Name: SOLUTION

Student Number: ____________________

ELEE 2317
Applied Electricity and Magnetism
Exam 2
April 24, 1999

1. This exam is closed book and closed notes. **No Calculators are allowed.**
2. For all solutions, ***no credit*** will be given if the work required to obtain the solution is not shown.
3. Perform all your work on the paper provided. If additional paper is needed, get them from the instructor.
4. You will have a total of 90 minutes.

Do not write below this line.

_____/20 Prob. 1	_____/20 Prob. 4
_____/20 Prob. 2	_____/20 Prob. 5
_____/20 Prob. 3	

Total _______/100

Useful Formulas

$$\nabla \times \mathbf{F} = \frac{1}{\rho}\begin{vmatrix} \hat{\boldsymbol{\rho}} & \rho\hat{\boldsymbol{\phi}} & \hat{\mathbf{z}} \\ \dfrac{\partial}{\partial \rho} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ F_\rho & \rho F_\phi & F_z \end{vmatrix}.$$

$$\nabla \times \mathbf{F} = \frac{1}{r^2 \sin\theta}\begin{vmatrix} \hat{\mathbf{r}} & r\hat{\boldsymbol{\theta}} & r\sin\theta\,\hat{\boldsymbol{\phi}} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ F_r & r F_\theta & r\sin\theta\, F_\phi \end{vmatrix}.$$

$$\nabla \cdot \mathbf{F} = \frac{1}{\rho}\frac{\partial(\rho F_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial F_\phi}{\partial \phi} + \frac{\partial F_z}{\partial z}$$

$$\nabla \cdot \mathbf{F} = \frac{1}{r^2}\frac{\partial(r^2 F_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta\, F_\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial \phi}$$

$$\nabla V = \frac{\partial V}{\partial \rho}\hat{\boldsymbol{\rho}} + \frac{\partial V}{\rho\,\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial V}{\partial z}\hat{\mathbf{z}}$$

$$\nabla V = \frac{\partial V}{\partial r}\hat{\mathbf{r}} + \frac{\partial V}{r\,\partial \theta}\hat{\boldsymbol{\theta}} + \frac{\partial V}{r\sin\theta\,\partial \phi}\hat{\boldsymbol{\phi}}$$

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$$

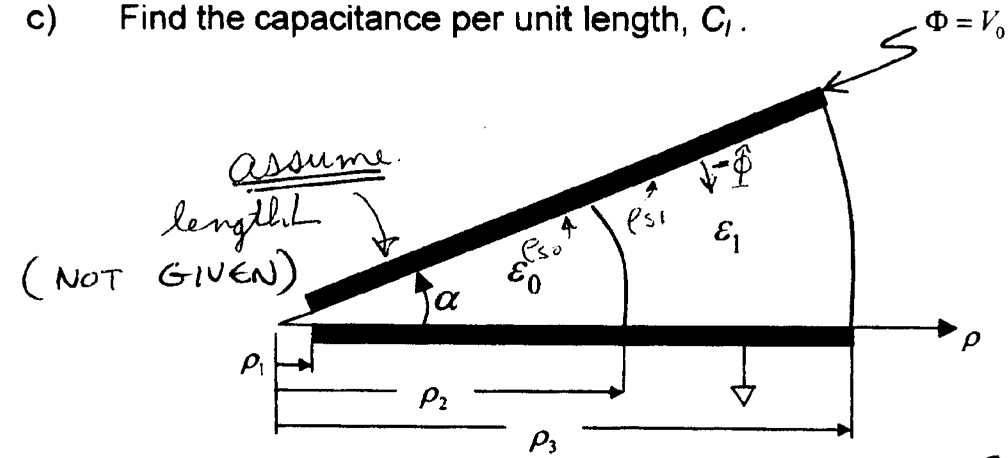
Problem 1 (20 pts):

Given the 2 region "angular capacitor" shown in the figure

a) Use Laplace's equation to find the potential Φ between the 2 conductors. Assume $\Phi = \Phi(\phi)$ and neglect fringing.

b) Find the total charge on each plate.

c) Find the capacitance per unit length, C_l.



a) $\nabla^2 \Phi(\phi) = \frac{1}{\rho^2}\frac{d^2\Phi}{d\phi^2} = 0 \Rightarrow \frac{d\Phi}{d\phi} = A, \Rightarrow \Phi = A\phi + B$

