ECE 3318
Applied Electricity and Magnetism

Spring 2023

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Dept. of ECE

Notes 9
Flux

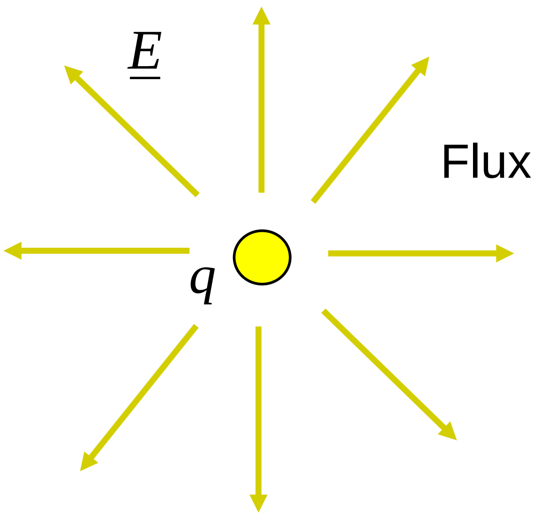
Notes prepared by the EM Group
University of Houston

Flux Density

From Coulomb's law:

$$\underline{E} = \frac{q}{4\pi\varepsilon_0 r^2}\,\underline{\hat{r}}$$

$$\varepsilon_0 \doteq 8.854187818\times 10^{-12} \quad [\mathrm{F/m}]$$



Flux lines (Flux lines show us the direction of the electric field vector.)

Charge in free space

Define:

$$\underline{D} \equiv \varepsilon_0\,\underline{E}$$

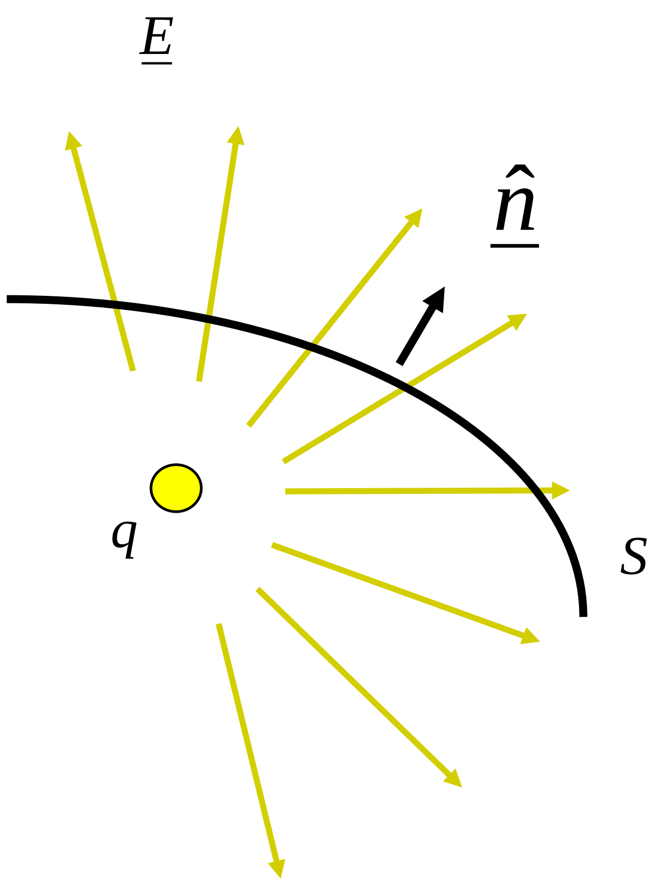
"flux density vector"

(This definition holds in free space.)

We then have

$$\underline{D} = \frac{q}{4\pi r^2}\,\underline{\hat{r}} \quad [\mathrm{C/m^2}]$$

Flux Through Surface



Cross-sectional view

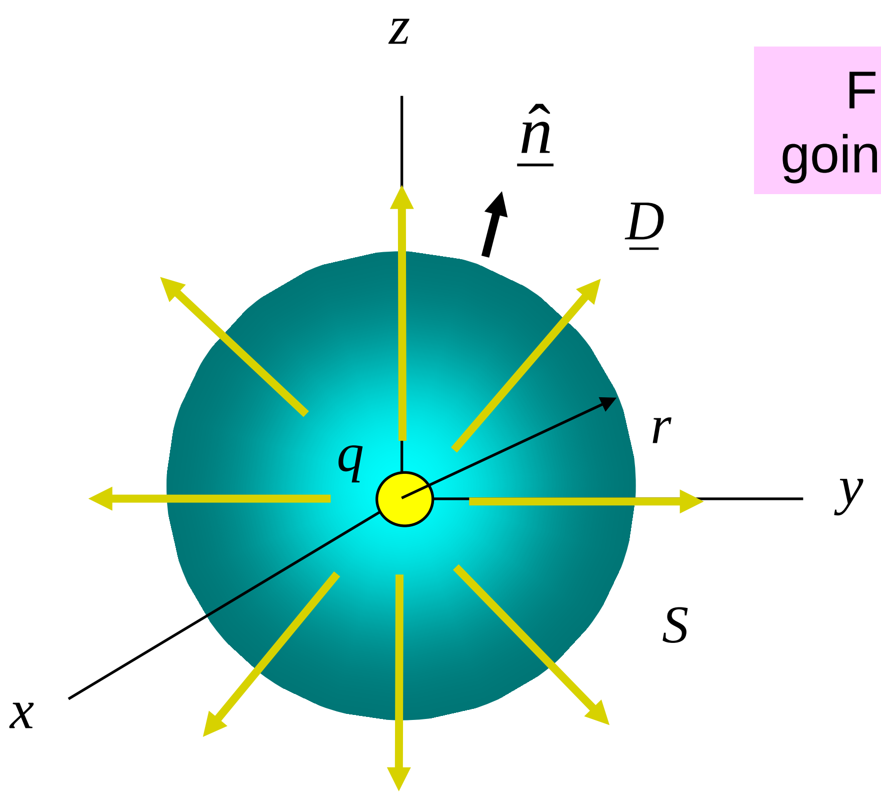
Define flux through a surface:

$$\psi \equiv \int_S \underline{D} \cdot \underline{\hat{n}} \, dS \quad [\mathrm{C}]$$

Note:
In this picture, flux is the flux crossing the surface in the outward sense.

$$\underline{D} = \frac{q}{4\pi r^2} \underline{\hat{r}} \quad [\mathrm{C/m^2}]$$

Example



$$\underline{\hat{n}} = +\underline{\hat{r}}$$

(We want the flux going out.)

Find the flux from a point charge going out through a spherical surface.

$$\psi \equiv \oint_S \underline{D} \cdot \underline{\hat{n}} \, dS$$

$$= \oint_S \underline{D} \cdot \underline{\hat{r}} \, dS$$

$$= \oint_S \left(\frac{q}{4\pi r^2} \underline{\hat{r}} \right) \cdot \underline{\hat{r}} \, dS$$

$$= \oint_S \frac{q}{4\pi r^2} dS$$

Example (cont.)

$$\psi = \int_0^{2\pi}\int_0^{\pi}\left(\frac{q}{4\pi r^2}\right) r^2 \sin\theta\, d\theta\, d\phi$$

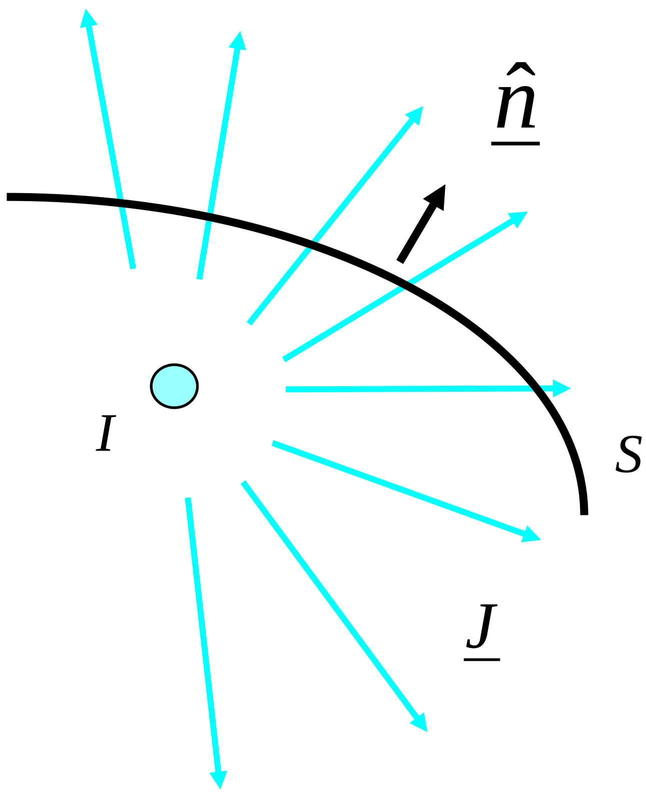
$$= \frac{q}{4\pi}\int_0^{2\pi}\int_0^{\pi} \sin\theta\, d\theta\, d\phi$$

