ECE 3318
Applied Electricity and Magnetism

Spring 2023

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Dept. of ECE

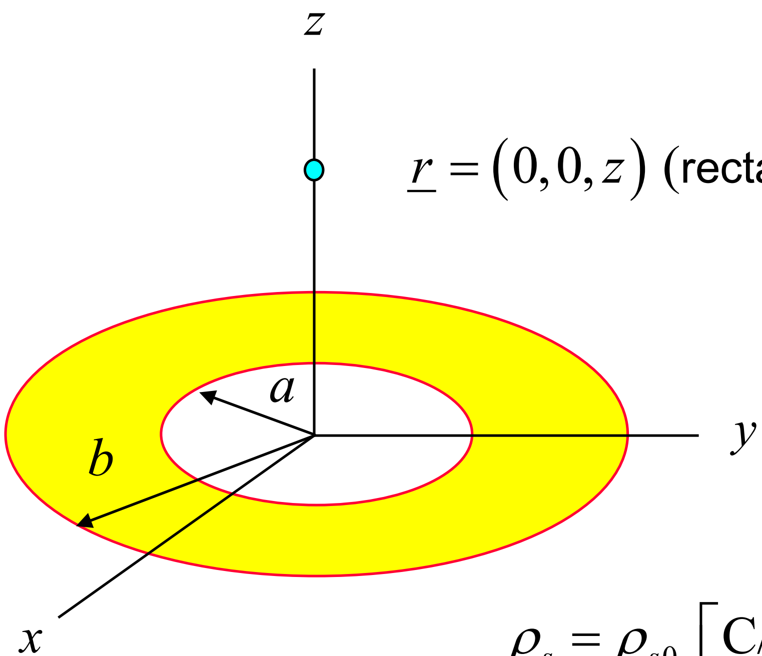
Notes 8
Coulomb's Law II

Notes prepared by the EM Group
University of Houston

Example

Annulus of uniform surface charge density

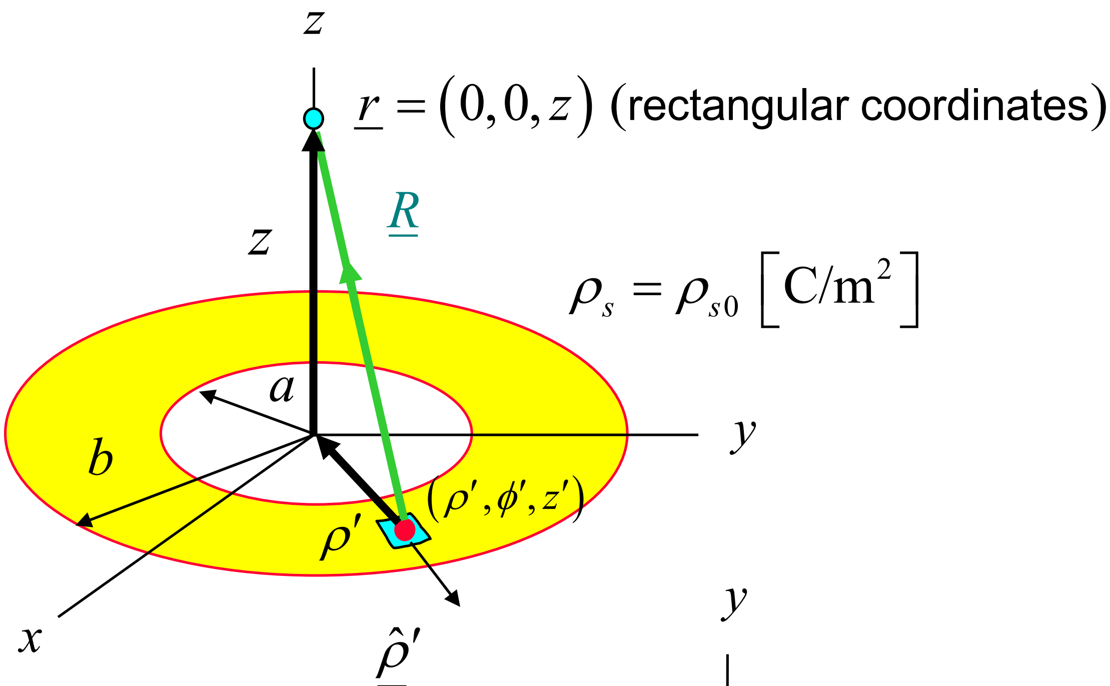
Find $\underline{E}(0,0,z)$



$\underline{r} = (0,0,z)$ (rectangular coordinates)

$$\rho_s = \rho_{s0} \left[\text{C/m}^2\right]$$

Example (cont.)



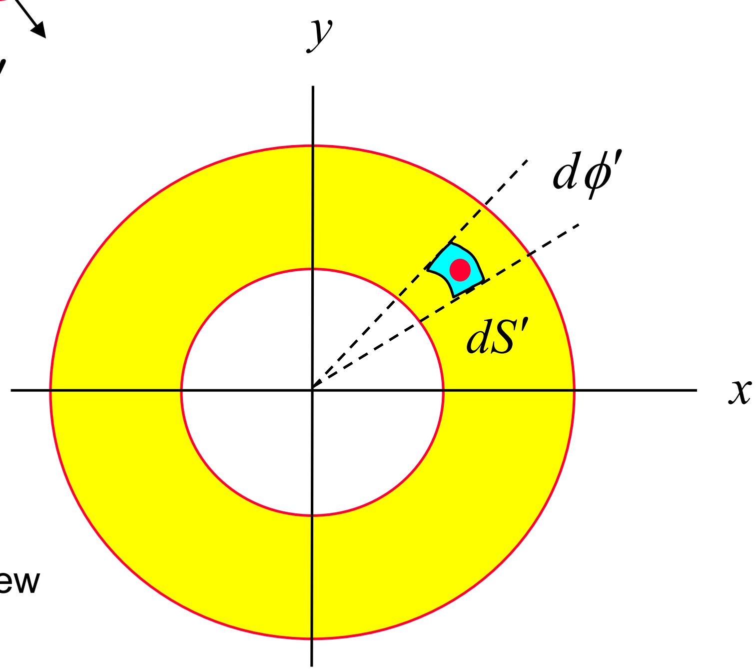
$$\underline{E} = \int_S \frac{\rho_s(\underline{r}')}{4\pi\varepsilon_0 R^2}\,\underline{\hat{R}}\;dS'$$

$$dS' = \rho'\,d\rho'\,d\phi'$$

$$\underline{R} = (-\underline{\hat{\rho}}')\,\rho' + (\underline{\hat{z}})\,z$$

$$R = \sqrt{\rho'^2 + z^2}$$

$$\underline{\hat{R}} = \frac{-\rho'\underline{\hat{\rho}}' + \underline{\hat{z}}z}{\sqrt{\rho'^2 + z^2}}$$



Top view

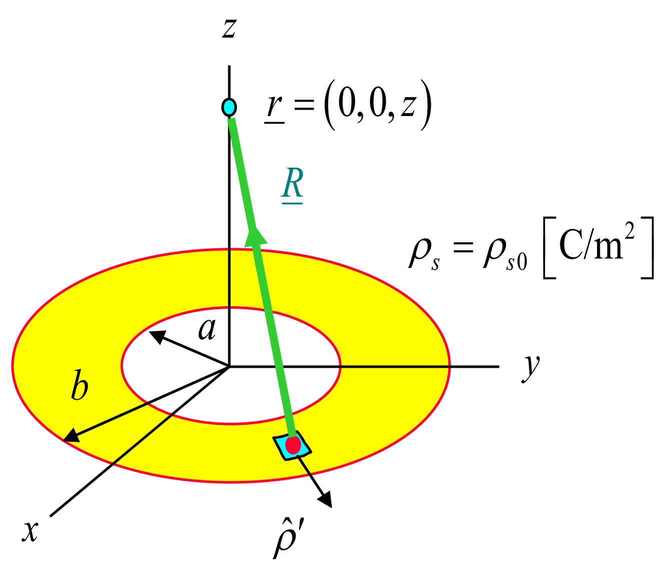
Example (cont.)

$$\underline{E} = \int_S \frac{\rho_s(\underline{r}')}{4\pi\varepsilon_0 R^2}\,\hat{\underline{R}}\;dS'$$

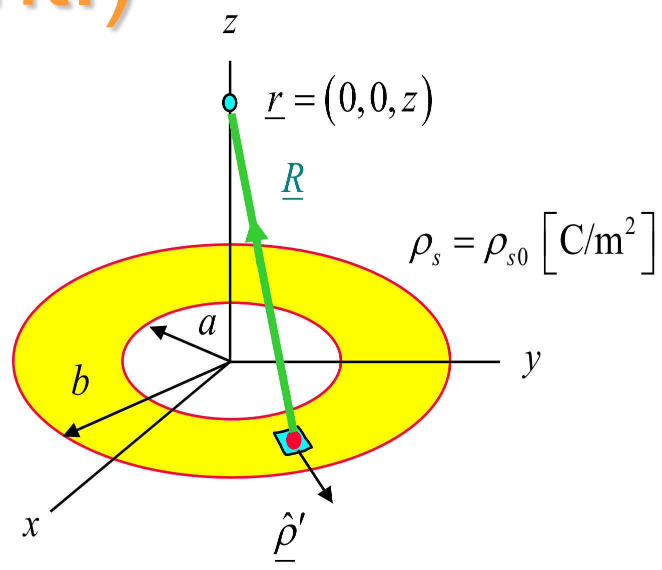
Note:
The upper limits are larger than the lower limits, in order to keep the differential lengths (defining the differential area dS') positive.