$@\ \phi = 0, \quad \Phi = 0 = A\cdot 0 + B \Rightarrow B = 0$

$@\ \phi = \alpha, \quad \Phi = V_0 = A\alpha \Rightarrow A = \frac{V_0}{\alpha}$

$\Rightarrow \Phi = V_0 \frac{\phi}{\alpha}$

b) $\underline{E} = -\nabla\Phi = -\frac{1}{\rho}\frac{d\Phi}{d\phi}\hat{\underline{\Phi}} = -\frac{V_0}{\alpha\rho}\hat{\underline{\phi}}, \quad \underline{D} = \epsilon\underline{E}, \quad \rho_s = D\cdot\hat{n}$

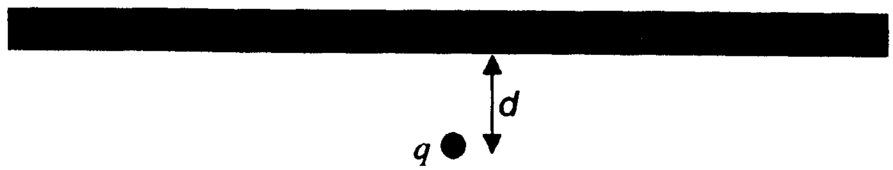
$$\Rightarrow Q = \int_0^L\int_{\rho_1}^{\rho_2} -\frac{\epsilon_0 V_0}{\alpha\rho}\hat{\underline{\phi}}\cdot(-\hat{\underline{\Phi}})\,d\rho\,dz + \int_0^L\int_{\rho_2}^{\rho_3} -\frac{\epsilon_1 V_0}{\alpha\rho}\hat{\underline{\phi}}\cdot(-\hat{\underline{\phi}})\,d\rho\,dz$$

$$= \frac{LV_0}{\alpha}\left[\epsilon_0 \ln \rho_2/\rho_1 + \epsilon_1 \ln \rho_3/\rho_2\right]$$

c) $C_\ell = \frac{C}{L} = \frac{Q}{LV_0} = \frac{\epsilon_0 \ln \rho_2/\rho_1 + \epsilon_1 \ln \rho_3/\rho_2}{\alpha}$

Problem 2 (20 pts):

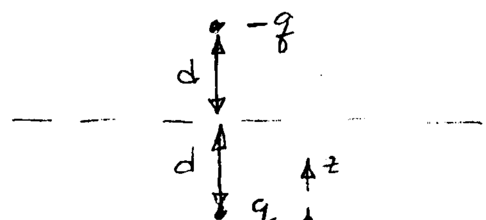
A charged particle q, with a mass m, is suspended a distance d below a flat conducting plate, as shown in the figure.



a) Find an expression for the location d in terms of q, m, and the gravitational force g. (Note that this balanced location is not stable.)

b) How does the answer in part a) change if q is replaced with $-q$?

a)



$$F_z = \frac{q \times q}{4\pi\epsilon_0 (2d)^2} = mg$$

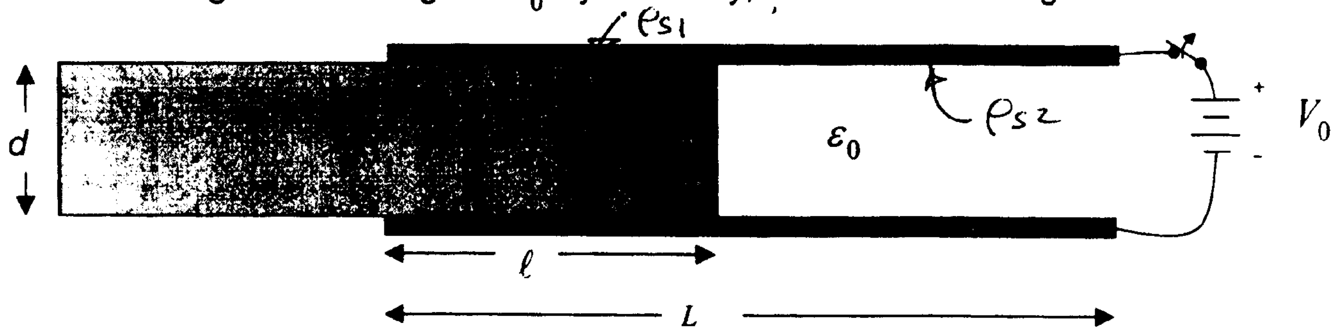
$$\Rightarrow d^2 = \frac{q^2}{16\pi\epsilon_0 m g}$$

$$d = \frac{q}{4\sqrt{\pi\epsilon_0 m g}}$$

b) Does not change.

