Name:____

Signature:_______________________________________

Student Number:__________________________________

ECE 2317
APPLIED ELECTRICITY AND MAGNETISM
EXAM 2
August 7, 1999

1. This exam is closed book and closed notes. Calculators may not be used.
2. No credit will be given if the work required to obtain the solution is not shown.
3. Perform all your work on the paper provided. The instructors will provide additional paper if needed.
4. A list of identities is provided at the end of the test. These identities may be needed in arriving at a final solution.
5. You will have a total of 90 minutes.

GOOD LUCK!

Do not write below this line.

Prob. 1: _____/25

Prob. 2: _____/25

Prob. 3: _____/25

Prob. 4: _____/25

TOTAL: _____/100

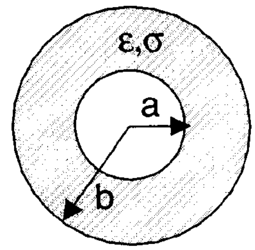
Problem 1 (25 points):

A lossy coaxial capacitor is comprised of an inner and outer conductor of radius *a* and *b*, respectively. The conductors are each of length *h* and the material between the two conductors has a permittivity ε and a conductivity σ. Assuming a constant line charge density of ρ_l on the inner conductor, find the

a.) capacitance of the coaxial capacitor.

b.) resistance of the coaxial capacitor.

c.) energy stored by the coaxial capacitor.



SOLUTION:

a.) The capacitance for a two-conductor system is defined as the absolute value of the *total* charge on each conductor divided by the voltage difference *between the two conductors.* Thus,

$$C = \frac{Q}{V}.$$

We are given the charge per unit length on the inner conductor, ρ_l.

⇒ The total charge on the inner conductor is

$$Q = \rho_l h.$$

(The total amount of charge on the outer conductor is equal to -Q.)

⇒ From Gauss's Law, the electric field between the outer and inner conductor is given by

$$E_\rho = \frac{\rho_l}{2\pi\varepsilon\rho}.$$

⇒The voltage difference (defined from the conductor with negative charge to the conductor with positive charge) is found using

$$V = -\int_b^a E_\rho \, d\rho = -\int_b^a \frac{\rho_l}{2\pi\varepsilon\rho} d\rho$$

$$V = \frac{\rho_l \ln(b/a)}{2\pi\varepsilon}$$

⇒ Thus, the capacitance is

$$\boxed{C = \frac{2\pi\varepsilon h}{\ln(b/a)}}$$

b.) Using the direct-current analogy, the conductance of the lossy capacitor is given by

$$\boxed{G = \frac{2\pi\sigma h}{\ln(b/a)}.} \qquad (\varepsilon \to \sigma)$$

c.) The energy stored in the capacitor can be found using the results found in part a. Since $U_E = \frac{1}{2}(CV^2)$,

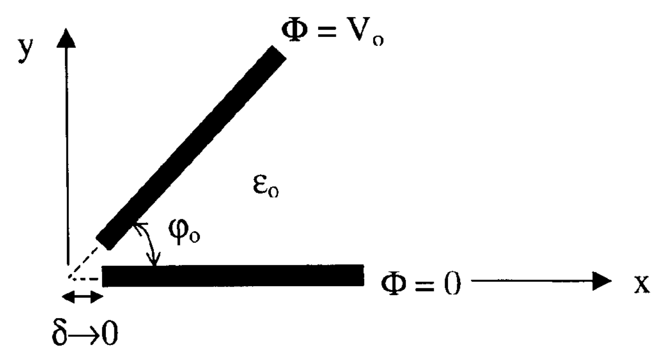
$$\boxed{U_E = \frac{(\rho_l)^2 \ln(b/a) h}{4\pi\varepsilon}}$$

Problem 2 (25 points):

Two semi-infinite conducting planes are arranged at an angle φ_o, as shown in the figure below. One plate is grounded and the other is charged to a potential V_o. A gap at the tip of the wedge (infinitesimally small) insulates one plate from the other. Assuming that a volume charge density given by $\rho_v = \rho_o \sin\varphi \,[\mathrm{C/m^3}]$ occupies the region $0 < \varphi < \varphi_o$,

a.) Find the potential in the region $0 < \varphi < \varphi_o$.

b.) Is your solution unique? Explain.



SOLUTION:

a.) Given that a source exists between the conducting plates, the potential between the plates must satisfy Poisson's equation:

$$\nabla^2\Phi = -\frac{\rho_v}{\varepsilon_o}.$$

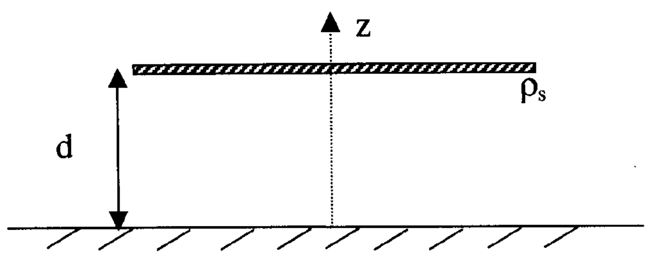
Due to the fact that the conducting planes are infinite in ρ and z, we expect that there will no variation in the potential with respect to either of these variables. Thus,