$$\underline{E} = \int_0^{2\pi}\int_a^b \left(\frac{\rho_{s0}}{4\pi\varepsilon_0}\right)\left(\frac{1}{\left(\rho'^2+z^2\right)}\right)\left(\frac{-\rho'\hat{\underline{\rho}}'+\hat{\underline{z}}z}{\sqrt{\rho'^2+z^2}}\right)\rho'\,d\rho'\,d\phi'$$

$$= \int_0^{2\pi}\int_a^b \left(\frac{\rho_{s0}}{4\pi\varepsilon_0}\right)\left(\frac{-\rho'\hat{\underline{\rho}}'+\hat{\underline{z}}z}{\left(\rho'^2+z^2\right)^{3/2}}\right)\rho'\,d\rho'\,d\phi'$$



Example (cont.)



$$\underline{E} = \int_0^{2\pi}\int_a^b \left(\frac{\rho_{s0}}{4\pi\varepsilon_0}\right)\left(\frac{-\rho'\hat{\underline{\rho}}' + \hat{\underline{z}}z}{\left(\rho'^2 + z^2\right)^{3/2}}\right)\rho'\, d\rho'\, d\phi'$$

Switch integration order and do ϕ' integral first:

function of ϕ'

$$\underline{E} = \int_a^b\int_0^{2\pi} \left(\frac{\rho_{s0}}{4\pi\varepsilon_0}\right)\left(\frac{-\rho'\hat{\underline{\rho}}' + \hat{\underline{z}}z}{\left(\rho'^2 + z^2\right)^{3/2}}\right)\rho'\, d\phi'\, d\rho'$$

Note that $$\int_0^{2\pi} \hat{\underline{\rho}}'\, d\phi' = 0$$

Note:
By doing the ϕ' integral first and realizing that it is zero for the first term, we save a lot of work.

Example (cont.)

Hence: $$\underline{E} = \underline{\hat{z}}\, z \int_a^b \int_0^{2\pi} \left(\frac{\rho_{s0}}{4\pi\varepsilon_0}\right) \left(\frac{\rho'}{\left(\rho'^2 + z^2\right)^{3/2}}\right) d\phi'\, d\rho'$$

Note:
We have a separable integrand with fixed limits.
(We can split this double integral into the product of two one-dimensional integrals.)

$$\underline{E} = \left(\frac{\rho_{s0}}{4\pi\varepsilon_0}\right) \underline{\hat{z}}\, z \int_a^b \left(\frac{\rho'}{\left(\rho'^2 + z^2\right)^{3/2}}\right) d\rho' \int_0^{2\pi} d\phi'$$

$$\underline{E} = \left(\frac{\rho_{s0}}{4\pi\varepsilon_0}\right) \underline{\hat{z}}\, z \int_a^b \frac{\rho'}{\left(\rho'^2 + z^2\right)^{3/2}} d\rho' \, (2\pi)$$

Example (cont.)

$$\underline{E} = 2\pi \left(\frac{\rho_{s0}}{4\pi\varepsilon_0} \right) \underline{\hat{z}}\, z \int_a^b \frac{\rho'}{\left(\rho'^2 + z^2 \right)^{3/2}} \, d\rho'$$

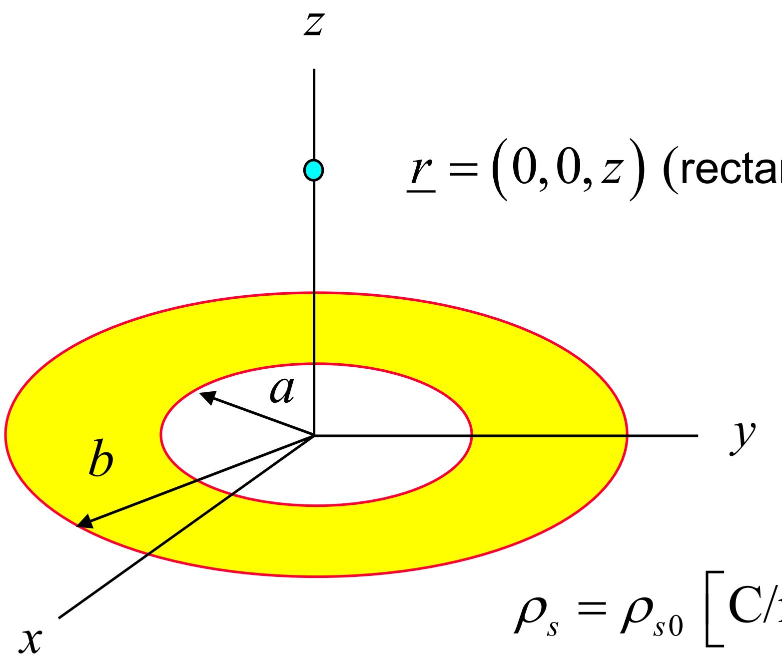
$$\underline{E} = 2\pi \left(\frac{\rho_{s0}}{4\pi\varepsilon_0} \right) \underline{\hat{z}}\, z \left[\frac{-1}{\sqrt{\rho'^2 + z^2}} \right]_a^b$$

$$\underline{E} = \left(\frac{\rho_{s0}}{2\varepsilon_0} \right) \underline{\hat{z}}\, z \left[\frac{1}{\sqrt{a^2 + z^2}} - \frac{1}{\sqrt{b^2 + z^2}} \right]$$

Example (cont.)

Summary

$$\underline{E} = \underline{\hat{z}}\left(\frac{\rho_{s0}}{2\varepsilon_0}\right)\left[\frac{z}{\sqrt{a^2+z^2}} - \frac{z}{\sqrt{b^2+z^2}}\right] \quad [\mathrm{V/m}]$$



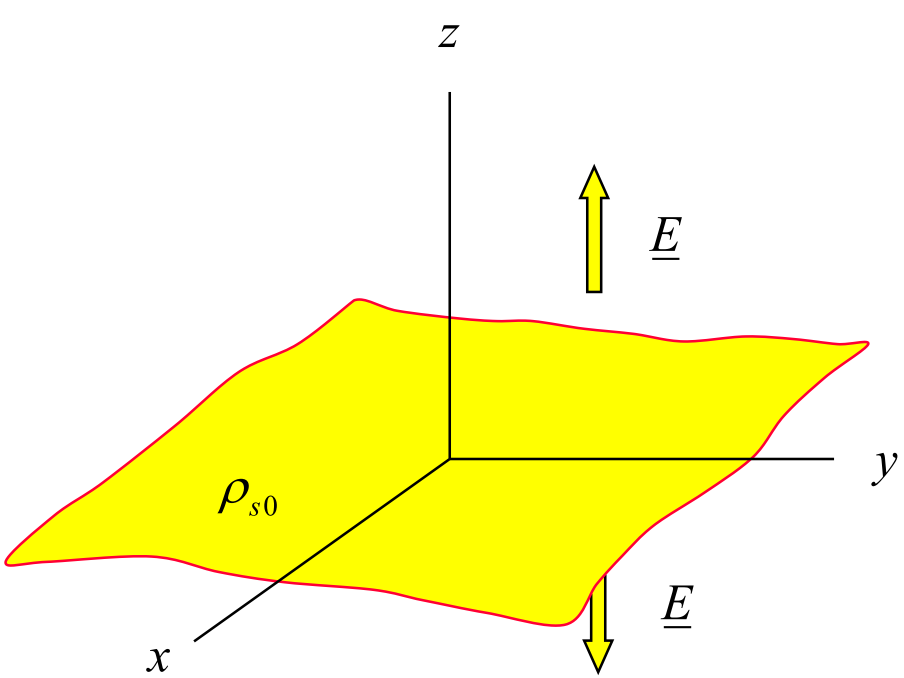
$\underline{r} = (0,0,z)$ (rectangular coordinates)

$\rho_s = \rho_{s0}\ [\mathrm{C/m^2}]$

Example (cont.)

Limiting case: $a \to 0,\ b \to \infty$ (infinite sheet of surface charge)

$$\underline{E} = \left(\frac{\rho_{s0}}{2\varepsilon_0}\right)\underline{\hat{z}}\left[\frac{z}{\sqrt{a^2+z^2}} - \frac{z}{\sqrt{b^2+z^2}}\right]$$



$a \to 0,\ b \to \infty$

$$\underline{E} = \underline{\hat{z}}\left(\frac{\rho_{s0}}{2\varepsilon_0}\right)\frac{z}{|z|}$$

or

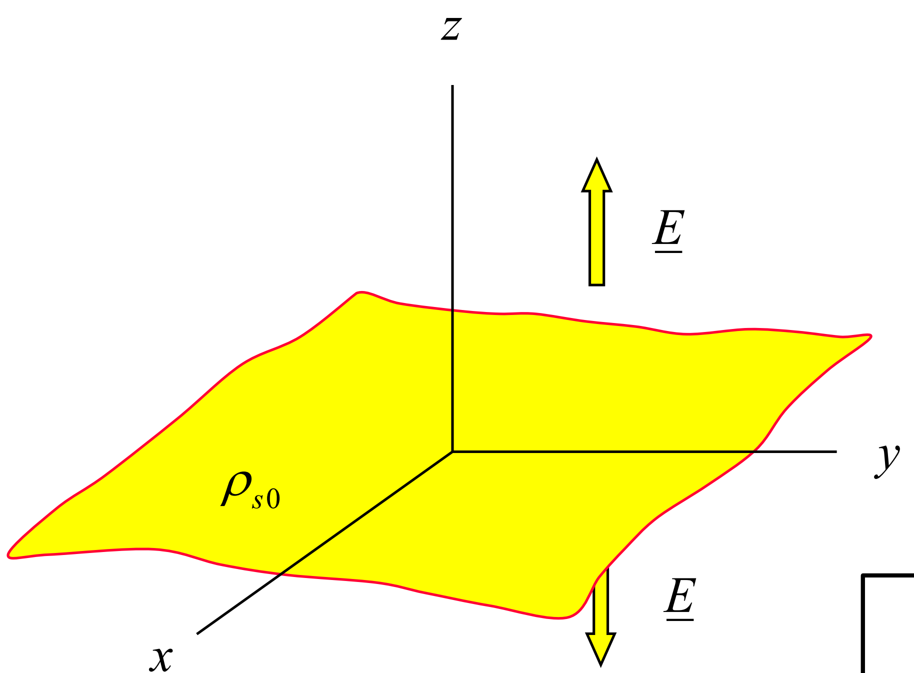
$$\underline{E} = \pm\,\underline{\hat{z}}\left(\frac{\rho_{s0}}{2\varepsilon_0}\right)$$

+ for $z > 0$

- for $z < 0$

Example (cont.)

