DO NOT BEGIN THIS EXAM UNTIL TOLD TO START

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Student Number: ______________________

Instructor: ______________________

ECE 2317

Applied Electricity and Magnetism
Exam 1
February 16, 2002

1. This exam is closed book and closed notes. A calculator and one crib sheet (one 8.5" X 11" piece of paper) are allowed.
2. Show all of your work. No credit will be given if the work required to obtain the solutions is not shown.
3. Perform all your work on the paper provided.
4. Write neatly. You will not be given credit for work that is not easily legible.
5. Leave answers in terms of the parameters given in the problem.
6. Show units in all of your final answers.
7. Circle your final answers.
8. If you have any questions, ask the instructors. You will not be given credit for work that is based on a wrong assumption.
9. You will have a total of 90 minutes to work the entire exam.

_____/25 Prob. 1	_____/25 Prob. 3
_____/25 Prob. 2	_____/25 Prob. 4

TABLE OF INTEGRALS

$$\int \frac{dx}{\left(x^2+a^2\right)^{1/2}} = \ln\left(x+\sqrt{x^2+a^2}\right)$$

$$\int \frac{dx}{\left(x^2+a^2\right)^{3/2}} = \frac{x}{a^2\sqrt{x^2+a^2}}$$

$$\int \frac{x\,dx}{\left(x^2+a^2\right)^{3/2}} = \frac{-1}{\sqrt{x^2+a^2}}$$

$$\int \frac{x\,dx}{\left(x^2+a^2\right)^{1/2}} = \sqrt{x^2+a^2}$$

$$\int \frac{x^2\,dx}{\left(x^2+a^2\right)^{3/2}} = \frac{-x}{\sqrt{x^2+a^2}} + \ln\left(x+\sqrt{x^2+a^2}\right)$$

$$\int \frac{x^2\,dx}{\left(x^2+a^2\right)^{1/2}} = \frac{x\sqrt{x^2+a^2}}{2} - \frac{a^2}{2}\ln\left(x+\sqrt{x^2+a^2}\right)$$

$$\int x\left(x^2+a^2\right)^{3/2}dx = \frac{1}{5}\left(x^2+a^2\right)^{5/2}$$

$$\int \left(x^2+a^2\right)^{3/2}dx = \frac{x\left(x^2+a^2\right)^{3/2}}{4} + \frac{3a^2x\sqrt{x^2+a^2}}{8} + \frac{3}{8}a^4\ln\left(x+\sqrt{x^2+a^2}\right)$$

TABLE OF COORDINATE SYSTEM FORMULAS

$$x = \rho\cos\phi$$
$$y = \rho\sin\phi$$
$$z = z$$

$$\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{x}} = \sin\theta\cos\phi = x/r$$
$$\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{y}} = \sin\theta\sin\phi = y/r$$
$$\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{z}} = \cos\theta = z/r$$

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

$$\rho = \sqrt{x^2 + y^2}$$
$$\phi = \tan^{-1}(y/x)$$
$$z = z$$

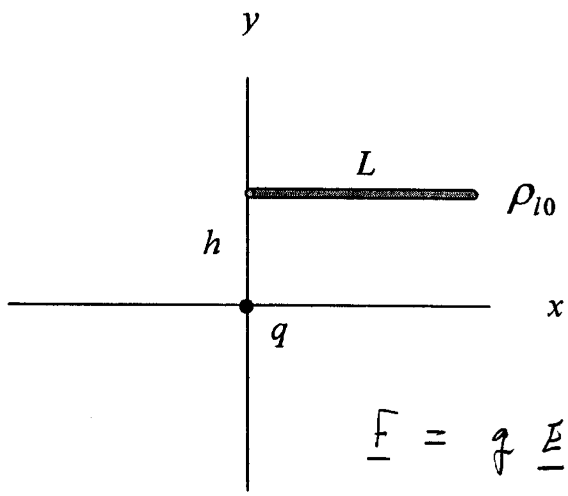
$$\rho = r\sin\theta$$
$$z = r\cos\theta$$
$$\phi = \phi$$

$$r = \sqrt{x^2 + y^2 + z^2}$$
$$\theta = \cos^{-1}\left(z/\sqrt{x^2 + y^2 + z^2}\right)$$
$$\phi = \tan^{-1}(y/x)$$

$$r = \sqrt{\rho^2 + z^2}$$
$$\theta = \tan^{-1}(\rho/z)$$
$$\phi = \phi$$

Problem 1 (25 pts)

A point charge having $q = 1$ [C] is at the origin. Find the vector force on this point charge due to a line segment of uniform charge density ρ_{l0} [C/m] that is parallel to the x axis as shown below. The line segment has length L and is located at a height h above the x axis.



$$\underline{F} = q\,\underline{E} = \underline{E}$$

$$\underline{E} = \int_C \frac{\rho_{l0}\,\hat{\underline{R}}}{4\pi\epsilon_0 R^2}\,dl' = \frac{\rho_{l0}}{4\pi\epsilon_0}\int_C \frac{\hat{\underline{R}}}{R}\,dl'$$

$$dl' = dx'$$

$$\underline{R} = (0,0,0) - (x', h, 0) = -\hat{\underline{x}}\,x' - \hat{\underline{y}}\,h$$

