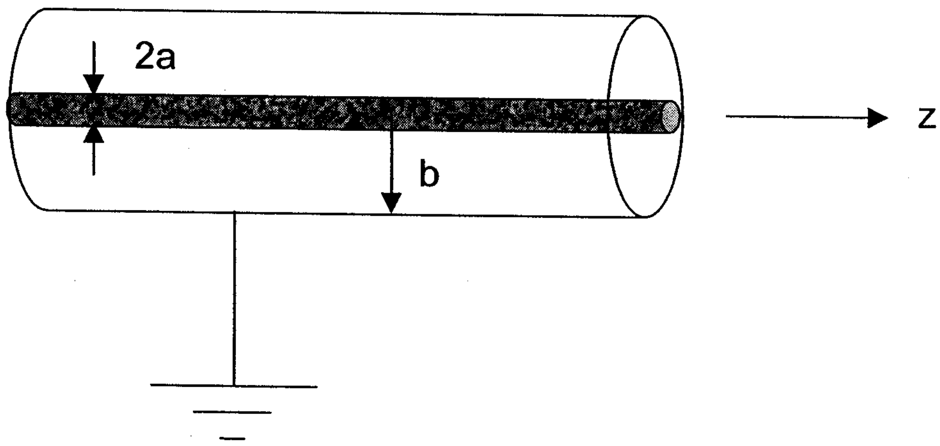
Problem 1 (25 pts)

An electron beam exists in air. The beam has a volume charge density $\rho_v = \rho^2$ [C/m^3] for $\rho < a$, and zero charge density for $\rho > a$. Surrounding this electron beam is a metal shell of radius $\rho = b$. Assume that the metal shell is grounded.

Determine the electric field vector in all three regions:

10 $\rho < a$
8 $a < \rho < b$
7 $\rho > b$



Gauss's law:

$$D_\rho (2\pi \rho h) = Q_{encl} \quad \text{so} \quad D_\rho = \frac{Q_{encl}}{2\pi \rho h}$$

$\rho < a$

$$Q_{encl} = \int_0^h \int_0^{2\pi} \int_0^\rho (\rho^2)\, \rho \, d\rho \, d\phi \, dz$$

$$= 2\pi h \int_0^\rho \rho^3 \, d\rho = 2\pi h \left(\frac{\rho^4}{4} \right),$$

ROOM FOR EXTRA WORK

Hence

$$D_\rho = \frac{\pi h \frac{\rho^4}{2}}{2\pi \rho h} = \frac{\rho^3}{4}$$

or

$$E_\rho = \frac{\rho^3}{4\epsilon_0}$$

(b) $\underline{a < \rho < b}$

$$D_\rho = \frac{\pi h \frac{a^4}{2}}{2\pi \rho h} = \frac{a^4}{4\rho}$$

or

$$E_\rho = \frac{a^4}{4\epsilon_0 \rho}$$

(c) $\underline{\rho > b}$

$E_\rho = 0$ (no field due to the grounding)

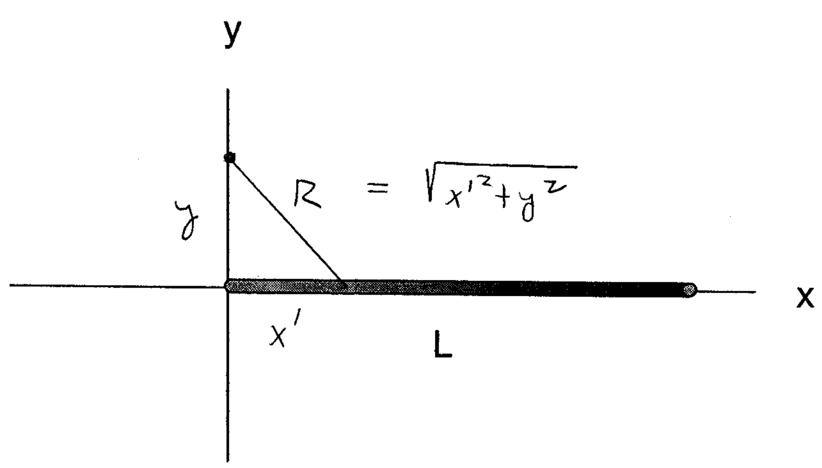
$$\boxed{\underline{E} = \begin{cases} \hat{\underline{\rho}} \, \dfrac{\rho^3}{4\epsilon_0} \ [\mathrm{V/m}], & \rho < a \\ \hat{\underline{\rho}} \, \dfrac{a^4}{4\epsilon_0 \rho} \ [\mathrm{V/m}], & a < \rho < b \\ \underline{0}, & \rho > b \end{cases}}$$

Problem 2 (25 pts)

A line segment of nonuniform charge density $\rho_l = x$ [C/m] is oriented along the x axis from $x = 0$ to $x = L$ as shown below.