Summary: Infinite sheet of uniform surface charge density



$$\underline{E} = \pm\,\underline{\hat{z}}\left(\frac{\rho_{S0}}{2\varepsilon_0}\right) \quad [\mathrm{V/m}]$$

$+$ for $z > 0$
$-$ for $z < 0$

Note:
The magnitude of the electric field does not depend on distance from the sheet!

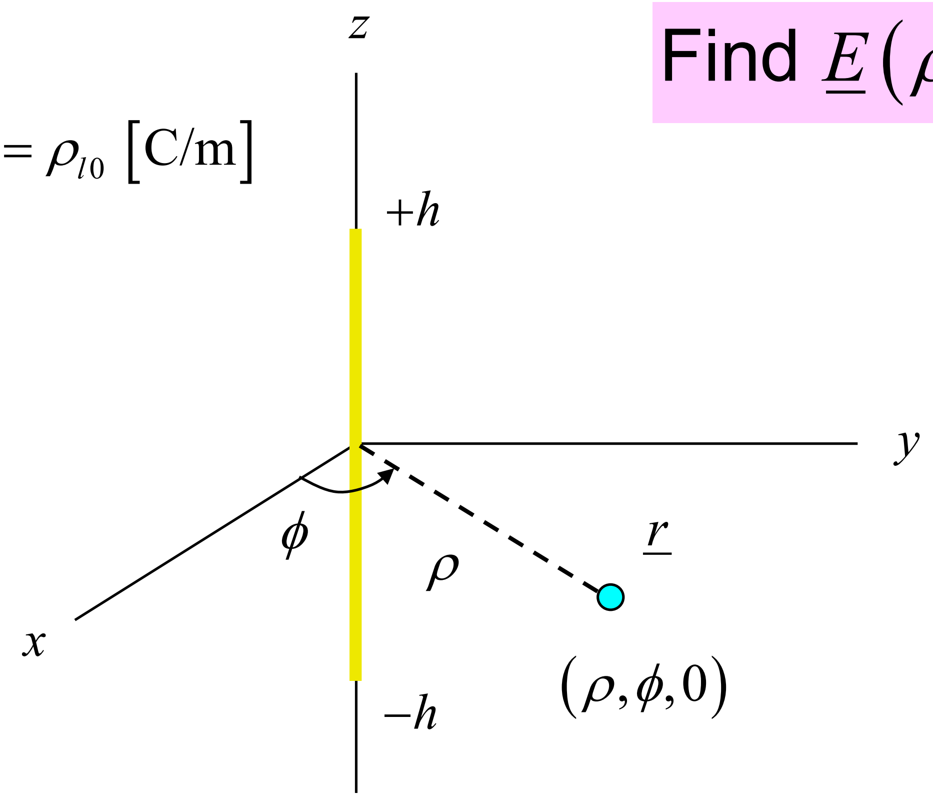
Note:
The electric field vector is ***discontinuous*** across a <u>sheet</u> of surface charge density.

Example

Finite-length uniform line charge

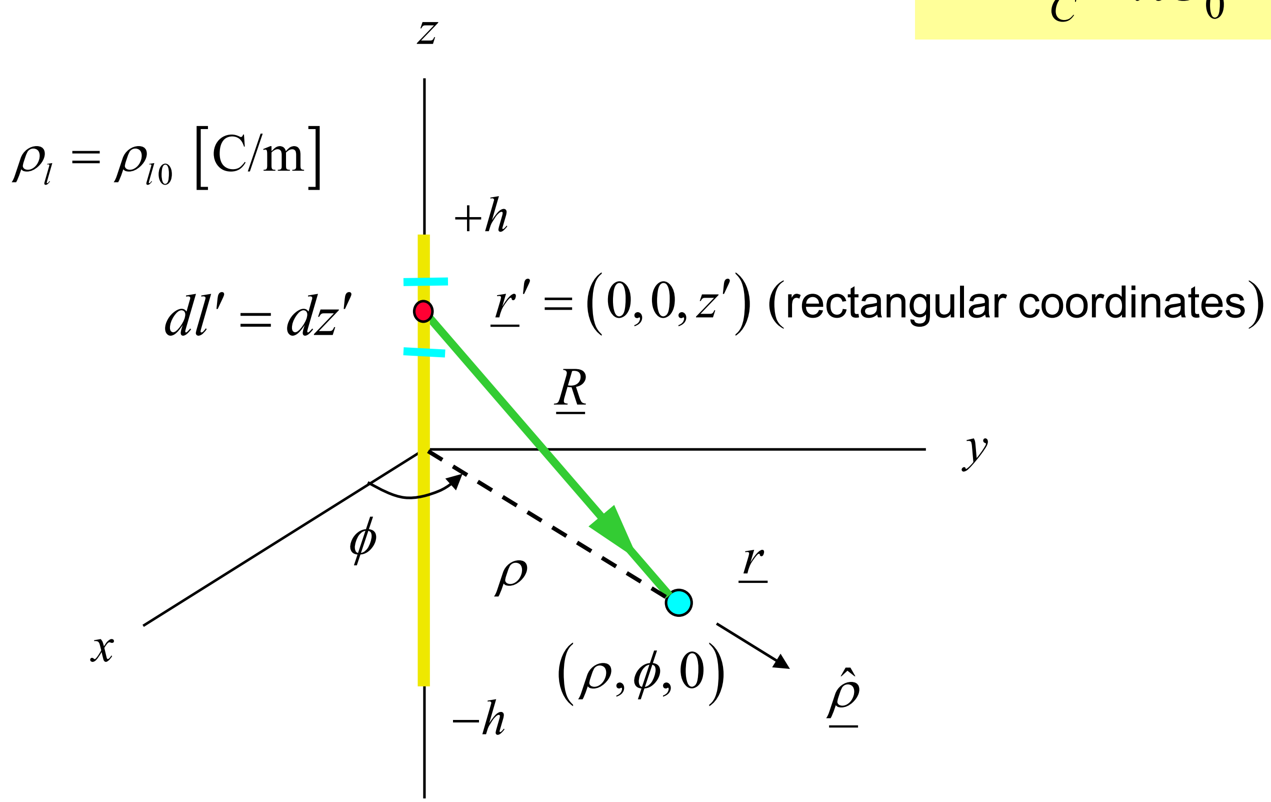
Find $\underline{E}(\rho, \phi, 0)$

$\rho_l = \rho_{l0}$ [C/m]



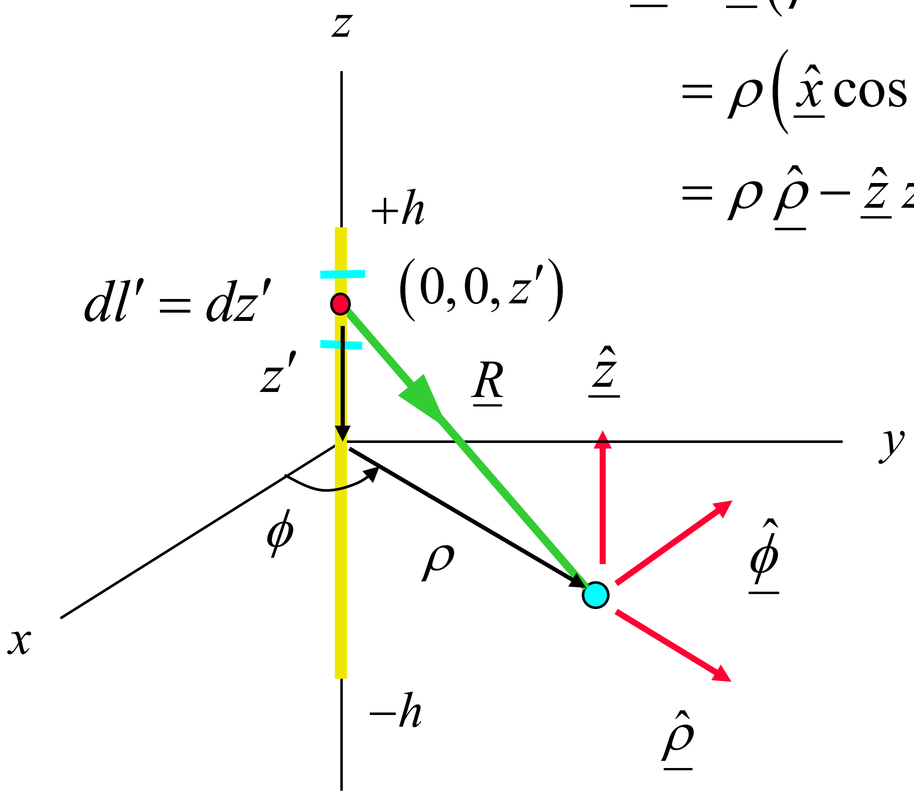
Example (cont.)

$$\underline{E} = \int_C \frac{\rho_l\left(\underline{r}'\right)}{4\pi\varepsilon_0 R^2}\,\hat{\underline{R}}\,dl'$$



Example (cont.)

