Name: Soltn

Student Number: ____________________

ELEE 2317
Applied Electricity and Magnetism
Exam 2
November 21, 1998

1. This exam is closed book and closed notes. Calculators may be used.
2. For all solutions, ***no credit*** will be given if the work required to obtain the solution is not shown.
3. Perform all your work on the paper provided. If additional paper is needed, get it from the instructor.
4. You will have a total of 90 minutes.

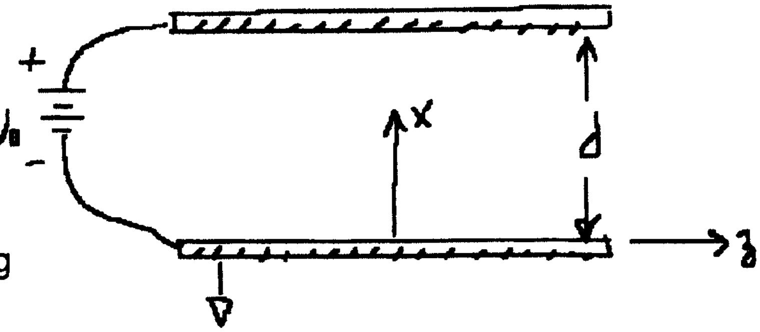
Do not write below this line.

_____/20 Prob. 1
_____/20 Prob. 2
_____/20 Prob. 3
_____/20 Prob. 4
_____/20 Prob. 5

Total _______/100

Problem 1 (20 pts):

For the parallel-plate structure shown in the figure,



Neglect Fringing

(a) Use Laplace's equation to find an expression for the potential as a function of x between the two parallel plates separated by a distance d as shown in the figure. The upper plate is at a potential V_0 and the lower plate is grounded. Assume that there is no charge between the plates.

(b) Now find the potential assuming that the region between the conductors is filled with a constant volume charge density ρ_V [C/m^3].

Room for additional work

a) Neglecting fringing, potential is a function of x only: $\Phi \equiv \Phi(x)$.

$$\nabla^2 \Phi = \left(\frac{\partial^2}{\partial x^2} + \cancel{\frac{\partial^2}{\partial y^2}}^{\,0} + \cancel{\frac{\partial^2}{\partial z^2}}^{\,0}\right)\Phi(x) = \frac{d^2\Phi}{dx^2} = 0$$

Since $\rho_v \equiv 0$ between conductors.

Integrating once;

$$\Rightarrow \frac{d\Phi}{dx} = A$$

Integrating a second time:

$$\Phi = Ax + B$$

Applying boundary conditions (B.C.'s):

@ $x = 0$, $\Phi = A \cdot 0 + B = 0 \Rightarrow B = 0$

Hence $\Phi = Ax$

@ $x = d$, $\Phi = Ad = V_o \Rightarrow A = \frac{V_o}{d}$

$$\Rightarrow \boxed{\Phi = V_o \frac{x}{d}}$$

b) Now $\frac{d^2\Phi}{dx^2} = \frac{-\rho_v}{\epsilon_0}$ (a const). Integrating twice yields

$\frac{d\Phi}{dx} = \frac{-\rho_v}{\epsilon_0} x + A$,

$$\Phi = \underbrace{\frac{-1}{2}\frac{\rho_v}{\epsilon_0} x^2}_{\text{part. soln}} + \underbrace{Ax + B}_{\text{homogeneous soln}}$$

Applying B.C.'s:

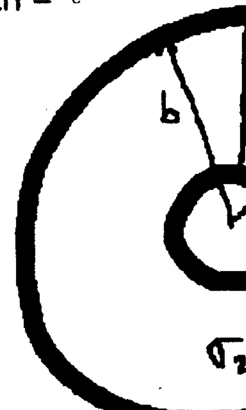
@ $x = 0$, $\Phi = 0 = 0 + A \cdot 0 + B \Rightarrow B = 0$

@ $x = d$, $\Phi = V_o = -\frac{1}{2}\frac{\rho_v}{\epsilon_0} d^2 + Ad \Rightarrow A = \frac{1}{d}\left(V_o + \frac{1}{2}\frac{\rho_v}{\epsilon_0}\right.$

$$\Rightarrow \Phi = -\frac{1}{2}\frac{\rho_v}{\epsilon_0}(x^2 - dx) + V_o\left(\frac{x}{d}\right)$$

or $\boxed{\Phi = -\frac{1}{2}\rho_v x(x-d) + V_o \frac{x}{d}}$

Length = ℓ



Problem 2 (20 pts):

Consider a coaxial resistor of length ℓ with a perfectly conducting inner conductor of radius a and a perfectly conducting outer conductor of radius b. One quadrant of the resistor's cross section is filled with a conducting material of conductance σ_1; the remaining region has a conductance σ_2. Neglect fringing at the ends of the coax.