$$= \frac{q}{4\pi} 2\pi \int_0^{\pi} \sin\theta\, d\theta$$

$$\psi = \frac{q}{4\pi}(2\pi)(2)$$

$$\psi = q \;\; [\mathrm{C}]$$

Current Analogy



Cross-sectional view

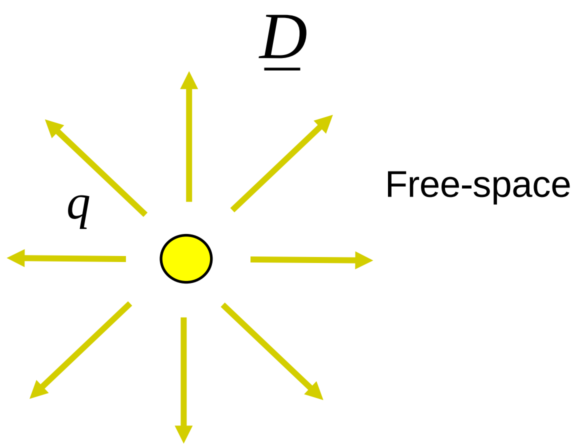
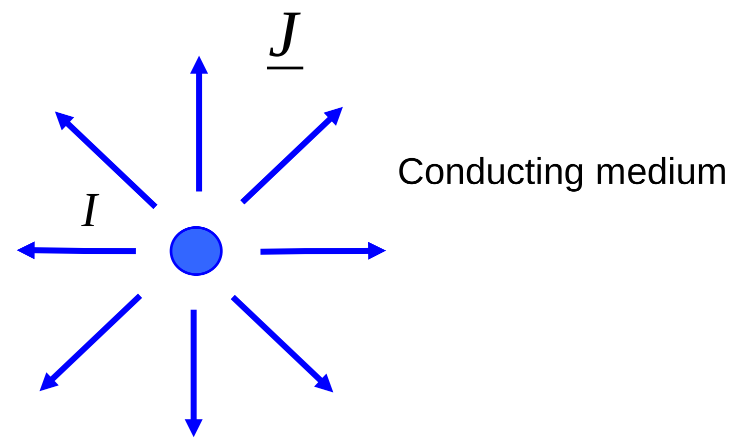
Analogy with electric current

$$I = \int_S \underline{J} \cdot \hat{\underline{n}}\, dS \quad [\mathrm{A}]$$

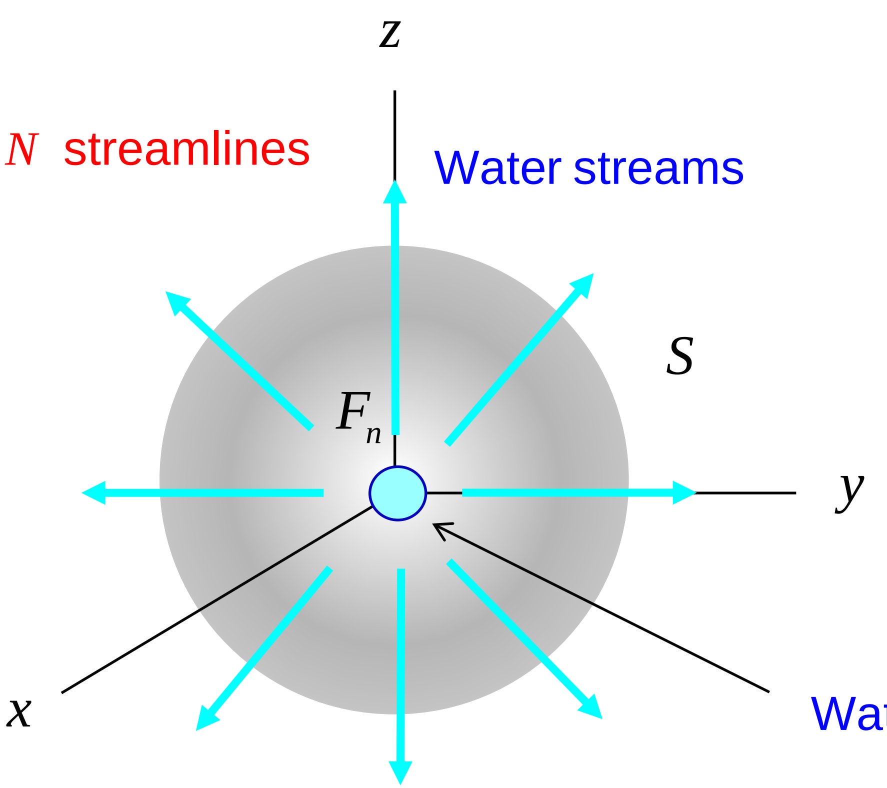
A small electrode in a conducting medium spews out current equally in all directions.

$$\underline{J} = \frac{I}{4\pi r^2} \hat{\underline{r}}$$

Current Analogy (cont.)

Electric Flux	Current
	
$\underline{D} = \dfrac{q}{4\pi r^2}\,\underline{\hat{r}} \quad [\mathrm{C/m^2}]$	$\underline{J} = \dfrac{I}{4\pi r^2}\,\underline{\hat{r}} \quad [\mathrm{A/m^2}]$
$\psi \equiv \int_S \underline{D}\cdot\underline{\hat{n}}\,dS \quad [\mathrm{C}]$	$I = \int_S \underline{J}\cdot\underline{\hat{n}}\,dS \quad [\mathrm{A}]$

Water Analogy



Each electric flux line is like a stream of water.

Each stream of water carries F_n / N [m^3/s].

F_n = flow rate of water out of nozzle [m^3/s]

F_s = flow rate of water out of surface [m^3/s]

Note that the flow rate through a closed surface is $F_s = F_n$.

Water Analogy (cont.)

Here is a real "flux fountain"
(Wortham fountain, a.k.a. the "Dandelion" on Allen Parkway).

Water Analogy (cont.)

Analogy example:

$$F_n = 15 \left[\mathrm{m}^3/\mathrm{s} \right] \text{ (flow rate from central node)}$$

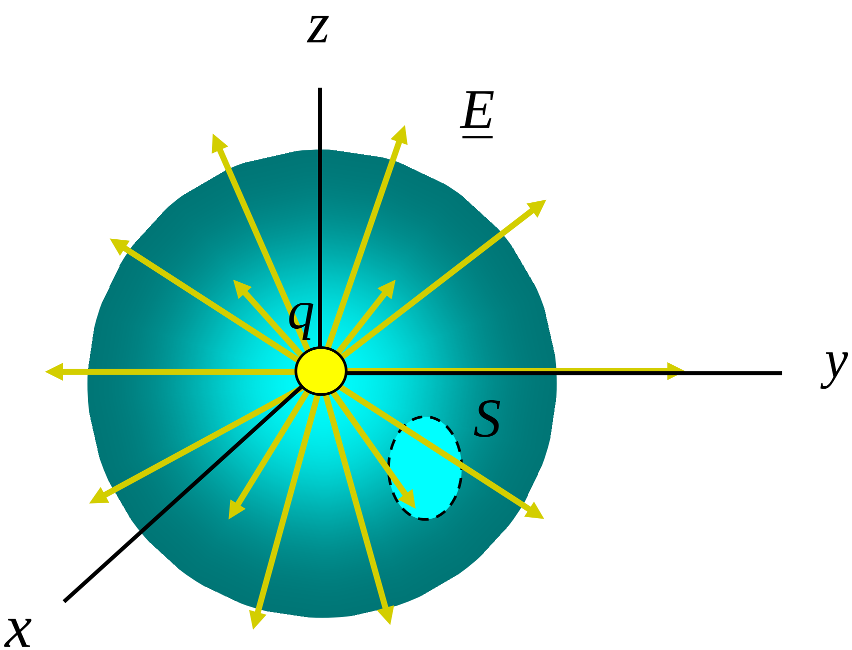
$$N = 1000 \text{ (number of pipes)}$$

$$N_s = 20 \text{ (number of flux lines through surface } S\text{)}$$

Find F_s (flow rate through surface S)

$$F_s = \left(\frac{15\left[\mathrm{m}^3/\mathrm{s}\right]}{1000\left[\mathrm{pipes}\right]} \right) (20 \text{ pipes}) = 0.3 \left[\mathrm{m}^3/\mathrm{s} \right]$$

Flux through a Surface using Flux Lines (Point Charge)



Example:

$$q = 15\ [\mathrm{C}]$$

$$N = 1000\ (\text{number of flux lines})$$

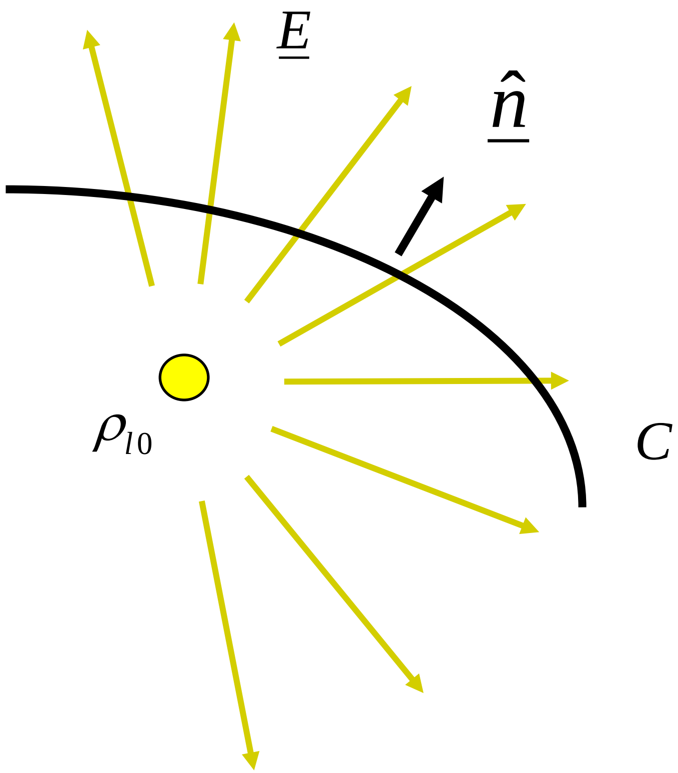
$$N_s = 20\ (\text{number of flux lines through surface } S)$$

