2} \quad \left(Q = \rho_{l0}\left(2\pi a\right)\right)$$

$$= \underline{\hat{r}}\,\frac{Q}{4\pi\varepsilon_0\,r^2} \quad \left(\underline{\hat{r}} = +\underline{\hat{z}} \text{ when } z>0, \quad \underline{\hat{r}} = -\underline{\hat{z}} \text{ when } z<0\right)$$

The loop looks like a point charge when we are very far away!