$$\underline{R} = \underline{\hat{x}}(x - x') + \underline{\hat{y}}(y - y') + \underline{\hat{z}}(z - z')$$



$$\begin{aligned}\underline{R} &= \underline{\hat{x}}(\rho\cos\phi - 0) + \underline{\hat{y}}(\rho\sin\phi - 0) + \underline{\hat{z}}(0 - z') \\ &= \rho\left(\underline{\hat{x}}\cos\phi + \underline{\hat{y}}\sin\phi\right) - \underline{\hat{z}}z' \\ &= \rho\,\underline{\hat{\rho}} - \underline{\hat{z}}\,z'\end{aligned}$$

Geometric way:

$$\underline{R} = \left(\underline{\hat{\rho}}\right)\rho + \left(-\underline{\hat{z}}\right)z'$$

$$R = \left|\underline{R}\right| = \sqrt{\rho^2 + (-z')^2}$$

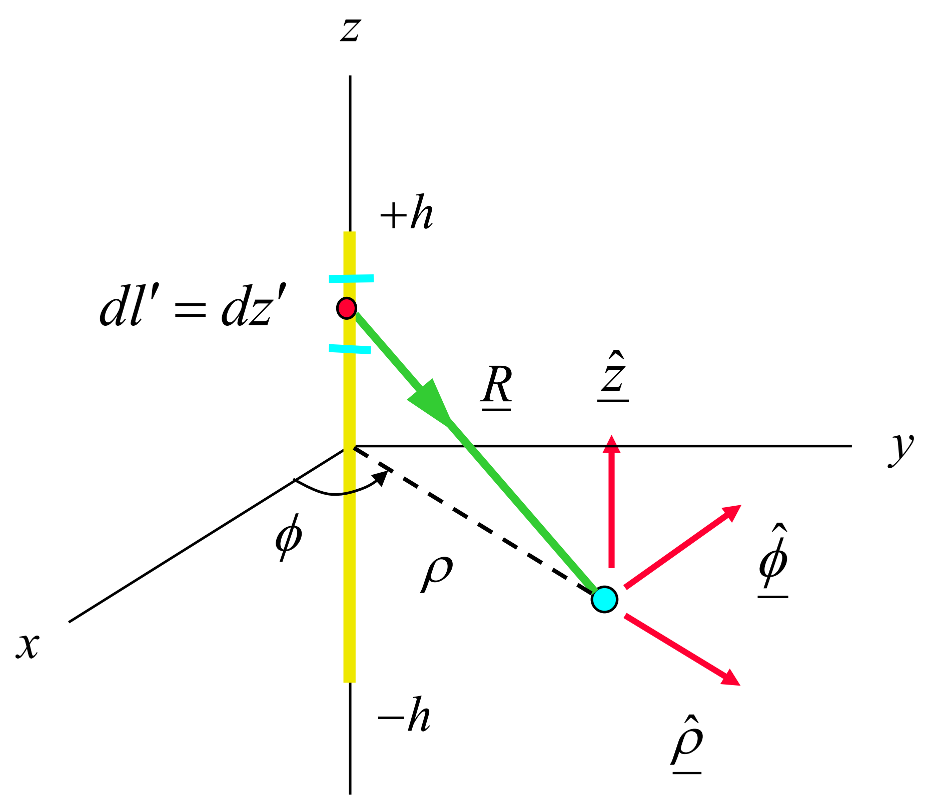
$$\underline{\hat{R}} = \frac{\underline{R}}{\left|\underline{R}\right|} = \frac{\underline{\hat{\rho}}\,\rho - \underline{\hat{z}}\,z'}{\sqrt{\rho^2 + z'^2}}$$

Example (cont.)

$$\underline{E} = \int_C \frac{\rho_l\left(\underline{r}'\right)}{4\pi\varepsilon_0 R^2}\,\hat{\underline{R}}\,dl'$$

$$\underline{E} = \int_{-h}^{+h} \left(\frac{\rho_{l0}}{4\pi\varepsilon_0}\right)\left(\frac{1}{\left(\rho^2 + z'^2\right)}\right)\left(\frac{\rho\,\hat{\underline{\rho}} - \hat{\underline{z}}\,z'}{\sqrt{\rho^2 + z'^2}}\right)dz'$$

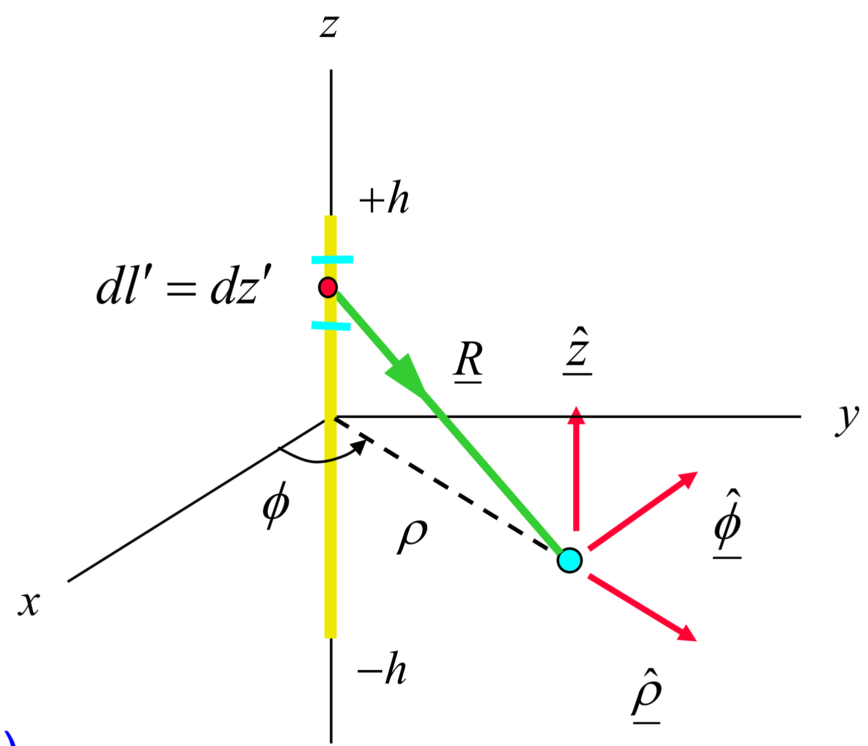
$$\underline{E} = \int_{-h}^{+h} \left(\frac{\rho_{l0}}{4\pi\varepsilon_0}\right)\left(\frac{\rho\,\hat{\underline{\rho}} - \hat{\underline{z}}\,z'}{\left(\rho^2 + z'^2\right)^{3/2}}\right)dz'$$



Note:
The upper limit is larger than the lower limit, in order to keep the differential length positive.

Example (cont.)

$$\underline{E} = \left(\frac{\rho_{l0}}{4\pi\varepsilon_0}\right)\int_{-h}^{+h}\frac{\rho\,\underline{\hat{\rho}} - \underline{\hat{z}}\,z'}{\left(\rho^2 + z'^2\right)^{3/2}}\,dz'$$



For the second term, note that by symmetry,

$$\int_{-h}^{+h}\left(\frac{z'}{\left(\rho^2 + z'^2\right)^{3/2}}\right)dz' = 0$$

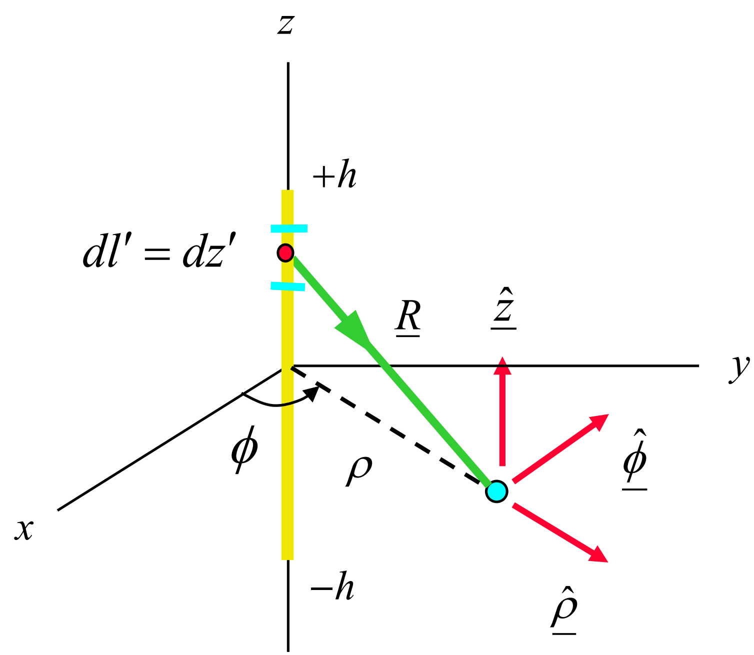
(The integrand is an <u>odd</u> function of z'.)