20 a) Find the potential at any point on the positive y axis, assuming that the potential is zero at infinity. Use the potential integral formula.

5 b) Modify your answer so that the potential is now V_0 volts at the origin.



(a)

$$\Phi = \int_C \frac{\rho_l(x')}{4\pi\epsilon_0 R}\, dl' \qquad dl' = dx'$$

$$= \int_0^L \frac{x'\, dx'}{4\pi\epsilon_0\sqrt{x'^2+y^2}} = \frac{1}{4\pi\epsilon_0}\left(x'^2+y^2\right)^{1/2}\Big|_0^L$$

$$= \frac{1}{4\pi\epsilon_0}\left[\sqrt{L^2+y^2} - y\right]$$

ROOM FOR EXTRA WORK

$$\boxed{\Phi = \frac{1}{4\pi\epsilon_0}\left[\sqrt{L^2+y^2} - y\right] \quad [V]}$$

(b)

$$\Phi' = \Phi + C$$

$$\Phi'(0,0,0) = \Phi(0,0,0) + C = V_0$$

So $\quad C = V_0 - \Phi(0,0,0)$

Hence

$$\boxed{\Phi' = \frac{1}{4\pi\epsilon_0}\left[\sqrt{L^2+y^2} - y\right] + V_0 - \frac{L}{4\pi\epsilon_0} \quad [V]}$$

Problem 3 (25 pts)

Two students propose the following two different flux densities as solutions to an electrostatics problem:

$$\mathbf{D}_1 = \varepsilon_0 \rho^2 \hat{\boldsymbol{\rho}},$$
$$\mathbf{D}_2 = \varepsilon_0 \rho^2 \sin^2\phi\, \hat{\boldsymbol{\rho}} + 3\varepsilon_0 \rho z \cos^2\phi\, \hat{\boldsymbol{\phi}},$$

a. Show that the volume charge densities derived from these two fields are identical. $\nabla \cdot \underline{D}_1 = \epsilon_0 \frac{1}{\rho}\frac{d}{d\rho}(\rho \cdot \rho^2) = \epsilon_0 3\rho$

$$\nabla \cdot \underline{D}_2 = \epsilon_0 \frac{1}{\rho}\frac{d}{d\rho}(\rho^3 \sin^2\phi) + \frac{3\epsilon_0}{\rho}\frac{\partial}{\partial\phi}(\rho z \cos^2\phi)$$
$$= 3\epsilon_0 \rho \sin^2\phi - 3\epsilon_0 z \cdot 2\cos\phi \sin\phi$$

ρ_v corresponding to $\mathbf{D}_1$ = $3\epsilon_0\rho$ [C/m^3]

ρ_v corresponding to $\mathbf{D}_2$ = $3\epsilon_0\rho \sin^2\phi - 6\epsilon_0 z \cos\phi \sin\phi$ [C/m^3]

b. Choose one of the following equivalent statements and, *showing all work*, use it to determine which of the flux densities in part **a** is a *valid* electric field:

i) $\oint_C \mathbf{E}\cdot d\boldsymbol{\ell} = 0$
ii) $\nabla \times \mathbf{E} = 0$
iii) $\mathbf{E} = -\nabla\Phi$

$$\nabla \times \frac{D_1}{\epsilon_0} = \nabla \times \underline{E}_1 = \hat{\phi}\, \overbrace{\cancel{\frac{\partial E_{1\rho}}{\partial z}}}^{0} - \hat{z}\, \overbrace{\cancel{\frac{1}{\rho}\frac{\partial E_{1\rho}}{\partial \phi}}}^{0} \quad \boxed{= 0}$$