Find ψ (flux through surface S)

$$\psi = \left(\frac{15[\mathrm{C}]}{1000[\text{flux lines}]}\right)(20\ \text{flux lines}) = 0.3\ [\mathrm{C}]$$

Flux in 2-D Problems

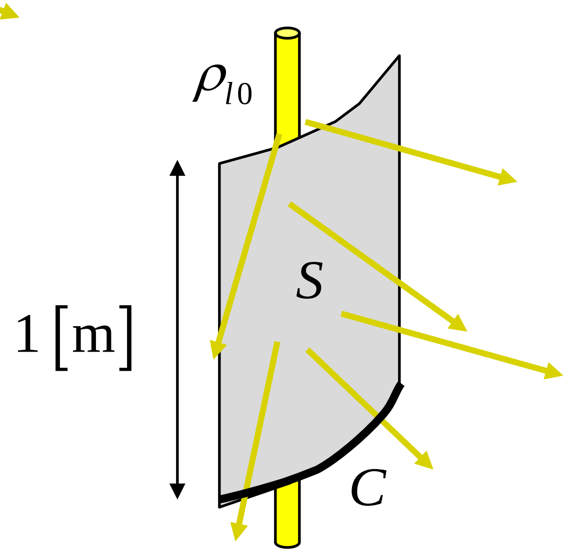
2-D problems: Everything is <u>infinite</u> and not changing in the z direction.



We now define the <u>flux per meter</u> in the z direction.

$$\psi_l \equiv \int_C \underline{D} \cdot \underline{\hat{n}}\, dl \quad [\mathrm{C/m}]$$

We can think of ψ_l as being the flux through a surface S that is the contour C extruded one meter in the z direction.



$$\psi = \int_S \underline{D} \cdot \underline{\hat{n}}\, dS \quad [\mathrm{C}]$$

$$\psi = \psi_l \times \left(1[\mathrm{m}]\right) = \psi_l$$

Flux Plot (2-D)

Rules for a 2-D flux plot:

1) Flux lines are in the direction of the electric field*.
2) The magnitude of the electric field is inversely proportional to the spacing between flux lines**.
3) Flux lines come out of positive charges and end on negative charges (they cannot stop or begin in free space)***.

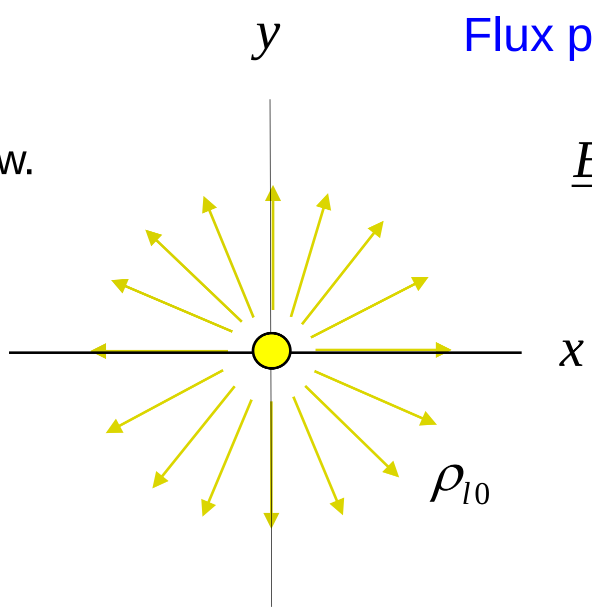
* A convention we adopt.

** A convention we adopt.

*** A consequence of Gauss's law.

Flux plot for a line charge

$$\underline{E} = \underline{\hat{\rho}}\left(\frac{\rho_{l0}}{2\pi\varepsilon_0 \rho}\right)$$

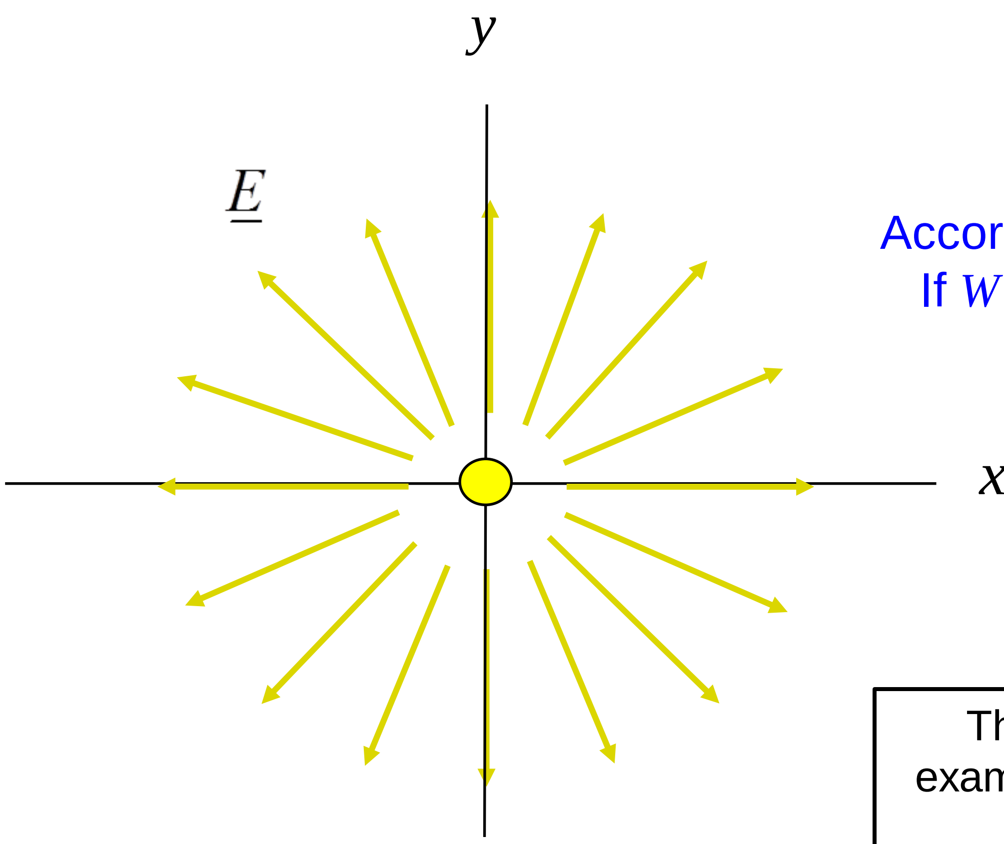


Example

Line charge

ρ_{l0} [C/m]

Notice how the flux lines get closer as we approach the line charge: there is a stronger electric field there.



$$\underline{E} = \underline{\hat{\rho}} \left(\frac{\rho_{l0}}{2\pi\varepsilon_0 \rho} \right)$$

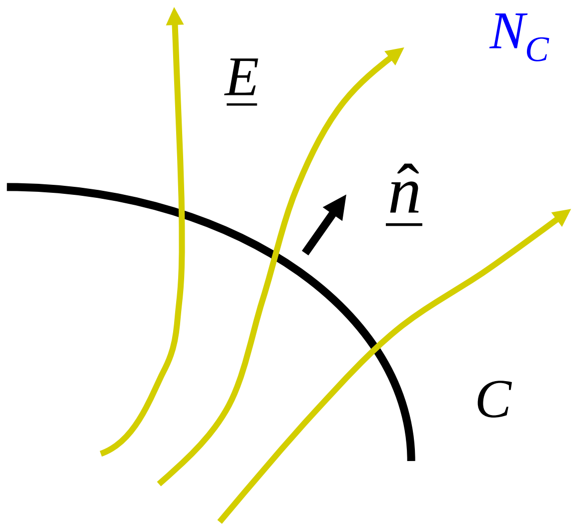
According to rule #2:
If W is the distance between flux lines:

$$W \propto \rho$$

This happens automatically in this example by choosing a fixed number of equally-spaced flux lines.

Flux Property in 2-D Flux Plots

The flux (per meter) ψ_l through a contour is proportional to the number of flux lines that cross the contour.