Hence, we have

$$\underline{E} = \left(\frac{\rho_{l0}}{4\pi\varepsilon_0}\right)\underline{\hat{\rho}}\,\rho\int_{-h}^{+h}\left(\frac{1}{\left(\rho^2 + z'^2\right)^{3/2}}\right)dz'$$

Example (cont.)

$$\underline{E} = \left(\frac{\rho_{l0}}{4\pi\varepsilon_0}\right)\underline{\hat{\rho}}\,\rho \int_{-h}^{+h}\left(\frac{1}{\left(\rho^2 + z'^2\right)^{3/2}}\right)dz'$$



Because the integrand is even,

$$\underline{E} = 2\left(\frac{\rho_{l0}}{4\pi\varepsilon_0}\right)\underline{\hat{\rho}}\,\rho \int_{0}^{+h}\left(\frac{1}{\left(\rho^2 + z'^2\right)^{3/2}}\right)dz'$$

$$= 2\left(\frac{\rho_{l0}}{4\pi\varepsilon_0}\right)\underline{\hat{\rho}}\,\rho \left[\frac{z'}{\rho^2\sqrt{\rho^2 + z'^2}}\right]_0^h$$

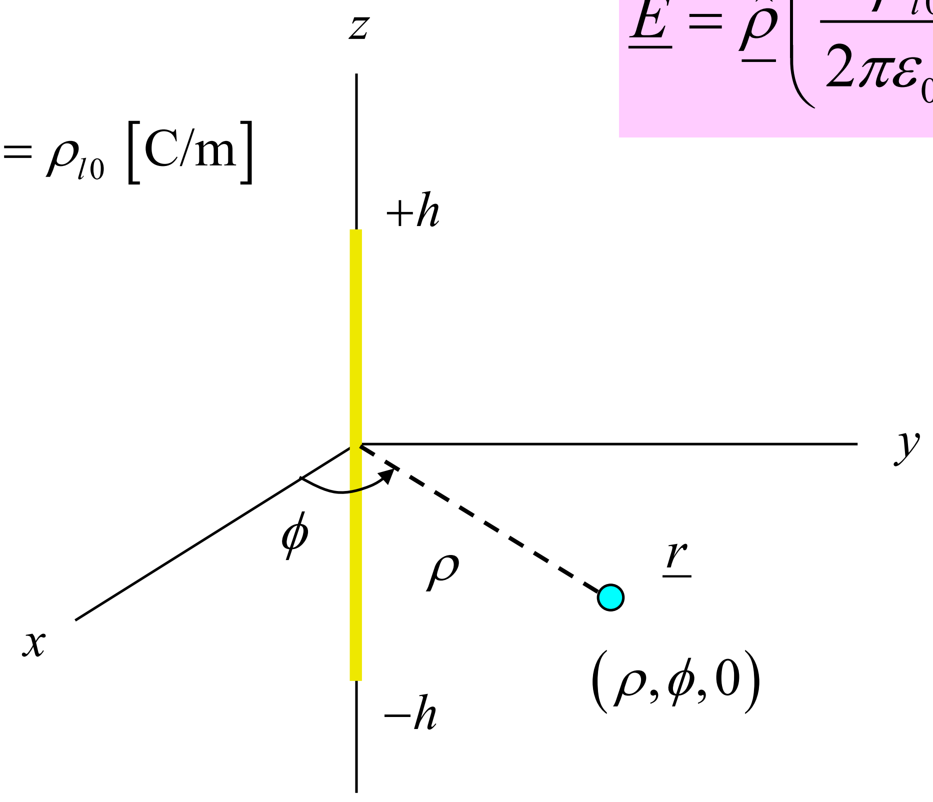
$$= 2\left(\frac{\rho_{l0}}{4\pi\varepsilon_0}\right)\underline{\hat{\rho}}\,\rho \left[\frac{h}{\rho^2\sqrt{\rho^2 + h^2}}\right]$$

Example (cont.)

Summary

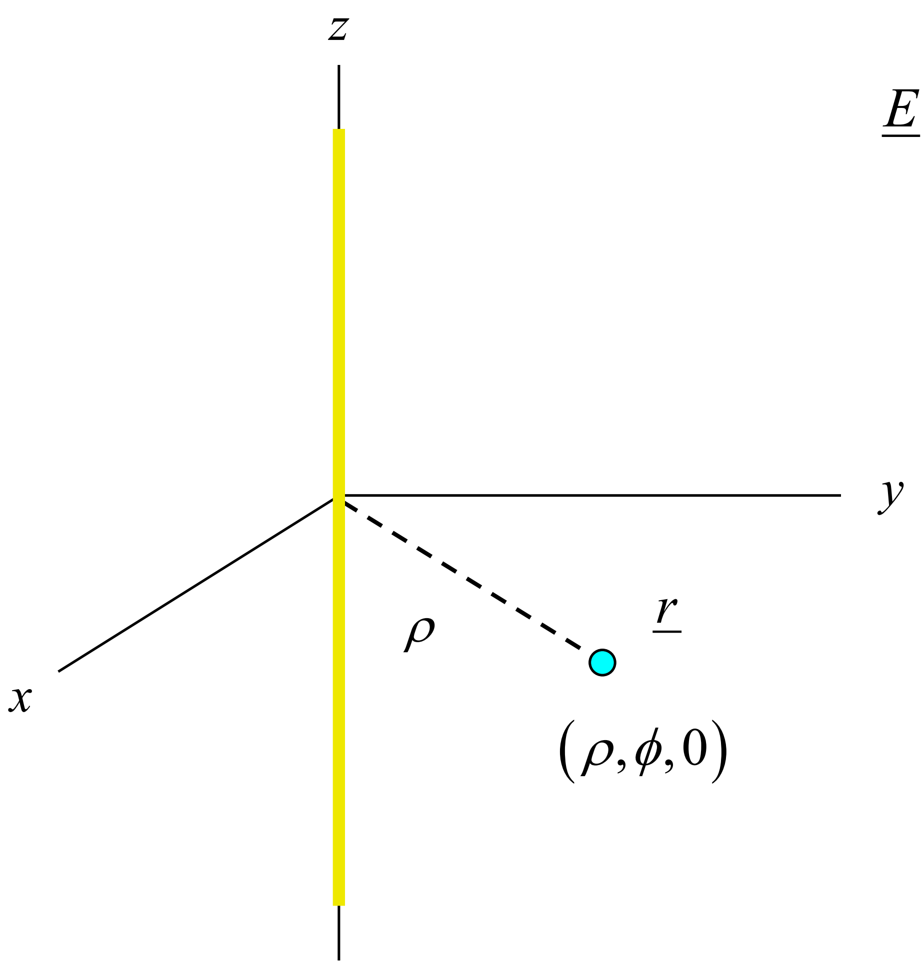
$$\underline{E} = \underline{\hat{\rho}}\left(\frac{\rho_{l0}}{2\pi\varepsilon_0\rho}\right)\left[\frac{h}{\sqrt{\rho^2+h^2}}\right] \quad [\text{V/m}]$$

$\rho_l = \rho_{l0}\ [\text{C/m}]$



Example (cont.)

Limiting case: let $h \to \infty$



$$\underline{E} = \underline{\hat{\rho}} \left(\frac{\rho_{l0}}{2\pi\varepsilon_0 \rho} \right) \frac{h}{h\sqrt{1+\left(\frac{\rho}{h}\right)^2}}$$

$h \to \infty$

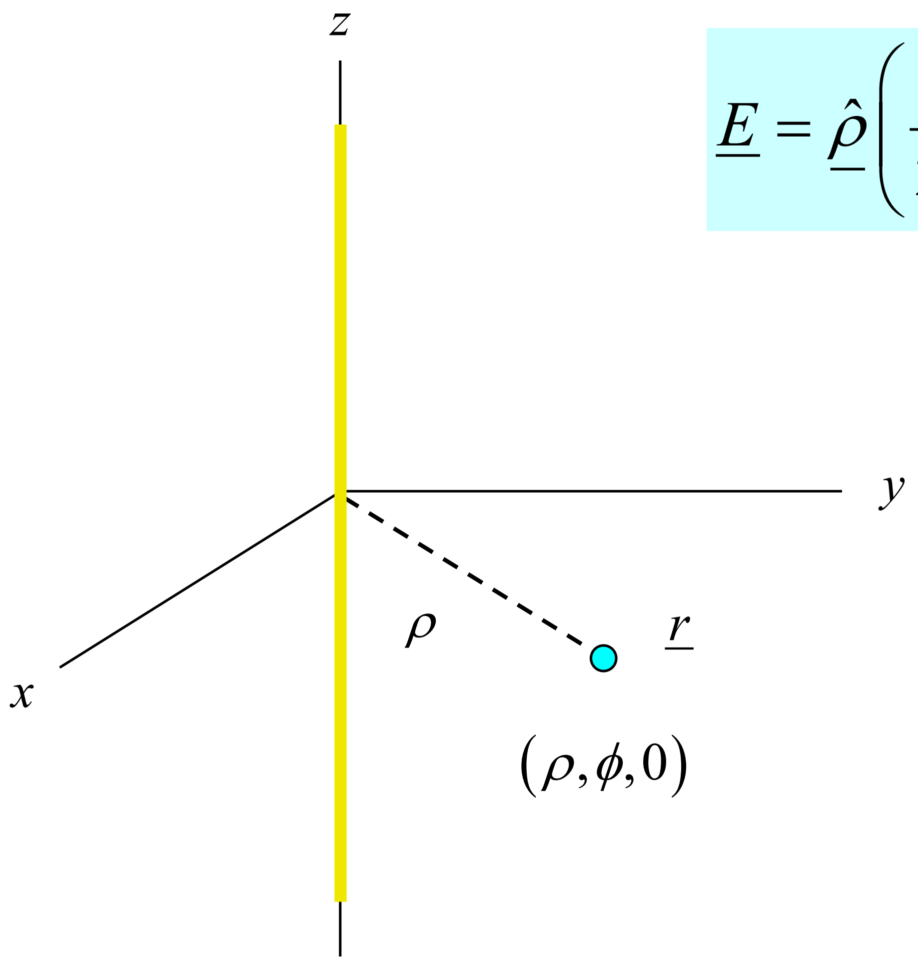
$$\underline{E} = \underline{\hat{\rho}} \left(\frac{\rho_{l0}}{2\pi\varepsilon_0 \rho} \right) \quad [\text{V/m}]$$

Infinite uniform line charge

Example (cont.)