Problem 3 (20 pts):

A parallel-plate capacitor of width W and length L is filled with a solid dielectric slab of permittivity ε and thickness d. The capacitor is charged to a voltage of V_0 by a battery, as shown in the figure.



Total Charge Q is conserved; $Q = \rho_s WL = \epsilon_r \epsilon_o \frac{V_o}{d} WL$

Assuming that the switch is opened and *then* the slab is withdrawn to the position shown,

a) Determine the surface charge distribution on the top plate after the slab has been withdrawn.

b) Determine the total charge on the top plate after the slab has been withdrawn.

c) Determine the voltage between the conductors after the slab has been withdrawn.

d) Determine the energy in the capacitor before *and* after the slab has been withdrawn. If these energies are different, why?

a) $Q = \rho_{s1} \ell W + \rho_{s2} (L-\ell) W = \epsilon_r \epsilon_o E \ell W + \epsilon_o E (L-\ell) W$

same in both regions

$\Rightarrow \epsilon_r \cancel{\epsilon_o} \frac{V_o}{d} \cancel{W} L = \cancel{\epsilon_o} E \left[\epsilon_r \ell + (L-\ell)\right] \cancel{W} \Rightarrow E = \frac{\epsilon_r V_o L}{d\left[\epsilon_r \ell + (L-\ell)\right]}$

so $\rho_{s1} = \epsilon_r \epsilon_o E = \frac{\epsilon_r^2 \epsilon_o V_o L}{d\left[\epsilon_r \ell + (L-\ell)\right]}$, $\rho_{s2} = \epsilon_o E = \frac{\epsilon_r \epsilon_o V_o L}{d\left[\epsilon_r \ell + (L-\ell)\right]}$

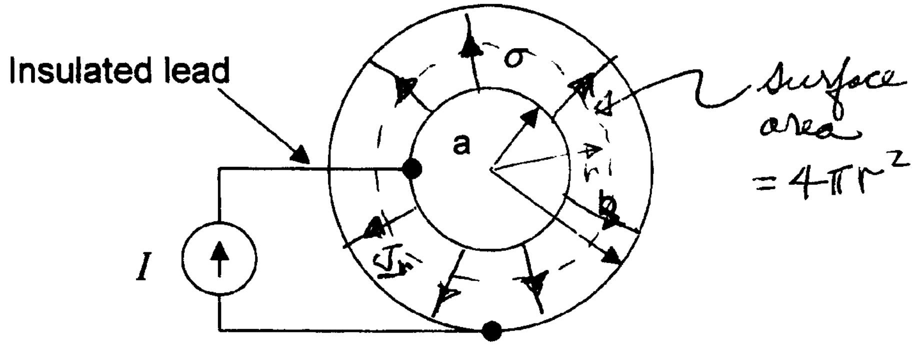
b) Should be Q; to check, $\rho_{s1} \ell W + \rho_{s2} (L-\ell) W = \frac{\epsilon_r \epsilon_o V_o L \left[\cancel{\epsilon_r \ell + (L-\ell)}\right] W}{d\left[\cancel{\epsilon_r \ell + (L-\ell)}\right]} =$

c) $V = Ed = \frac{\epsilon_r V_o L}{\epsilon_r \ell + (L-\ell)}$

d) $U_{E0} = \frac{1}{2} \epsilon_r \epsilon_o \left(\frac{V_o}{d}\right)^2 WLd$; $U_{E1} = \frac{1}{2} \frac{\epsilon_r^2 V_o^2 L^2 \epsilon_o}{d^2 \left[\epsilon_r \ell + (L-\ell)\right]^2} \left[\cancel{\epsilon_r \ell + (L-\ell)}\right] d$ $\neq U$

Problem 4 (20 pts):

Consider a "spherical resistor" consisting of concentric conducting spheres, the inner sphere of radius *a* and the outer sphere of radius *b*. The medium between the spheres is filled with a material of conductivity σ. The resistor is connected to a current source of I [A] and the lead to the inner sphere is insulated.



a) What is the total current flowing through any concentric spherical surface of radius r, $a<r<b$? *Justify your answer.*

Total current = I [A]

b) What is the volume current density $\mathbf{J}$ at radius r, $a<r<b$?