4 (a) Circle the quantity that will exist between the perfect conductors and will be continuous at the boundary between the conducting regions 1 and 2:

$$\underbrace{J_\rho, \quad E_\rho,}_{\text{only components present}} \quad J_\phi, \quad E_\phi, \quad J_z, \quad E_z$$

E_ρ is continuous at boundary (between σ_1 & σ_2) since it is a tangential component. Hence

$J_{\rho 1} = \sigma_1 E_\rho \neq J_{\rho 2} = \sigma_2 E_\rho$

4 (b) Circle the form of the dependence on ρ that the quantity of part (a) will have in the region between the perfect conductors:

$$A\rho, \quad A \ln \rho, \quad \frac{A}{\rho}, \quad \frac{A}{\rho^2}$$

Same form as Gauss's law analog.
OR note potential is $\Phi(\rho)$, so
$\nabla^2 \Phi = \frac{1}{\rho}\frac{d}{d\rho}(\rho \Phi') = 0 \Rightarrow \Phi = A' \ln \rho + B$
$E = -\nabla\Phi = -\frac{A'}{\rho} = \frac{A}{\rho}$ (absorb sign of A' into A)

4 (c) Using the results from parts (a) and (b), find the voltage between the inner and outer conductors and total current leaving the inner conductor in terms of the constant A:

$$V = \int_a^b E_\rho \, d\rho = A \int_a^b \frac{d\rho}{\rho} = A \ln b/a$$

[Note: now we know that $A = \frac{V}{\ln b/a}$]

4 (d) Find the resistance of the partially filled coaxial resistor.

$$\frac{1}{R} = G = \frac{I}{V} \; ; \quad I = \underbrace{\ell \int_0^{\pi/2} J_{\rho 1}\, a \, d\phi}_{I_1} + \underbrace{\ell \int_{2\pi}^{\pi/2} J_{\rho 2}\, a\, d\phi}_{I_2} = \ell \sigma_1 \int_0^{\pi/2} \frac{A}{\rho}\Big|_{\rho=a} a \, d\phi + \ell \sigma_2 \int_{2\pi}^{\pi/2} \frac{A}{\rho}\Big|_{\rho=a} a\, d\phi$$

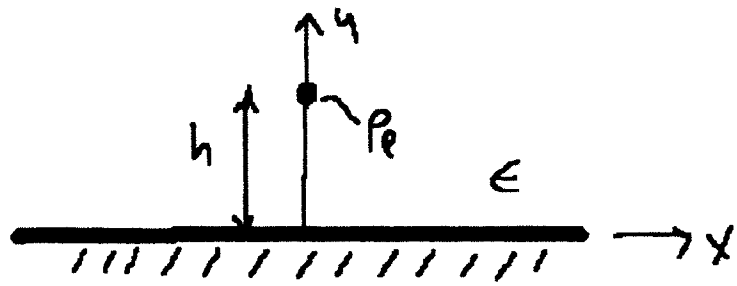
$$= \ell A \left[\sigma_1 \frac{\pi}{2} + \sigma_2 \frac{3\pi}{2} \right]$$

$$G = \frac{\ell A [\sigma_1 \pi/2 + \sigma_2 3\pi/2]}{A \ln b/a} \quad \left(\xrightarrow[\sigma_1 \to \sigma_2 = \sigma]{} \frac{2\pi\sigma\ell}{\ln b/a} \text{ (check!)} \right)$$

$$R = \frac{\ln b/a}{\ell [\sigma_1 \pi/2 + \sigma_2 3\pi/2]}$$

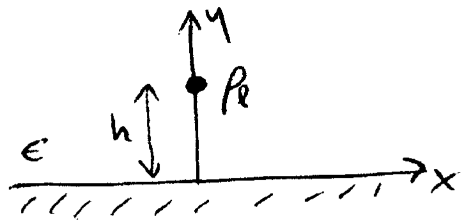
Problem 4 (20 pts):

Consider an infinitely long line charge, characterized by ρ_ℓ, a distance h above a conducting ground plane.