General $\underline{E}$ field

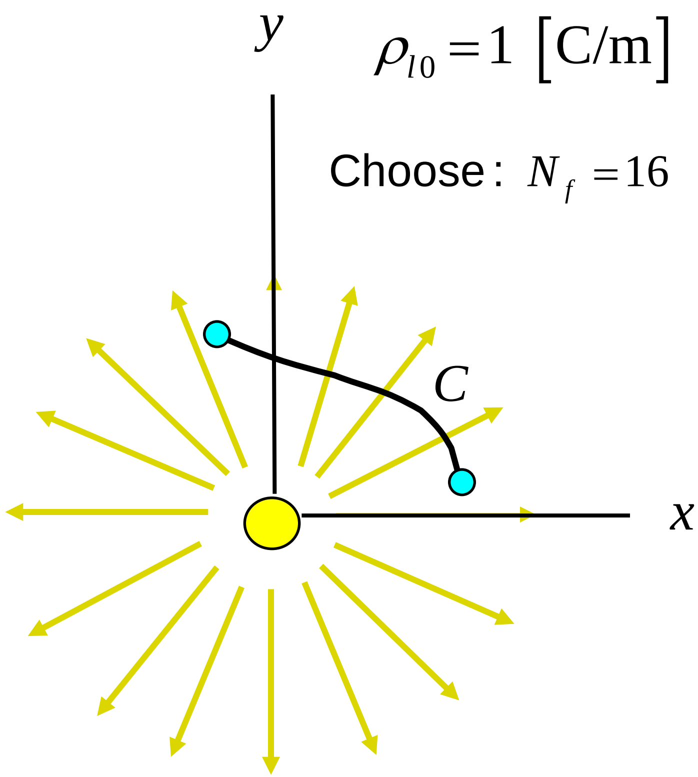
N_C is defined as the number of flux lines through C.

$$\psi_l \equiv \int_C \underline{D} \cdot \underline{\hat{n}} \, dl \propto N_C$$

Note:
The constant of proportionality depends on how many flux lines you decide to draw.

Please see the Appendix for a proof of this flux property.

Example



$\rho_{l0} = 1\ [\text{C/m}]$

Choose: $N_f = 16$

Goal:

Graphically evaluate

$$\psi_l = \int_C \underline{D} \cdot \underline{\hat{n}}\, dl$$

$$\psi_l = (4\ \text{lines})\left(\frac{1[\text{C/m}]}{16[\text{lines}]}\right)$$

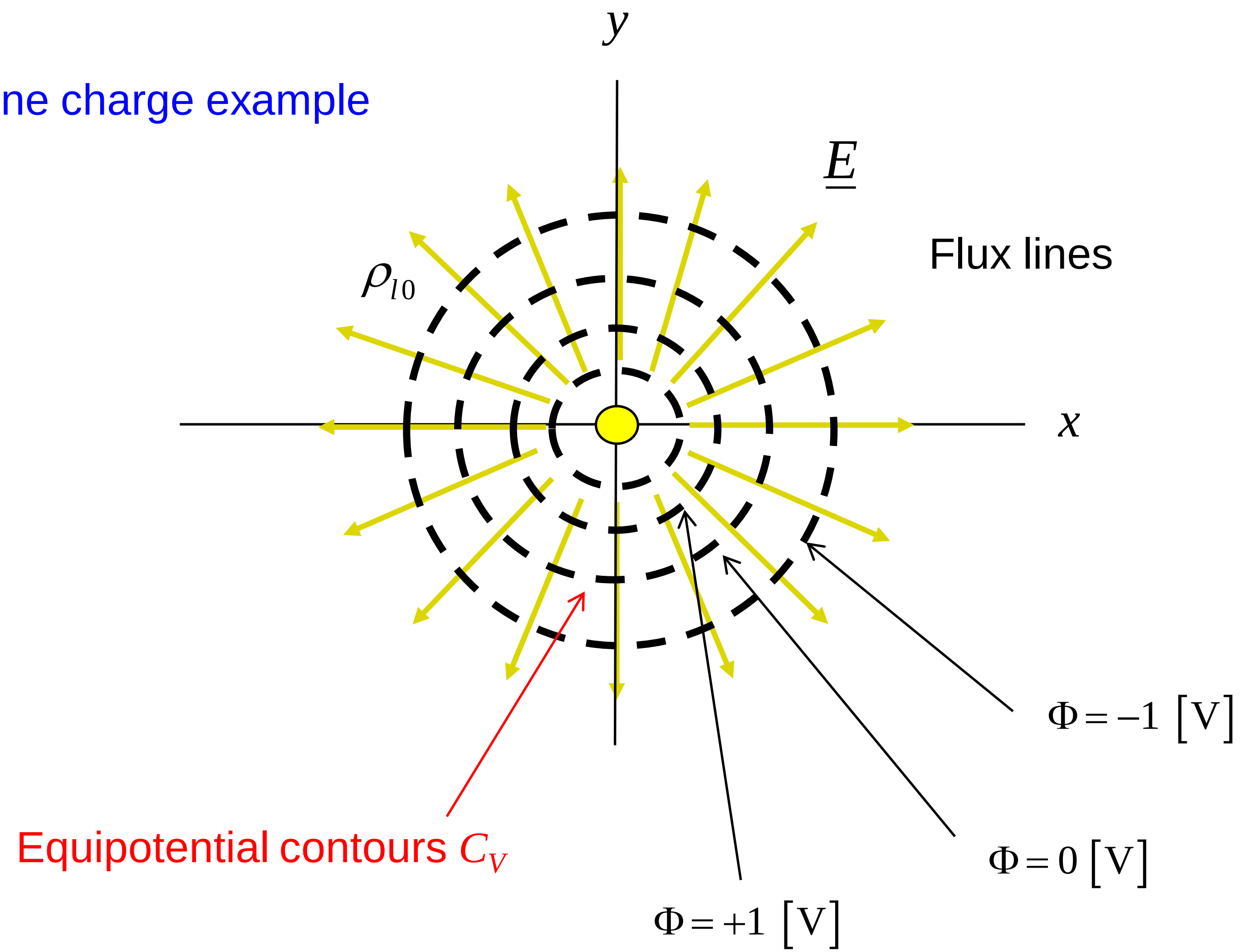
$$\psi_l = \frac{1}{4}\ [\text{C/m}]$$

Note:
The answer will be more accurate if we use a plot with more flux lines!

Equipotential Contours

An equipotential contour C_V is a contour on which the potential is constant.

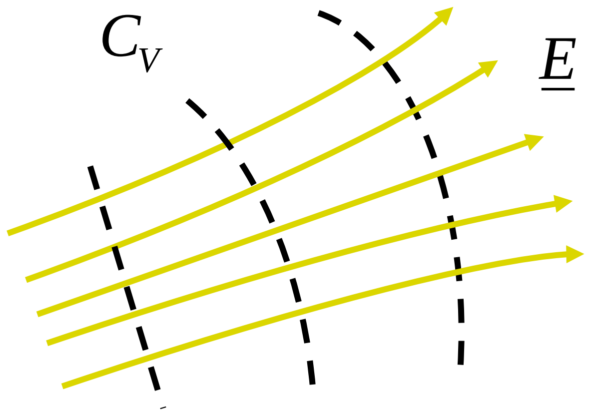
Line charge example



Equipotential Contours (cont.)

Property:

$$\underline{E} \perp C_V$$



(Φ = constant)

The flux line are always perpendicular to the equipotential contours.

(proof on next slide)

Equipotential Contours (cont.)

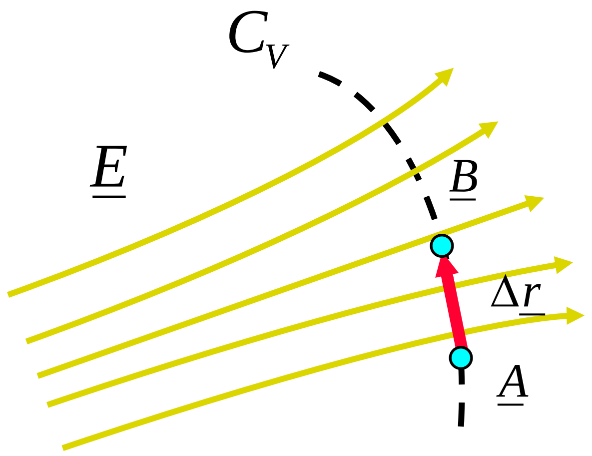
Proof of perpendicular property:

Two nearby points on an equipotential contour are considered.

$$\Delta\Phi \equiv V_{AB} = \int_{\underline{A}}^{\underline{B}} \underline{E} \cdot d\underline{r}$$

$$\approx \underline{E} \cdot \int_{\underline{A}}^{\underline{B}} d\underline{r}$$

$$= \underline{E} \cdot \Delta\underline{r}$$



Proof:

On C_V:

$$\Delta\Phi = 0$$

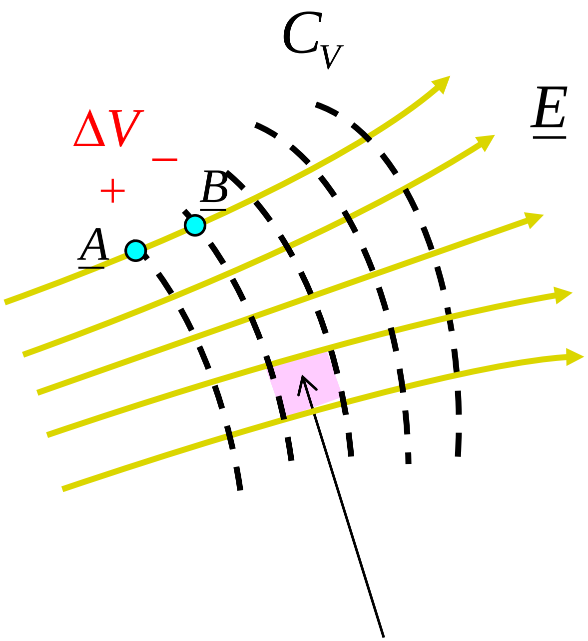
$$\Rightarrow \quad \underline{E} \cdot \Delta\underline{r} = 0$$

$$\Rightarrow \quad \underline{E} \perp \Delta\underline{r}$$

Conclusion: The $\Delta\underline{r}$ vector is perpendicular to the flux lines.