So E_1 is a valid field

$$\nabla \times \frac{D_2}{\epsilon_0} = \nabla \times \underline{E}_2 = \hat{\rho}\left[-\frac{\partial E_{2\phi}}{\partial z}\right] + 3\hat{\phi}\left[\frac{\partial E_{2\rho}}{\partial z}\right] + \hat{z}\,\frac{1}{\rho}\left[\frac{\partial(\rho E_{2\phi})}{\partial \rho} - \frac{\partial E_{2\rho}}{\partial \phi}\right]$$
$$= \hat{\rho}\,\underbrace{(-3\epsilon_0\rho\cos^2\phi)}_{\neq 0} + \ldots$$

Circle the valid field: $\mathbf{D}_1$ $\mathbf{D}_2$

c. Assume that the potential associated with the charge distribution in part **a** depends on ρ alone, that is, we may write the potential as $\Phi(\rho)$. Show that setting $\mathbf{E} = -\nabla\Phi$ results in a simple differential equation for the potential. Integrate the differential equation to obtain the potential using the *valid* field from parts **a** and **b.** You need not choose a reference point.

$$E_1 = \frac{D_1}{\epsilon_0} = = \rho^2 \hat{\rho} = -\overbrace{\frac{d\Phi}{d\rho}\hat{\rho}}^{\nabla\Phi}$$

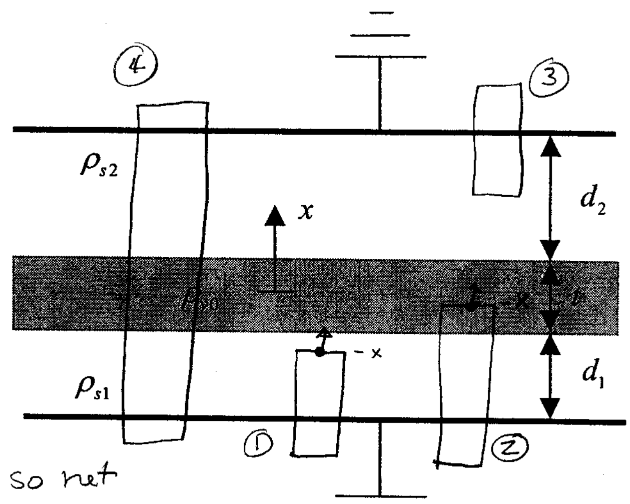
$$\Rightarrow \frac{d\Phi}{d\rho} = -\rho^2$$

$$\Phi = -\frac{\rho^3}{3} + C$$

$\Phi =$ $-\rho^3/3 + C$ ($\leftarrow$ not needed) [V]

Problem 4 (25 pts)

A slab of charge of thickness t having a uniform volume charge density ρ_{v0} is placed between two grounded conductors at distances of d_1 and d_2 from the opposite edges of the slab as shown.



a. Determine the surface charge densities on the two conducting plates. Flux in 4 is zero, so net charge is zero: $\rho_{s1} A + \rho_{s2} A + \rho_{v0} A t = 0$

$\Rightarrow \rho_{s1} + \rho_{s2} = -\rho_{v0} t.$

SEE EXTRA WORK PAGE. IN FOLLOWING PROBS. ASSUME ρ_{s1}, ρ_{s2} KNOWN.

ρ_{s1} = ______ [C/m²]

ρ_{s2} = ______ [C/m²]

b. Find the electric field in the three regions between the grounded conductors.

Refer to Gaussian surf. above

1: $\epsilon_0 E_x \cancel{A} = \rho_{s1} \cancel{A} \Rightarrow E_x = \frac{\rho_{s1}}{\epsilon_0}$

2: $\epsilon_0 E_x \cancel{A} = \rho_{s1} \cancel{A} + \rho_{v0} \cancel{A} (x + t/2)$

3: $-\epsilon_0 E_x \cancel{A} = \rho_{s2} \cancel{A}$

$\mathbf{E} = \frac{\rho_{s1}}{\epsilon_0} \hat{x}$ [V/m], $-d_1 - \frac{t}{2} < x < -\frac{t}{2}$

$\mathbf{E} = \left[\frac{\rho_{s1}}{\epsilon_0} + \frac{\rho_{v0}}{\epsilon_0}(x + \frac{t}{2})\right] \hat{x}$ [V/m], $-\frac{t}{2} < x < \frac{t}{2}$

$\mathbf{E} = -\frac{\rho_{s2}}{\epsilon_0} \hat{x}$ [V/m], $\frac{t}{2} < x < \frac{t}{2} + d_2$

c. Find the potential in the three regions between the grounded conductors. Assume the potential is zero on the bottom plate.