Total force vector =

$$\frac{\rho_{l0}}{4\pi\epsilon_0}(-\hat{\underline{x}})\left[\frac{1}{h} - \frac{1}{\sqrt{L^2+h^2}}\right] + \frac{\rho_{l0}}{4\pi\epsilon_0}(h)(-\hat{\underline{y}})\left[\frac{L}{h^2\sqrt{L^2+h^2}}\right]$$ [N]

ROOM FOR EXTRA WORK

Hence

$$R = \sqrt{x'^2 + h^2}$$

$$\hat{\underline{R}} = \frac{-\hat{\underline{x}}\, x' - \hat{\underline{y}}\, h}{\sqrt{x'^2 + h^2}}$$

So

$$\underline{E} = \frac{\rho_{l0}}{4\pi\epsilon_0} \int_0^L \frac{-\hat{\underline{x}}\, x' - \hat{\underline{y}}\, h}{(x'^2 + h^2)^{3/2}}\, dx'$$

$$= \frac{\rho_{l0}}{4\pi\epsilon_0} (-\hat{\underline{x}}) \int_0^L \frac{x'\, dx'}{(x'^2 + h^2)^{3/2}}$$

$$+ \frac{\rho_{l0}}{4\pi\epsilon_0} (-\hat{\underline{y}})\, h \int_0^L \frac{dx'}{(x'^2 + h^2)^{3/2}}$$

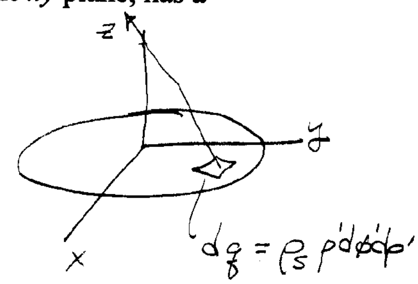
So

$$\underline{E} = \frac{\rho_{l0}}{4\pi\epsilon_0} (-\hat{\underline{x}}) \left[-(x'^2 + h^2)^{-1/2} \right]_0^L$$

$$+ \frac{\rho_{l0}}{4\pi\epsilon_0} (h) (-\hat{\underline{y}}) \left[\frac{x'}{h^2 \sqrt{x'^2 + h^2}} \right]_0^L$$

Problem 2 (25 pts)

A circular disk of radius a centered at the coordinate origin, and lying in the xy plane, has a surface charge density $\rho_s = \left(x^2 + y^2 + 3^2\right)^{\frac{3}{2}}$ [C/m^2].



(a) Find the total charge on the disk.

$$Q = \iint_S \rho_s \, dS$$

$$= \int_0^a \int_0^{2\pi} (x'^2 + y'^2 + 3^2)^{3/2} \rho' \, d\phi' d\rho' = 2\pi \int_0^a (\rho'^2 + 3^2)^{3/2} \rho' d\rho'$$

$$= \frac{2\pi}{5} (\rho'^2 + 3^2)^{5/2} \Big|_{\rho'=0}^{a}$$

$$= \frac{2\pi}{5} \left[(a^2 + 9)^{5/2} - \underbrace{3^5}_{243} \right]$$

Total charge = $\frac{2\pi}{5}\left[(a^2+9)^{5/2} - \overset{3^5}{\overbrace{243}}\right]$ [C]

(b) Find the electric field at (0,0,3). $= \underline{r}$; $\underline{r}' = \rho' \hat{\rho}'$, $\hat{\underline{R}} = \frac{\underline{r} - \underline{r}'}{R} = \frac{3\hat{z} - \rho'\hat{\rho}'}{\sqrt{9 + \rho'^2}}$

$$\underline{E} = \int_0^{2\pi} \int_0^a \frac{\cancel{(\rho'^2 + 9)^{3/2}} (3\hat{z} - \rho'\hat{\rho}')}{4\pi\epsilon_0 \cancel{(9 + \rho'^2)^{3/2}}} \rho' d\rho' d\phi'$$

integrates to zero

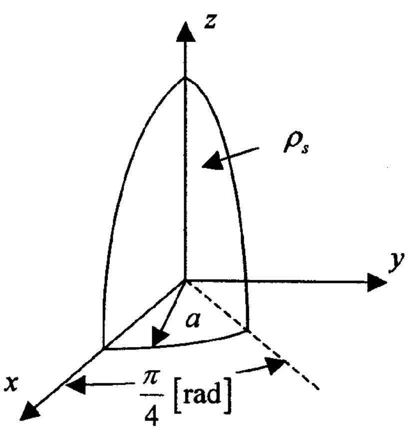
$$= \frac{3\,\cancel{6}\cancel{\pi}}{2\,\cancel{4}\cancel{\pi}\epsilon_0} \hat{z} \int_0^a \rho' d\rho' = \hat{z}\frac{3}{2\epsilon_0} \frac{a^2}{2} = \hat{z}\frac{3a^2}{4\epsilon_0} \; [V/m]$$