Method of Curvilinear Squares

2-D Flux Plot



"Curvilinear square"

Assumption:

Assume a <u>constant voltage difference</u> ΔV between adjacent equipotential lines in a 2-D flux plot.

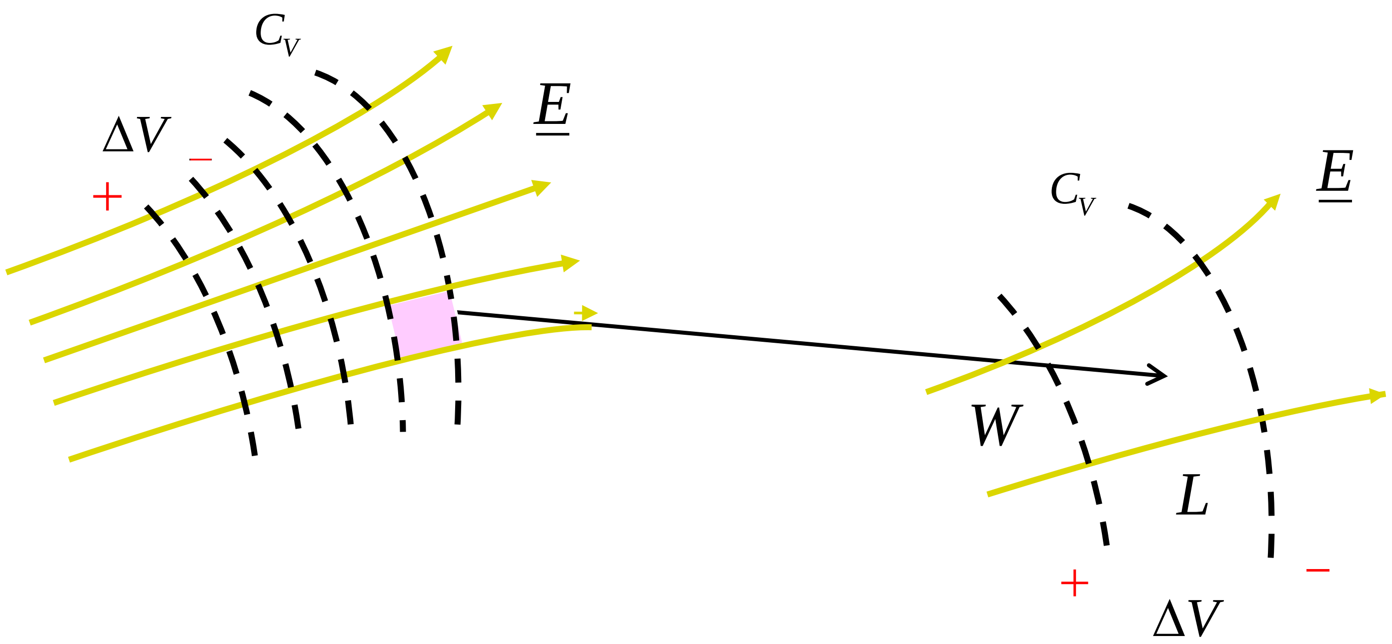
Note:
Along a flux line, the voltage always <u>decreases</u> as we go in the direction of the flux line.

Note:
It is called a curvilinear "square" even though the shape may be rectangular.

Method of Curvilinear Squares (cont.)

Theorem: *The shape (aspect ratio L/W) of the "curvilinear squares" is preserved throughout the flux plot.*

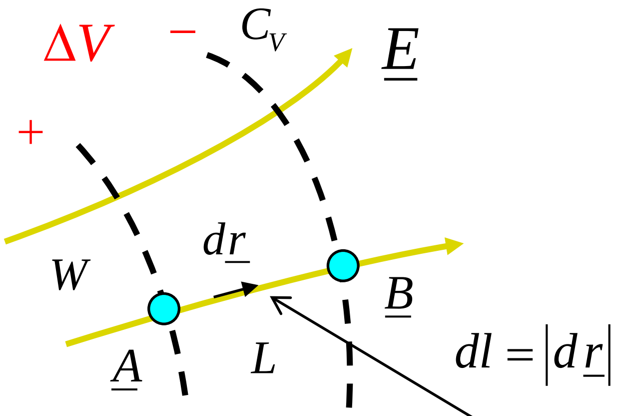
Assumption: ΔV is constant throughout plot.



$$\frac{L}{W} = \text{constant}$$

Method of Curvilinear Squares (cont.)

Proof of constant aspect ratio property



$$V_{AB} = \int_{\underline{A}}^{\underline{B}} \underline{E} \cdot d\underline{r} = \Delta V$$

If we integrate along the flux line, $\underline{E}$ is in the same direction as $d\underline{r}$.

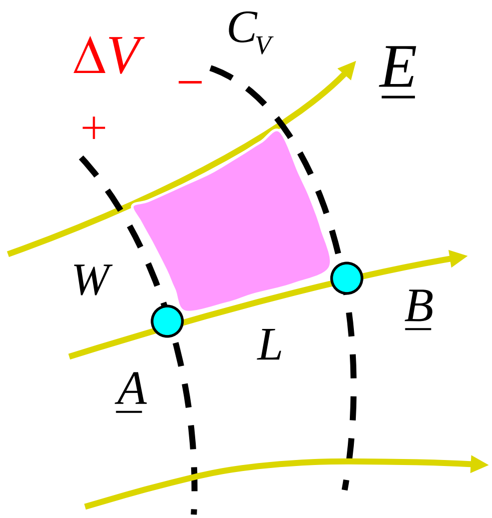
$$\Rightarrow \underline{E} \cdot d\underline{r} = \left|\underline{E}\right| \left|d\underline{r}\right| \cos\left(0^{\circ}\right) = \left|\underline{E}\right| dl$$

Hence, $$V_{AB} = \int_{\underline{A}}^{\underline{B}} \left|\underline{E}\right| \, dl = \Delta V$$

so $$\left|\underline{E}\right| \int_{\underline{A}}^{\underline{B}} dl \approx \Delta V$$

Therefore $$\left|\underline{E}\right| \, L \approx \Delta V$$

Method of Curvilinear Squares (cont.)



Hence,

$$L = \frac{\Delta V}{\left|\underline{E}\right|}$$

Also,

$$\left|\underline{E}\right| \propto \frac{1}{W}$$

Reminder:
In a flux plot the magnitude of the electric field is inversely proportional to the spacing between the flux lines.

so

$$W = C_1 \frac{1}{\left|\underline{E}\right|}$$

The constant C_1 is some constant of proportionality.

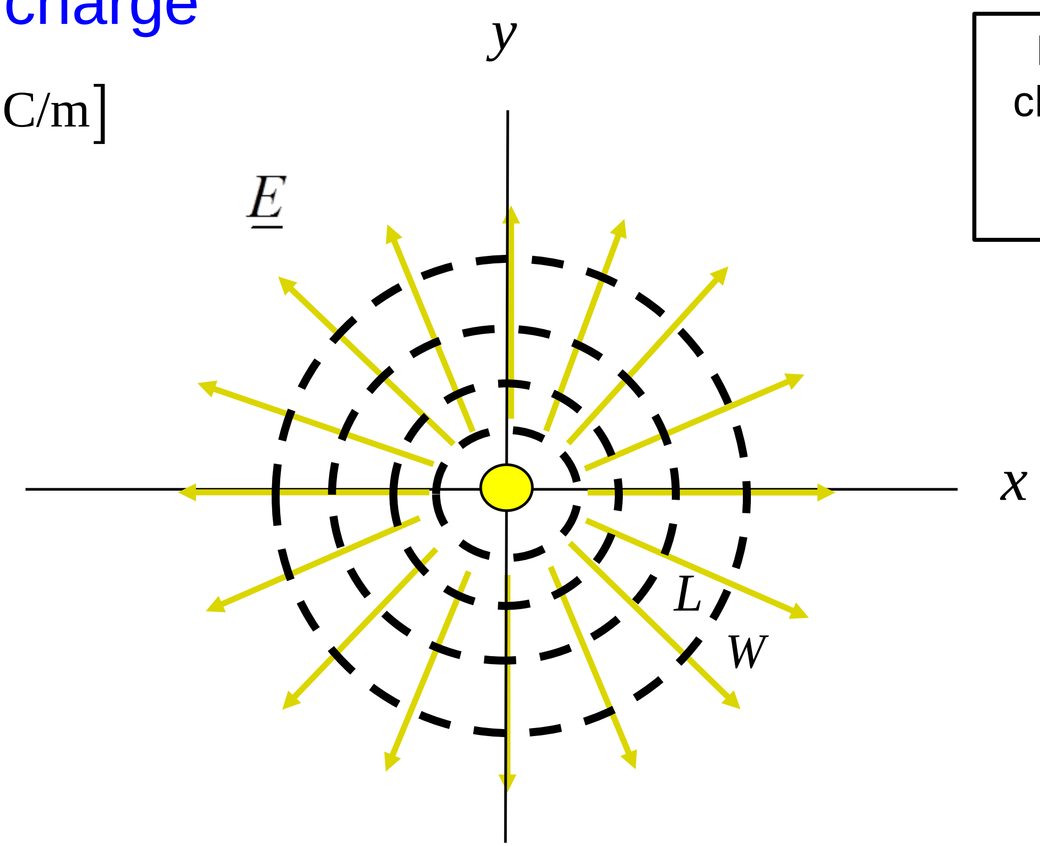
Hence,

$$\frac{L}{W} = \Delta V \left(\frac{1}{C_1} \right) = \text{constant}$$ (proof complete)