$-\frac{t}{2} - d_1 < x < -\frac{t}{2}$:

$$\Phi(x) = \cancel{\Phi\left(-\frac{t}{2} - d_1\right)}^{\,0 \text{ (Ground)}} - \int_{-\frac{t}{2}-d_1}^{x} \frac{\rho_{s1}}{\epsilon_o}\, dx = -\frac{\rho_{s1}}{\epsilon_o}\left(x + \frac{t}{2} + d_1\right)$$

$-\frac{t}{2} < x < \frac{t}{2}$:

$$\Phi(x) = \underbrace{\Phi(-t/2)}_{\text{use above result}} - \int_{-t/2}^{x} \left[\frac{\rho_{vo}}{\epsilon_o}\left(x + \frac{t}{2}\right) + \frac{\rho_{s1}}{\epsilon_o}\right] dx$$

$$= -\frac{\rho_{s1} d_1}{\epsilon_o} - \frac{\rho_{vo}}{2\epsilon_o}\left(x + \frac{t}{2}\right)^2 - \frac{\rho_{s1}}{\epsilon_o}\left(x + \frac{t}{2}\right)$$

$$= -\frac{\rho_{s1}}{\epsilon_o}\left(x + \frac{t}{2} + d_1\right) - \frac{\rho_{vo}}{2\epsilon_o}\left(x + \frac{t}{2}\right)^2$$

$\frac{t}{2} < x < \frac{t}{2} + d_2$:

$$\Phi(x) = \cancel{\Phi(t/2 + d_2)}^{\,0 \text{ (Ground)}} - \int_{t/2+d_2}^{x} -\frac{\rho_{s2}}{\epsilon_o}\, dx = \frac{\rho_{s2}}{\epsilon_o}\left(x - \frac{t}{2} - d_2\right)$$

$\Phi = -\frac{\rho_{s1}}{\epsilon_o}\left(x + d_1 + \frac{t}{2}\right)$ [V] $\quad -d_1 - \frac{t}{2} < x < -\frac{t}{2}$

$\Phi = -\frac{\rho_{s1}}{\epsilon_o}\left(x + \frac{t}{2} + d_1\right) - \frac{\rho_{vo}}{2\epsilon_o}\left(x + \frac{t}{2}\right)^2$ [V] $\quad -\frac{t}{2} < x < \frac{t}{2}$

$\Phi = \frac{\rho_{s2}}{\epsilon_o}\left(x - d_2 - t/2\right)$ [V] $\quad \frac{t}{2} < x < \frac{t}{2} + d_2$

ROOM FOR EXTRA WORK

BY CONSTRUCTION, Φ IS CONTINUOUS @ $X = -\frac{t}{2}$. WE FORCE CONTINUITY @ $x = +\frac{t}{2}$:

$$\Phi(t/2) = -\frac{\rho_{s1}}{\cancel{\epsilon_0}}(t+d_1) - \frac{\rho_{vo}}{2\cancel{\epsilon_0}}t^2 = \frac{\rho_{s2}}{\cancel{\epsilon_0}}(-d_2)$$

Using $-\rho_{vo}t = \rho_{s1} + \rho_{s2}$,

$$\Rightarrow -\rho_{s1}(t+d_1) + \frac{(\rho_{s1}+\rho_{s2})t}{2} = -\rho_{s2}d_2$$

$$\Rightarrow \rho_{s1}\left(\frac{t}{2}+d_1\right) = \rho_{s2}\left(\frac{t}{2}+d_2\right)$$

Solve for ρ_{s2} in terms of ρ_{s1} and substitute into $-\rho_{vo}t = \rho_{s1}+\rho_{s2}$:

$$-\rho_{vo}t = \rho_{s1}\left[1 + \frac{t/2+d_1}{t/2+d_2}\right]$$

$$\Rightarrow \rho_{s1} = \frac{-\rho_{vo}t\left(\frac{t}{2}+d_2\right)}{t+d_1+d_2}$$

$$\Rightarrow \rho_{s2} = \frac{-\rho_{vo}t\left(\frac{t}{2}+d_1\right)}{t+d_1+d_2}$$