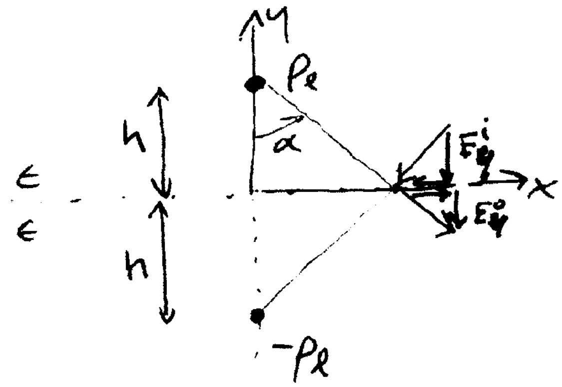


Find the electric field along the ground plane (y>0).

(#4)



$\Rightarrow$



$\bar{E} = \bar{E}^o + \bar{E}^i$ ← image line source 4

↑ original line source

Along ground plane $(y=0)$ $\bar{E} = E_y \hat{y}$

$\cancel{E_y = E_y^o + E_y^i = 2}$

5 5

$$\bar{E} = \frac{\rho_\ell}{2\pi R_o \epsilon_o} \frac{(+x\hat{x} + (-h)\hat{y})}{R_o} + \frac{-\rho_\ell}{2\pi R_i \epsilon_o} \frac{(x\hat{x} + h\hat{y})}{R_i}$$

$$R = R_o = R_i = [x^2 + h^2]^{1/2}$$

$$= \frac{-\rho_\ell (2h)}{2\pi\epsilon_o R} \frac{\hat{y}}{R} = \frac{-2\rho_\ell}{2\pi\epsilon_o R} \frac{h}{R} \hat{y} = \frac{-\rho_\ell}{\pi\epsilon_o [x^2+h^2]^{1/2}} \cos\alpha \, \hat{y}$$

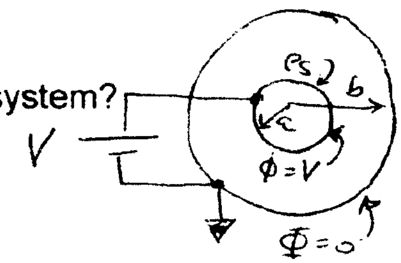
4

$$= \frac{-\rho_\ell h}{\pi\epsilon_o (x^2+h^2)} \hat{y}$$

Problem 5 (20 pts):

Consider a spherical capacitor constructed with two concentric conducting spheres with radii 1 [cm] and 5 [cm], respectively. The region between the conductors is free space. A potential difference of 3600 [V] is maintained between the conductors. The surface charge density on the inner conductor is 4 [μC/m^2].

What is the total electric energy stored in this system?



$$U_E = \frac{1}{2}\int_V \rho_v \Phi \, dV$$

$$\longrightarrow \frac{1}{2}\int_S \underbrace{\rho_s \Phi}_{\text{constant } r=a!} ds$$

$$= \frac{1}{2}\rho_s V \underbrace{4\pi a^2}_{\text{surf. area}} - \underbrace{0}_{\text{potential at } r=b!}$$

$$= \frac{1}{2} \times (4\times10^{-6})(3600)\,4\pi\,(10^{-2})^2$$

$$= 90.5\times10^{-7} = 9.05\,[\mu J]$$

alternate soln.

Find capacitance using Gauss' Law to be $C = \frac{4\pi\epsilon}{1/a - 1/b}$

$$U_E = \frac{1}{2}CV^2 = \frac{1}{2}\left(\frac{4\pi\epsilon}{1/a - 1/b}\right)(V^2)$$

$$= \frac{1}{2}\,\frac{4\pi\times\left(\frac{1}{36\pi}\times10^{-9}\right)}{\frac{1}{1\times10^{-2}} - \frac{1}{5\times10^{-2}}}\,(3600)^2$$

$$\cong 9.0\,[\mu J]$$

Quiz #2

Name: Solution
Student #:

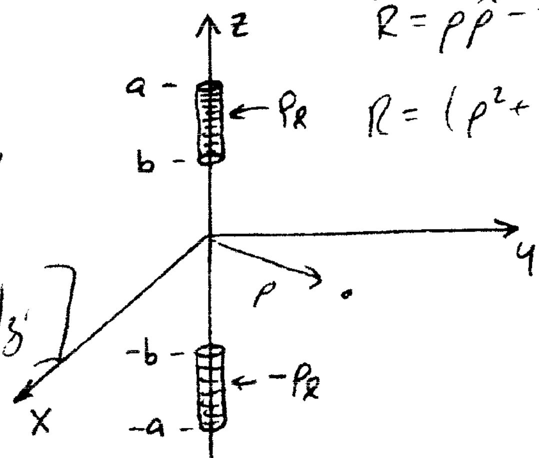
ELEE 2317
Spring 1998
Sec. 10211

Please put your *name* and *student number* at the top of each page. Note that this quiz is closed book and closed notes; calculators are allowed. Perform all of your work on the paper provided. The instructor will provide any additional paper that may be required. Note that no credit will be given for answers in which the work required to develop the solution is not shown. If there are any questions concerning the quiz, ask the instructor for clarification. *Good Luck!*