Summary: Infinite uniform line charge density

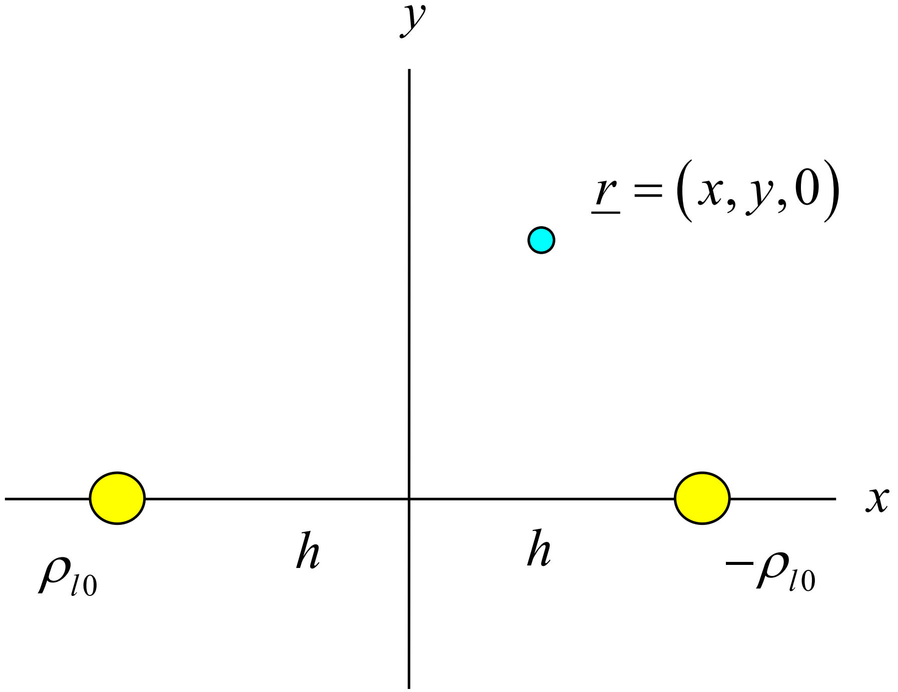
$$\underline{E} = \underline{\hat{\rho}}\left(\frac{\rho_{l0}}{2\pi\varepsilon_0\rho}\right) \quad [\mathrm{V/m}]$$



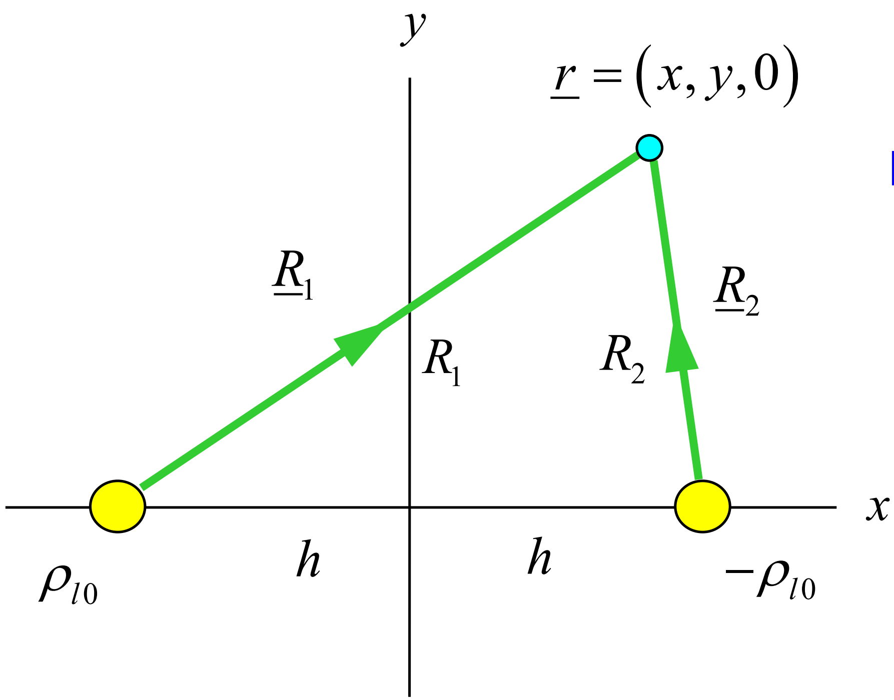
Example

Two infinite uniform line charges

Find $\underline{E}(x, y, 0)$



Example (cont.)



For a single line charge ρ_{l0} on the z axis:

$$\underline{E} = \underline{\hat{\rho}} \left(\frac{\rho_{l0}}{2\pi\varepsilon_0 \rho} \right)$$

Let

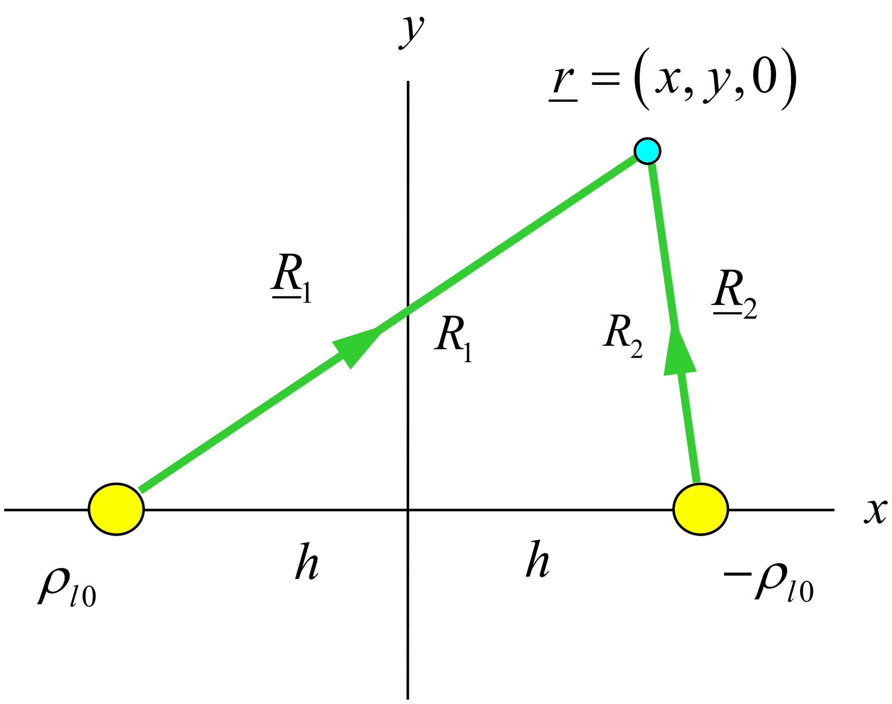
$$\underline{\hat{\rho}} \rightarrow \underline{\hat{R}}_i \quad (i = 1, 2)$$

$$\rho \rightarrow R_i = \left| \underline{R}_i \right| \quad (i = 1, 2)$$

From superposition:

$$\underline{E} = \underline{\hat{R}}_1 \left(\frac{\rho_{l0}}{2\pi\varepsilon_0 R_1} \right) + \underline{\hat{R}}_2 \left(\frac{-\rho_{l0}}{2\pi\varepsilon_0 R_2} \right)$$

Example (cont.)



$$\underline{E} = \hat{\underline{R}}_1 \left(\frac{\rho_{l0}}{2\pi\varepsilon_0 R_1} \right) + \hat{\underline{R}}_2 \left(\frac{-\rho_{l0}}{2\pi\varepsilon_0 R_2} \right)$$

$$\underline{R}_1 = \hat{\underline{x}}\left(x - (-h)\right) + \hat{\underline{y}}\left(y - 0\right)$$

$$\underline{R}_2 = \hat{\underline{x}}\left(x - h\right) + \hat{\underline{y}}\left(y - 0\right)$$

or

$$\underline{R}_1 = \hat{\underline{x}}\left(x + h\right) + \hat{\underline{y}}\left(y\right)$$

$$\underline{R}_2 = \hat{\underline{x}}\left(x - h\right) + \hat{\underline{y}}\left(y\right)$$

Hence, we have:

$$R_1 = \sqrt{(x+h)^2 + y^2}$$

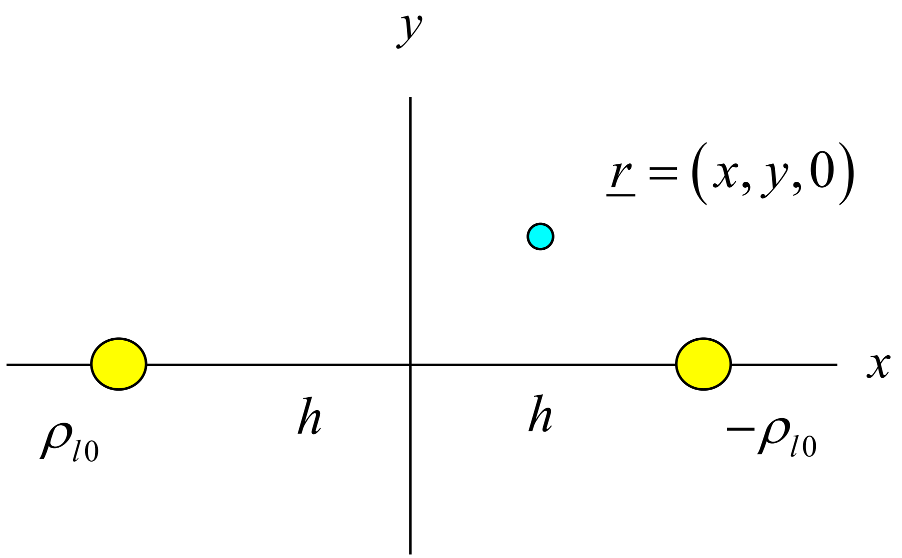
$$R_2 = \sqrt{(x-h)^2 + y^2}$$

$$\hat{\underline{R}}_1 = \frac{\hat{\underline{x}}(x+h) + \hat{\underline{y}}\, y}{\sqrt{(x+h)^2 + y^2}}$$

$$\hat{\underline{R}}_2 = \frac{\hat{\underline{x}}(x-h) + \hat{\underline{y}}\, y}{\sqrt{(x-h)^2 + y^2}}$$

Example (cont.)