E = $\hat{z}\,\frac{3a^2}{4\epsilon_0}$ [V/m]

ROOM FOR EXTRA WORK

Problem 3 (25 pts)

A surface charge density $\rho_s = \rho_{s0} \dfrac{\cos\phi}{\sin\theta}$ [C/m^2], where ρ_{s0} is a constant, exists on the portion $0 < \theta < \dfrac{\pi}{2}$, $0 < \phi < \dfrac{\pi}{4}$ of a sphere of radius a as shown. Find the total charge on this part of the sphere.



$$Q = \int_S \rho_s \, dS = \int_0^{\frac{\pi}{4}}\int_0^{\pi/2} \frac{\rho_{s0}\cos\phi}{\cancel{\sin\theta}} \, \overbrace{a^2 \cancel{\sin\theta}\, d\theta\, d\phi}^{dS}$$

$$= a^2 \rho_{s0} \frac{\pi}{2} \int_0^{\frac{\pi}{4}} \cos\phi \, d\phi = a^2 \rho_{s0} \frac{\pi}{2} \sin\phi \Big|_0^{\frac{\pi}{4}} = a^2 \rho_{s0} \frac{\pi}{2} \frac{\sqrt{2}}{2}$$

$$= a^2 \rho_{s0} \frac{\pi\sqrt{2}}{4} \; [C]$$

Total charge = $\rho_{s0} \dfrac{\pi\sqrt{2}}{4} a^2$ [C]

ROOM FOR EXTRA WORK

Problem 4 (25 pts)

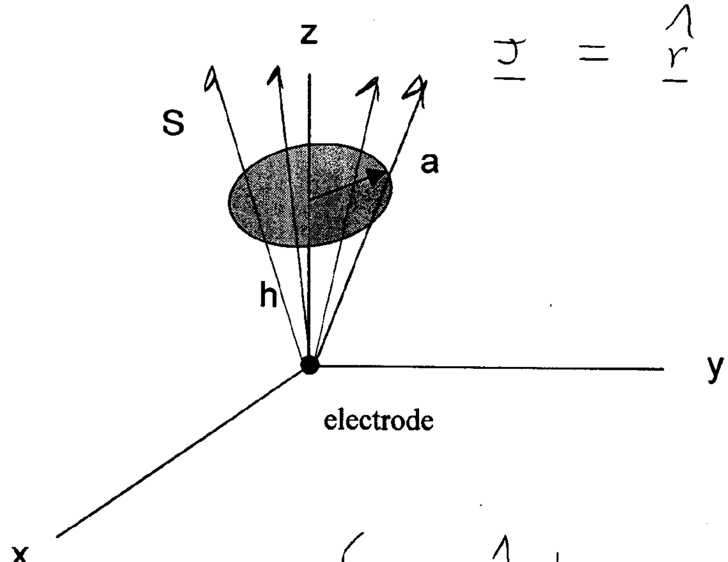
A point electrode is at the origin inside a large tank of saltwater. The electrode injects a total current of I_0 [A], which spreads equally in all directions. This produces a current density vector in space that is described in spherical coordinates by

$$J = \hat{r}\left(\frac{I_0}{4\pi r^2}\right)\ [\mathrm{A/m^2}]$$

Calculate the total current I that crosses the circular area S shown below, flowing in the upward direction. Do this by performing an integration in cylindrical coordinates, over the area of the circle.

The circle lies in a plane that is parallel to the xy plane, and is centered on the z axis. The radius of the circle is a, and it lies at a height h above the xy plane.

$$\underline{J} = \underline{\hat{r}}\left(\frac{I_0}{4\pi r^2}\right)$$



$$I = \int_S \underline{J}\cdot\underline{\hat{n}}\,ds \ ; \quad \underline{\hat{n}} = \underline{\hat{z}}$$

Total current = $\dfrac{I_0}{2}\left(1 - \dfrac{h}{\sqrt{h^2+a^2}}\right)$ [A]

ROOM FOR EXTRA WORK

So
$$I = \int_S J_z \, ds$$

$$J_z = \underline{J} \cdot \hat{\underline{z}} = \frac{I_0}{4\pi r^2} \left(\hat{\underline{r}} \cdot \hat{\underline{z}} \right)$$

$$= \frac{I_0}{4\pi r^2} \cos\theta$$

$$= \frac{I_0}{4\pi r^2} \left(\frac{h}{r} \right)$$

$$= \frac{I_0 h}{4\pi r^3} = \frac{I_0 h}{4\pi (p^2 + z^2)^{3/2}}$$

So
$$I = \int_0^{2\pi} \int_0^a \frac{I_0 h}{4\pi (p^2 + z^2)^{3/2}} \, p \, dp \, d\phi$$

$$= (2\pi) \left(\frac{I_0}{4\pi} \right) h \int_0^a \frac{p \, dp}{(p^2 + z^2)^{3/2}}$$

$$= \frac{I_0 h}{2} \left[- (p^2 + h^2)^{-1/2} \right]_0^a$$