Problem 1: (25 pts)
Find the electric field in the x-y plane due to the pair of uniformly charged line sources shown in the figure.

$$\hat{R} = \rho\hat{\rho} - z'\hat{z}$$

$$R = (\rho^2 + z'^2)^{1/2}$$



$$\bar{E} = \frac{1}{4\pi\epsilon_0}\int_\ell \frac{\rho_\ell}{R^2}\hat{R}\,d\ell'$$

$$= \frac{1}{4\pi\epsilon_0}\left[\int_b^a \frac{\rho_\ell(\rho\hat{\rho} - z'\hat{z})}{(\rho^2+z'^2)^{3/2}}dz' + \int_{-a}^{-b}\frac{(-\rho_\ell)(\rho\hat{\rho} - z'\hat{z})}{(\rho^2+z'^2)^{3/2}}dz'\right]$$

$$= \frac{1}{4\pi\epsilon_0}\left[\int_b^a \frac{\rho_\ell(\rho\hat{\rho} - z'\hat{z})}{(\rho^2+z'^2)^{3/2}}dz' + \int_b^a \frac{\rho_\ell(\rho\hat{\rho} + z'\hat{z})}{(\rho^2+z'^2)^{3/2}}(-dz')\right]$$

$$\Rightarrow \bar{E} = \frac{1}{4\pi\epsilon_0}(-2\hat{z})\rho_\ell\int_b^a \frac{z'}{(\rho^2+z'^2)}dz'$$

Useful integral relationships:

$$\int \frac{u\,du}{(u^2+b^2)^{3/2}} = -(u^2+b^2)^{-1/2}$$

$$\int \frac{u\,du}{(u^2+b^2)^{1/2}} = (u^2+b^2)^{1/2}$$

$$\int \frac{du}{(u^2+b^2)^{1/2}} = \ln\left[u + (u^2+b^2)^{1/2}\right]$$

$$\int \frac{du}{(u^2+b^2)^{3/2}} = \frac{u}{b^2}(u^2+b^2)^{-1/2}$$

$$\bar{E} = -\hat{z}\frac{\rho_\ell}{2\pi\epsilon_0}\left[\frac{1}{(b^2+\rho^2)^{1/2}} - \frac{1}{(a^2+\rho^2)^{1/2}}\right]$$

in x-y plane

Quiz # 4

Name: Sol'n

ELEE 2317

Student #:____________________

Fall 1998

Please put your *name* and *student number* at the top of each page. Note that this quiz is closed book and closed notes; calculators are allowed. Perform all of your work on the paper provided. The instructor will provide any additional paper that may be required. Note that no credit will be given for answers in which the work required to develop the solution is not shown. If there are any questions concerning the quiz, ask the instructor for clarification. *Good Luck!*

The following formulas may be useful:

$$\nabla\Phi = \frac{\partial\Phi}{\partial\rho}\hat{\boldsymbol{\rho}} + \frac{1}{\rho}\frac{\partial\Phi}{\partial\phi}\hat{\boldsymbol{\phi}} + \frac{\partial\Phi}{\partial z}\hat{\mathbf{z}} \qquad \nabla^2\Phi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial z^2}$$

Problem 1: (25 pts)

A coaxial conductor has inner and outer conductors of radii a and b, respectively, with $b > a$. The region between the conductors is free space ($\varepsilon_0 \approx \frac{1}{36\pi} \times 10^{-9}$). A potential difference $\Phi_{ab} = V$ [V] is maintained between the conductors. The outer conductor is grounded.

a) Using Laplace's equation, find an expression for the potential between the conductors in terms of $a, b, V,$ and ε_0.

Potential:____________________

b) Find an expression for the surface charge density on the conductor of radius a.

Expression for surface charge density:____________________

c) Evaluate the surface charge density of part b) if $a = 1\,[\text{cm}],\ b = 4\,[\text{cm}],\ V = 5\,[\text{V}]$.

Value for surface charge density:____________________