Summary

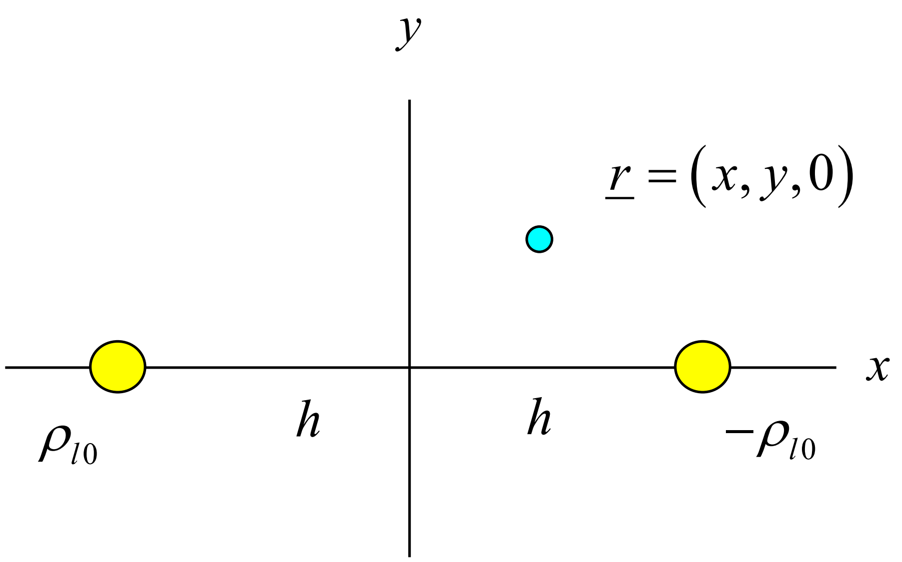


Answer in vector form:

$$\underline{E} = \left(\frac{\rho_{l0}}{2\pi\varepsilon_0} \right) \left[\frac{\underline{\hat{x}}\left(x+h\right) + \underline{\hat{y}}\, y}{\left(x+h\right)^2 + y^2} - \frac{\underline{\hat{x}}\left(x-h\right) + \underline{\hat{y}}\, y}{\left(x-h\right)^2 + y^2} \right] \quad [\text{V/m}]$$

Example (cont.)

Summary



Answer in component form:

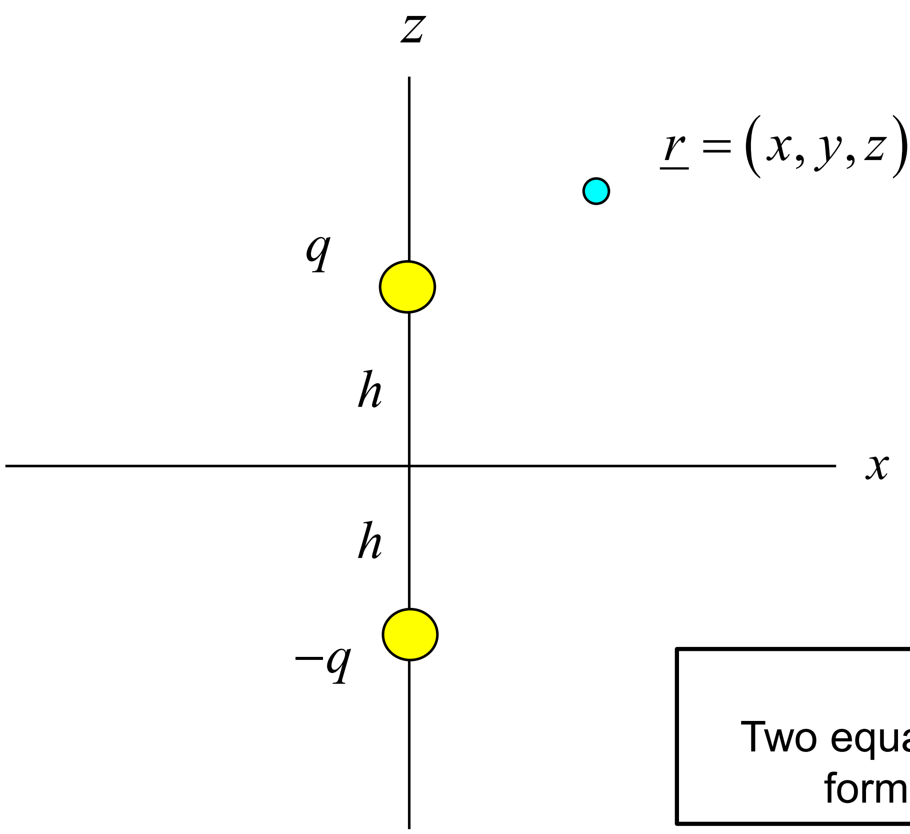
$$E_x = \left(\frac{\rho_{l0}}{2\pi\varepsilon_0}\right)\left[\frac{x+h}{(x+h)^2+y^2} - \frac{x-h}{(x-h)^2+y^2}\right] \quad [\mathrm{V/m}]$$

$$E_y = \left(\frac{\rho_{l0}}{2\pi\varepsilon_0}\right)\left[\frac{y}{(x+h)^2+y^2} - \frac{y}{(x-h)^2+y^2}\right] \quad [\mathrm{V/m}]$$

Example

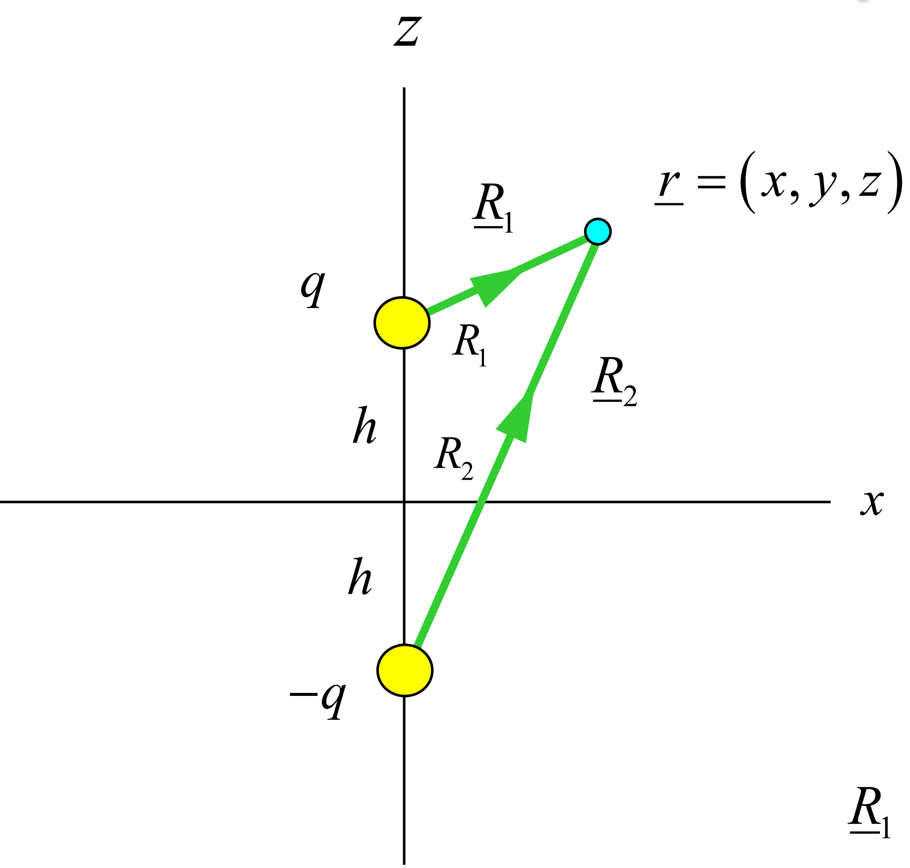
Two point charges (dipole)

Find $\underline{E}(x, y, z)$



Note:
Two equal and opposite point charges form an "electrostatic dipole".

Example (cont.)



Single charge at origin:

$$\underline{E} = \frac{q}{4\pi\varepsilon_0 r^2}\,\underline{\hat{r}}$$

From superposition:

$$\underline{E} = \underline{\hat{R}}_1\left(\frac{q}{4\pi\varepsilon_0 R_1^2}\right) + \underline{\hat{R}}_2\left(\frac{-q}{4\pi\varepsilon_0 R_2^2}\right)$$

$$\underline{R}_1 = \underline{\hat{x}}(x-0) + \underline{\hat{y}}(y-0) + \underline{\hat{z}}(z-h)$$

$$\underline{R}_2 = \underline{\hat{x}}(x-0) + \underline{\hat{y}}(y-0) + \underline{\hat{z}}(z+h)$$