Example

Line charge

ρ_{l0} [C/m]



Notice how the flux lines get closer as we approach the line charge: there is a stronger electric field there.

$$\left|\underline{E}\right| \propto \frac{1}{W} \propto \frac{1}{\rho}$$

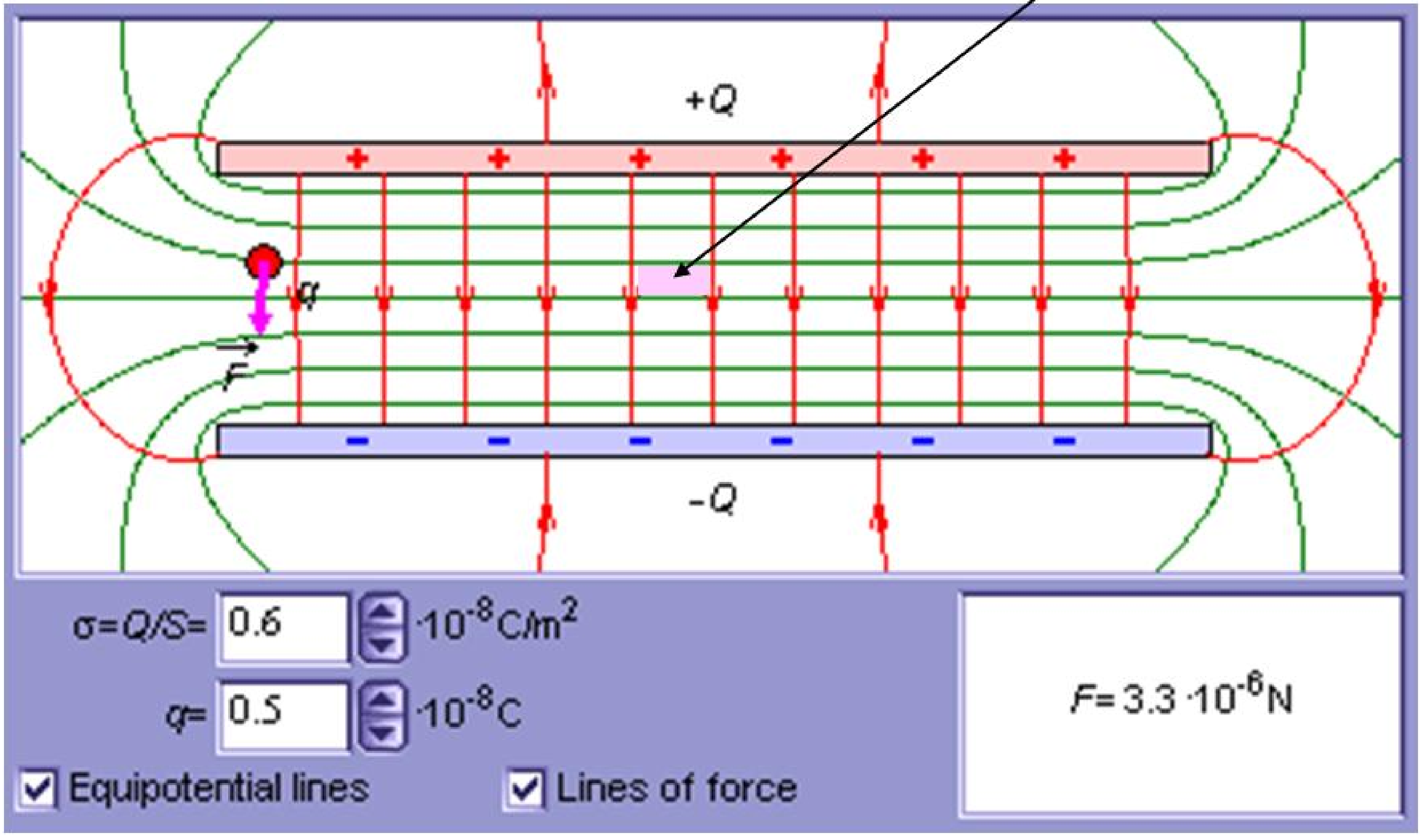
The aspect ratio L/W has been chosen to be unity in this plot.

In this example L and W are both proportional to the radius ρ.

Example

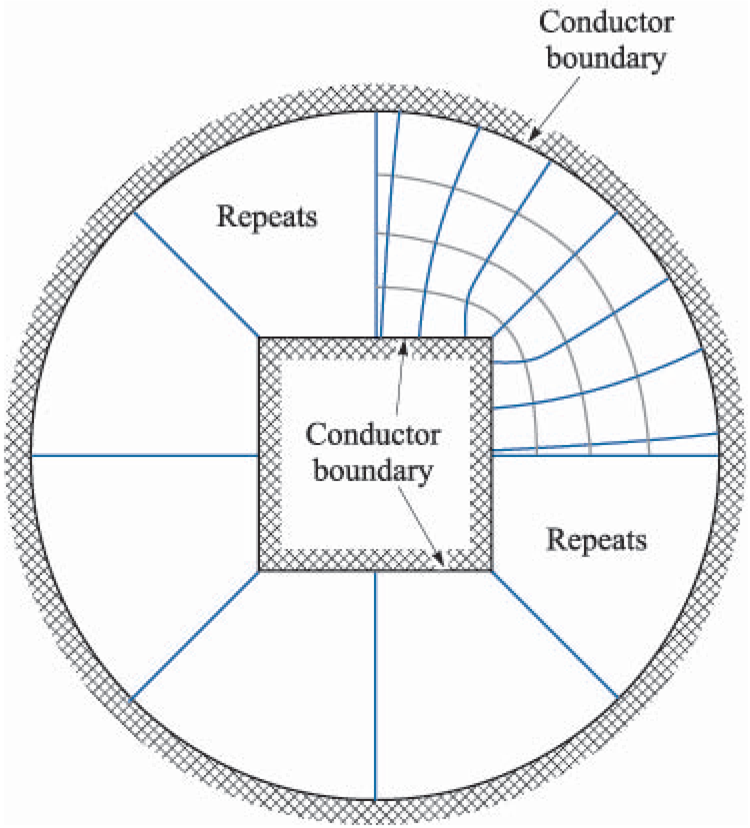
A parallel-plate capacitor

Note: $L / W \approx 0.5$



Example

Coaxial cable with a square inner conductor



$$\frac{L}{W} = 1$$

Figure 6-8 in the Hayt and Buck book (9th Ed.).

Making a Flux Plot

Here are the rules for making a 2-D flux plot, assuming that we start with equipotential contours separated by a fixed value of ΔV *:

Rule 1: We start with equipotential contours having a fixed ΔV between.

Rule 2: Flux lines are drawn perpendicular to the equipotential contours.

Rule 3: $L\,/\,W$ is kept constant throughout the plot.

If all of these rules are followed, then we have the following properties:

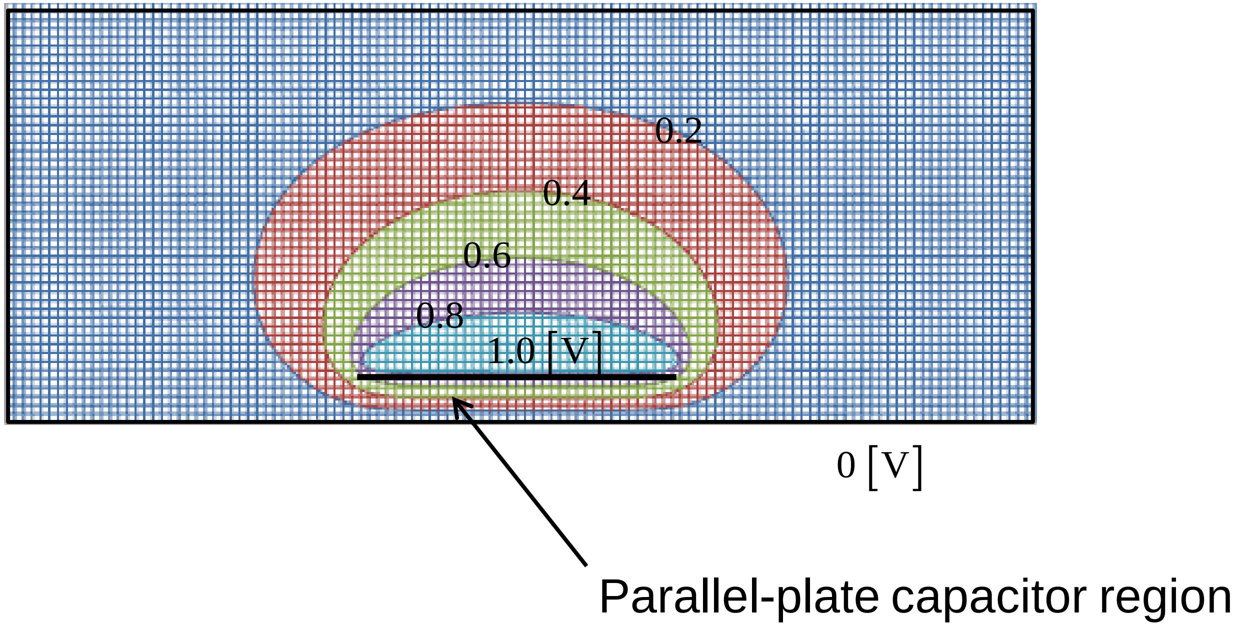
- ❖ Flux lines are in direction of $\underline{E}$.
- ❖ The magnitude of the electric field is inversely proportional to the spacing between the flux lines.