$\mathbf{J}$ = $\dfrac{I}{4\pi r^2}\hat{r}$ [A/m²]

$$I = \int J_r\, r^2 \sin\theta\, d\theta\, d\phi = 4\pi r^2 J_r \Rightarrow \underline{J} = \frac{I}{4\pi r^2}\hat{r}$$

c) Find the conductance of the spherical resistor.

Conductance = $\dfrac{4\pi\sigma}{\left[\frac{1}{a}-\frac{1}{b}\right]} = \dfrac{4\pi\sigma ab}{b-a}$ [S or mhos]

$$\sigma E_r = J_r$$

$$E_r = J_r/\sigma$$

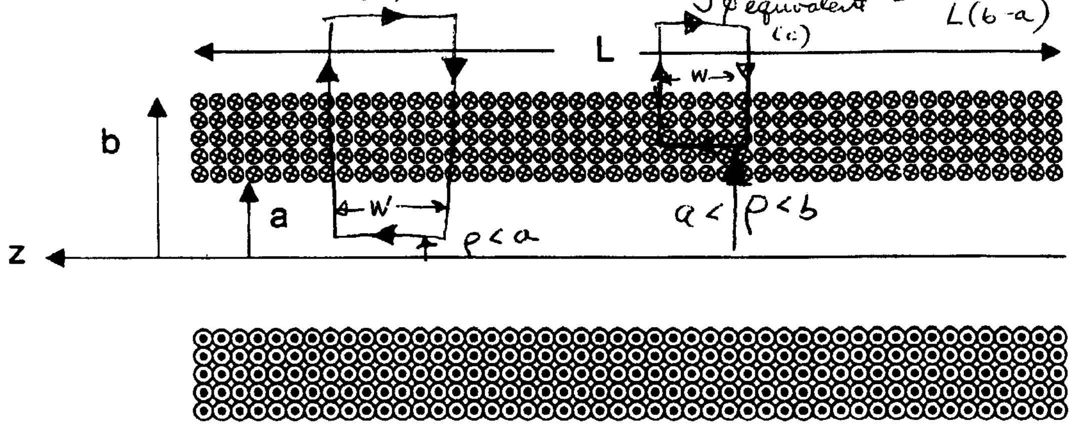
$$V_{ab} = \int_a^b E_r\, dr = \frac{I}{4\pi\sigma}\int_a^b \frac{dr}{r^2} = \frac{I}{4\pi\sigma}\left[\frac{1}{a}-\frac{1}{b}\right]$$

$$G = \frac{I}{V_{ab}} = \frac{4\pi\sigma}{\left[\frac{1}{a}-\frac{1}{b}\right]} = \frac{4\pi\sigma ab}{b-a}$$

Problem 5 (20 pts):

A long solenoid of length L is wound with many layers of thin wire. A cross section of the solenoid is shown in the figure. There are a total of N turns, each carrying a current I. The inner radius of the solenoid is a and the outer radius is b. Neglect fringing of the fields near the end of the solenoid.

$\bar{J}_\phi$ equivalent $= \frac{NI}{L(b-a)}$



a) What is the magnetic field **H** in the region $\rho > b$?

H = 0 [A/m]

Since solenoid of volume current, is like many layers of solenoids composed of sheet currents, each producing zero field outside sheet, sum of fields is 0.

b) Find the magnetic field **H** in the region $\rho < a$.

H = $\frac{NI}{L}\hat{z}$ [A/m]

Using path (b) as shown, $H_z \neq 0$

$$\oint \underline{H} \cdot d\underline{\ell} = H_z W = J_\phi^{\text{equivalent}} \times W(b-a) = \frac{NIW}{L}$$

c) Find the magnetic field **H** in the region $a < \rho < b$.

H = $\frac{NI}{L}\frac{b-\rho}{b-a}\hat{z}$ [A/m]

Using path (c) shown, $H_z \neq 0$,

$$\oint \underline{H} \cdot d\underline{\ell} = H_z W = J_\phi^{\text{equivalent}} W(b-\rho) = \frac{NIW}{L}\frac{(b-\rho)}{(b-a)}$$