$$R_1 = \sqrt{x^2 + y^2 + (z-h)^2}$$

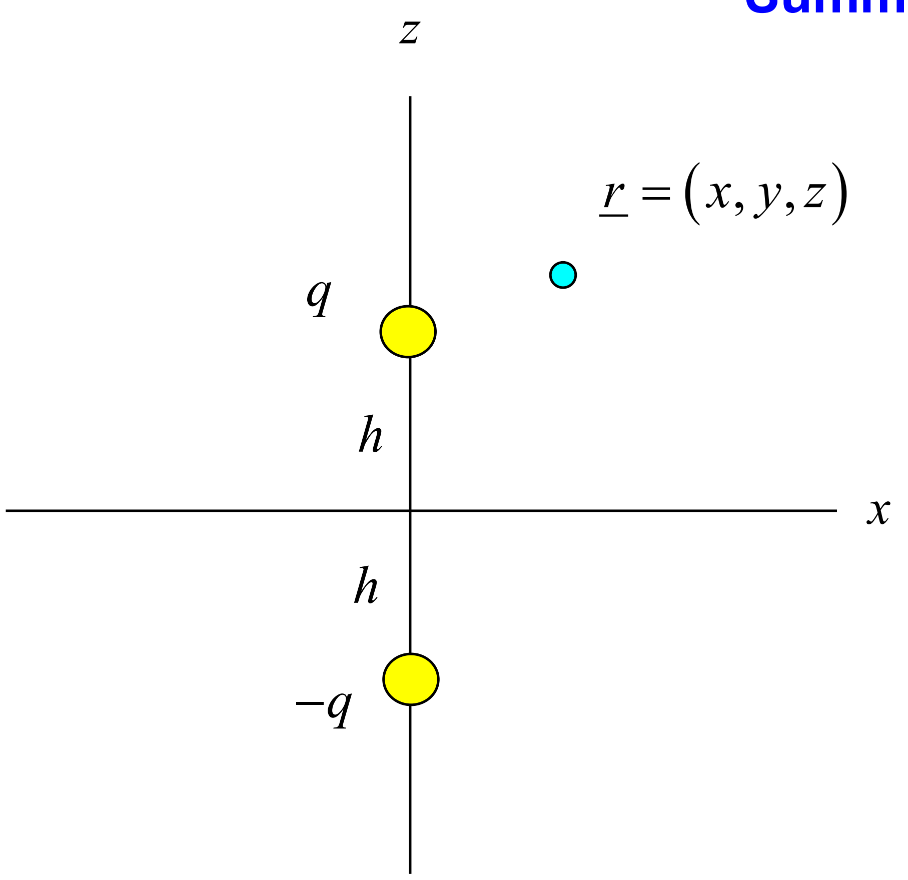
$$R_2 = \sqrt{x^2 + y^2 + (z+h)^2}$$

$$\underline{\hat{R}}_1 = \frac{\underline{\hat{x}}x + \underline{\hat{y}}\,y + \underline{\hat{z}}(z-h)}{\sqrt{x^2 + y^2 + (z-h)^2}}$$

$$\underline{\hat{R}}_2 = \frac{\underline{\hat{x}}x + \underline{\hat{y}}\,y + \underline{\hat{z}}(z+h)}{\sqrt{x^2 + y^2 + (z+h)^2}}$$

Example (cont.)

Summary



$$\underline{r} = (x, y, z)$$

$$\underline{E} = \underline{\hat{R}}_1 \left(\frac{q}{4\pi\varepsilon_0 R_1^2} \right) + \underline{\hat{R}}_2 \left(\frac{-q}{4\pi\varepsilon_0 R_2^2} \right)$$

$$R_1 = \sqrt{x^2 + y^2 + (z-h)^2}$$

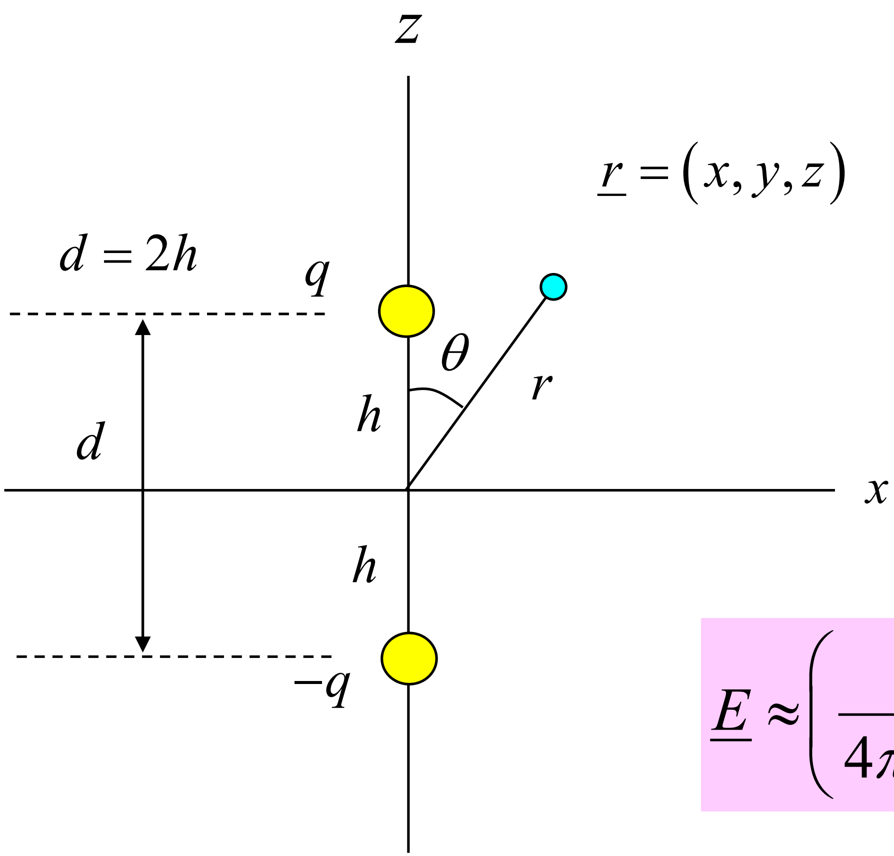
$$R_2 = \sqrt{x^2 + y^2 + (z+h)^2}$$

$$\underline{\hat{R}}_1 = \frac{\underline{\hat{x}}x + \underline{\hat{y}}\,y + \underline{\hat{z}}(z-h)}{\sqrt{x^2 + y^2 + (z-h)^2}} \qquad \underline{\hat{R}}_2 = \frac{\underline{\hat{x}}x + \underline{\hat{y}}\,y + \underline{\hat{z}}(z+h)}{\sqrt{x^2 + y^2 + (z+h)^2}}$$

Example (cont.)

Dipole Approximation

$$r >> h$$



$$\underline{r} = (x, y, z)$$

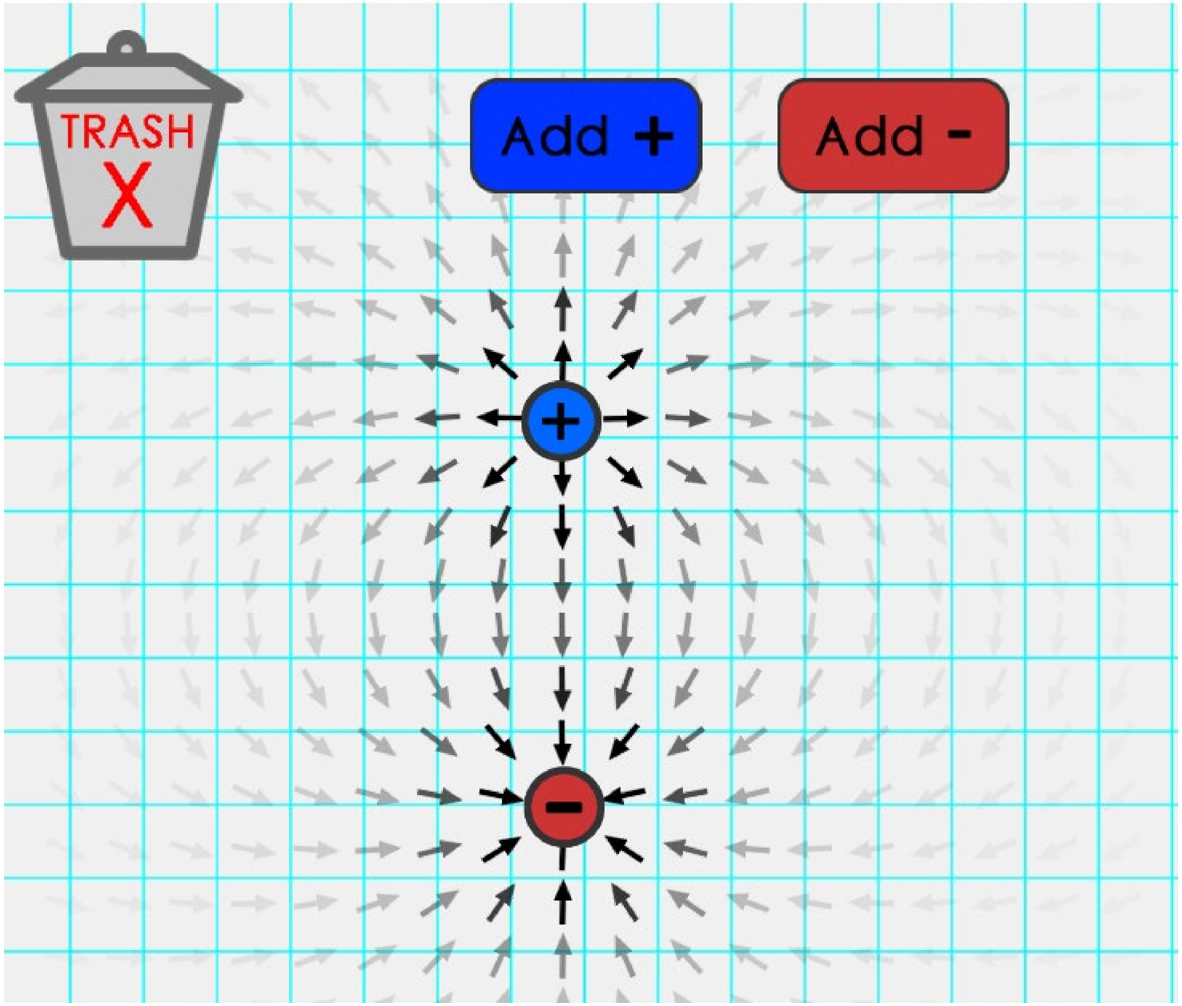
$$p \equiv qd$$

The term p is the "dipole moment".

$$\underline{E} \approx \left(\frac{p}{4\pi\varepsilon_0 r^3} \right) \left[\underline{\hat{r}} \left(2\cos\theta \right) + \underline{\hat{\theta}} \left(\sin\theta \right) \right]$$

(The derivation is omitted.)

Electric Field Applet



https://www.physicsclassroom.com/Physics-Interactives/Static-Electricity/Electric-Field-Lines/Electric-Field-Lines-Interactive