* This is what you will be doing in the class project.

Example

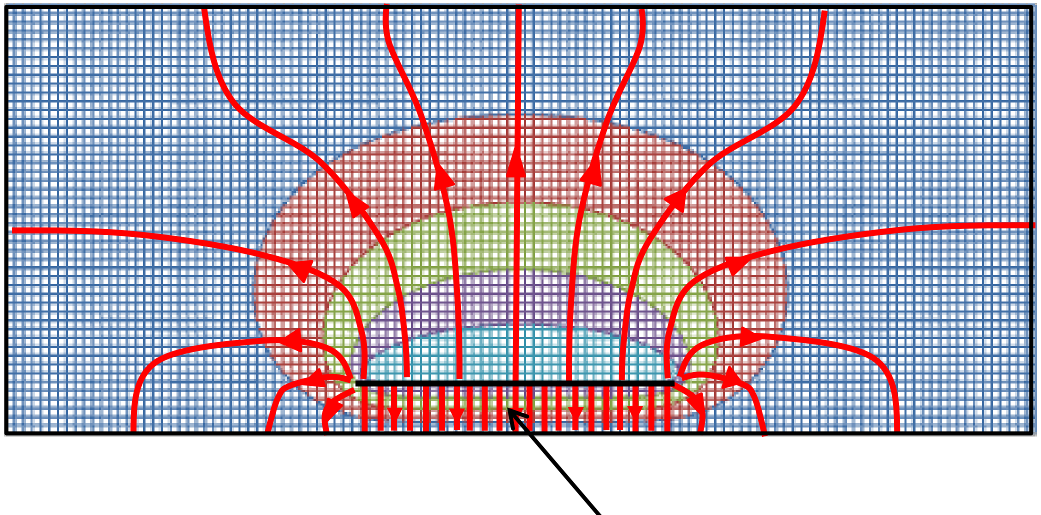
In the class project, you will be drawing in flux lines.

(The equipotential contours come from Excel.)



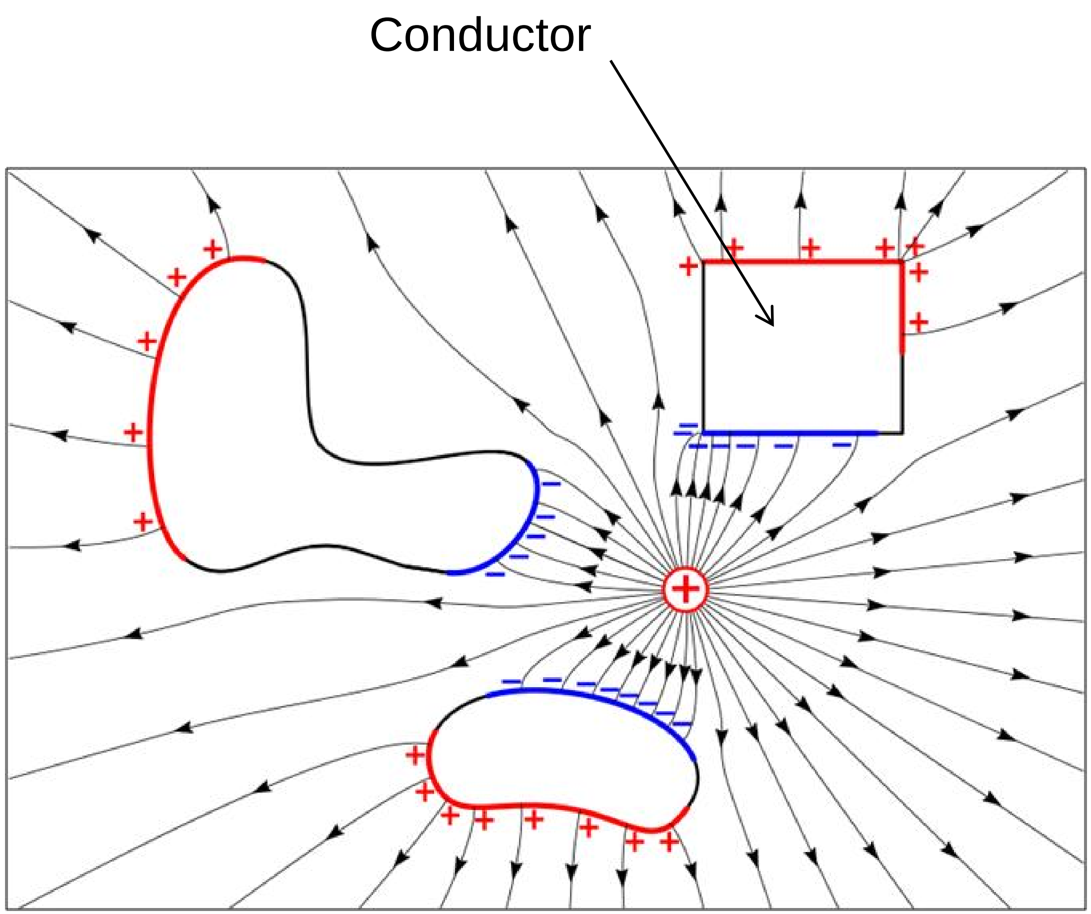
Example (cont.)

In the class project, you will be drawing in flux lines.



Parallel-plate capacitor region

Flux Plot with Conductors

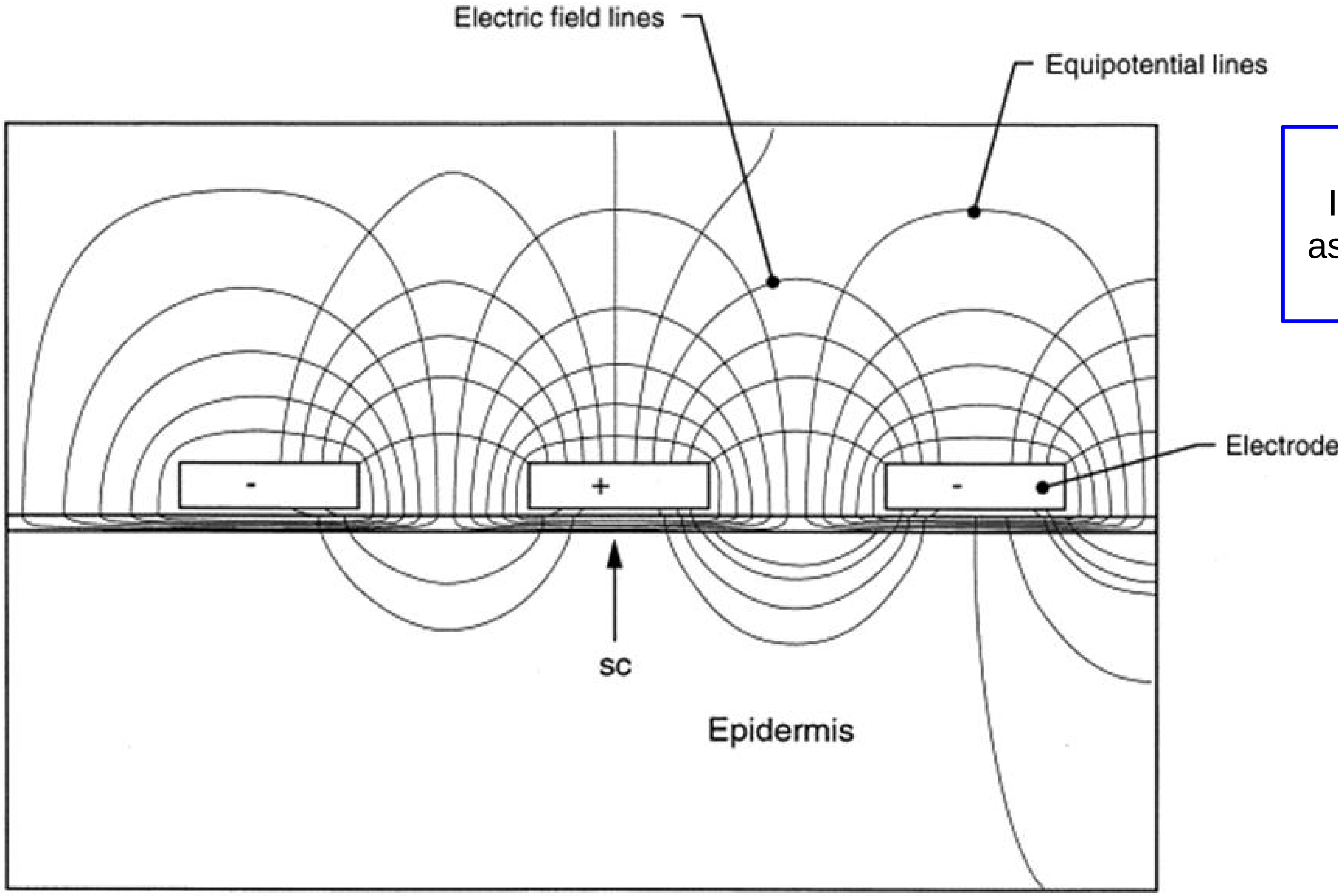


Some observations:

- Flux lines are closer together where the field is stronger.
- The field is strong near a sharp conducting corner.
- Flux lines begin on positive charges and end on negative charges.
- Flux lines enter a conductor perpendicular to it.

http://en.wikipedia.org/wiki/Electrostatics

Example of Electric Flux Plot



Note:
In this example, the aspect ratio L/W is not held constant!

Electroporation-mediated topical delivery of vitamin C for cosmetic applications

Lei Zhang[a], Sheldon Lerner[b], William V Rustrum[a], Günter A Hofmann[a]

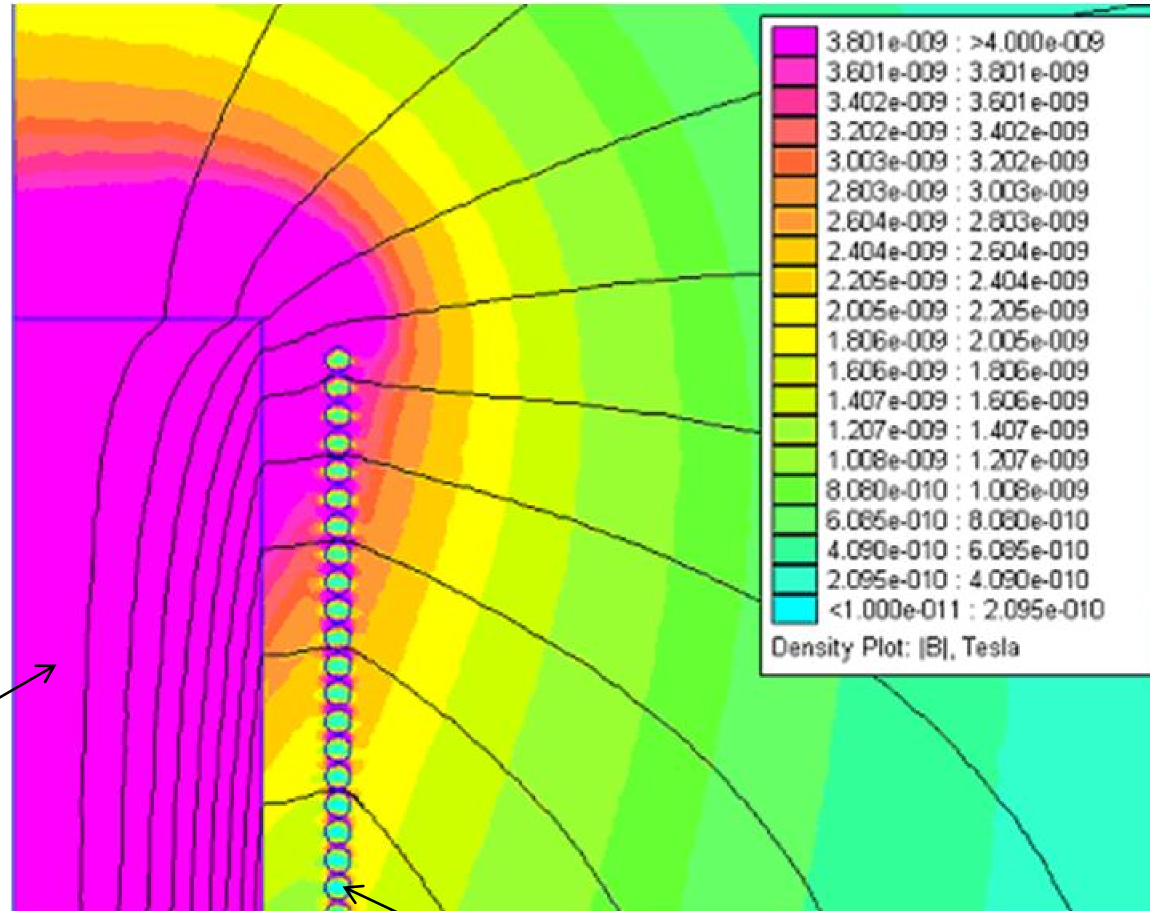
[a] Genetronics Inc., 11199 Sorrento Valley Rd., San Diego, CA 92121, USA

[b] Research Institute for Plastic, Cosmetic and Reconstructive Surgery Inc., 3399 First Ave., San Diego, CA 92103, USA.

Example of Magnetic Flux Plot

Solenoid wrapped around a ferrite core (cross sectional view)

Flux plots are often used to display the results of a numerical simulation, for either the electric field or the magnetic field.



Magnetic flux lines

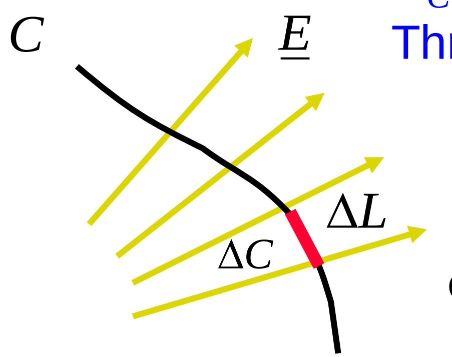
Ferrite core

Solenoid windings

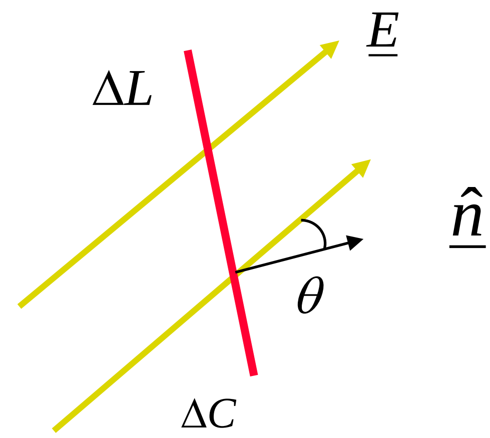
Appendix:
Proof of Flux Property

Proof of Flux Property

N_C : flux lines
Through C



One small piece of the contour
(the length is ΔL)

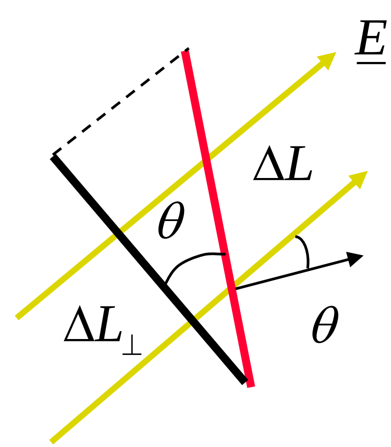


$\Delta N_C =$ number of flux lines

$$\Delta \psi_l \approx \left(\underline{D} \cdot \underline{\hat{n}}\right) \Delta L = \left|\underline{D}\right| \cos\theta \, \Delta L$$

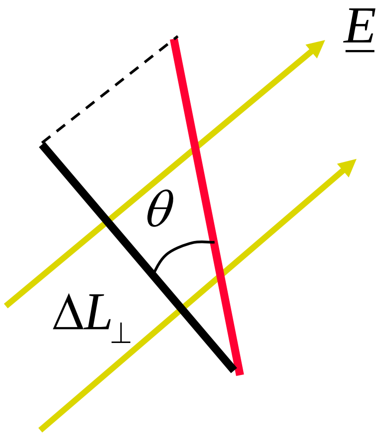
so $$\Delta \psi_l \approx \left|\underline{D}\right| \left(\Delta L \cos\theta\right)$$

or $$\Delta \psi_l \approx \left|\underline{D}\right| \left(\Delta L_\perp\right)$$



Flux Property Proof (cont.)

$$\Delta \psi_l \approx \left| \underline{D} \right| \left(\Delta L_\perp \right)$$



Also,

$$\left| \underline{D} \right| \propto 1 / \left(\frac{\Delta L_\perp}{\Delta N_C} \right)$$

(from the property of a flux plot)

Hence, substituting into the above equation, we have

$$\Delta \psi_l \propto \left| \underline{D} \right| \Delta L_\perp \propto \left(\frac{\Delta N_C}{\Delta L_\perp} \right) \left(\Delta L_\perp \right) = \Delta N_C$$

Therefore, $\psi_l \propto N_C$ (proof complete)