$$\frac{\partial\Phi}{\partial\rho} = \frac{\partial\Phi}{\partial z} = 0.$$

The Laplacian operation becomes

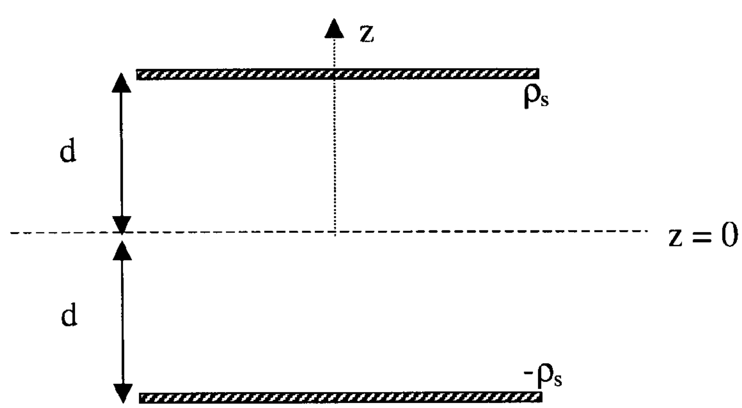
Problem 3 (25 points):

A metal plate of area A [m^2] lies a distance d above an infinite ground plane, as shown in the figure below. Assume that positive charge has been distributed uniformly on the surface of the metal plate. Find the capacitance between the metal plate and the ground plane. Neglect fringing effects.



<u>**SOLUTION:**</u>

According to image theory, the configuration given below will produce the same fields as those produced by the geometry given above, **in the region z>0.**



Since we can neglect fringing effects (ie. the bending of the electric field near the edges of the plates), the field can assumed to be constant between the two conductors.

According to Gauss's Law, the electric field *due to one of the sheets* is given by

$$\iint_{S_{top}} \varepsilon \mathbf{E}\cdot \mathbf{ds}_{top} + \iint_{S_{bot}} \varepsilon \mathbf{E}\cdot \mathbf{ds}_{bot} = \iint_{S} \rho_s \, ds$$

$$\iint_{S_{top}} \varepsilon E_z \hat{z}\cdot \hat{z}\, ds + \iint_{S_{bot}} \varepsilon(-E_z \hat{z})\cdot(-\hat{z}) ds = \iint_{S} \rho_s \, ds$$

$$2\varepsilon E_z A = \rho_s A$$

$$|E_z| = \frac{\rho_s}{2\varepsilon} \; .$$

Notice that the electric field is in the positive z-direction above the sheet and in the negative z-direction below the sheet. Thus, by superposition (accounting for the image), the electric field between the two conductors is given by

$$\mathrm{E}_z = -\frac{\rho_s}{\varepsilon} \text{ for } 0 \le z \le d \, .$$

We recognize that the electric field is equal to zero everywhere outside the plates (the electric field due to the top sheet is of equal magnitude but in opposite direction to the field due to the bottom sheet).

From the electric field, the potential difference between the ground plane and the conducting sheet becomes

$$V = -\int_0^d E_z \, \hat{\mathbf{z}}\cdot\hat{\mathbf{z}} \, dz$$

$$\Rightarrow \; -\int_0^d \left(\frac{-\rho_s}{\varepsilon}\right) dz$$

$$\Rightarrow \; V = \frac{\rho_s d}{\varepsilon}$$

Notice that, for convenience, the line integral is carried out along the z-axis. However, it is important to recognize that due to the conservative nature of the electric field, the integral is path independent.

Having computed the voltage drop across the conductors it is next necessary to calculate the total charge contained on the conducting plane. Recall that the capacitance is given by

$$C = \frac{Q}{V}.$$

Since the surface charge is distributed uniformly on the conducting sheet (given), the total charge becomes

$$Q = \int_S \rho_s \, ds$$

$$\Rightarrow \; Q = \rho_s A$$

Substituting Q and V into the expression for the capacitance,

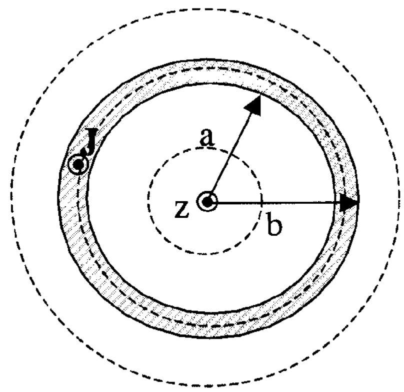
$$\boxed{C = \frac{\varepsilon A}{d}.}$$

Problem 4 (25 points):

An infinitely long cylindrical conductor lies along the *z*-axis and has an inner and outer radii equal to *a* and *b*, respectively. A current density given by

$$\mathbf{J} = 2\rho\,\hat{\mathbf{z}}\left[\frac{A}{m^2}\right]$$

flows in the conductor. Determine the magnetic field intensity in each region.



<u>**SOLUTION:**</u>

Applying Ampere's Law to each region defined by the boundaries of the conductors (Amperian loops shown above):

ρ**<a:**

$$\oint_C \mathbf{H}\cdot\mathbf{dl} = \int_S \mathbf{J}\cdot\mathbf{ds}$$

$$\Rightarrow \oint_C \mathbf{H}\cdot\mathbf{dl} = 0$$

$$\Rightarrow \boxed{\mathbf{H} = 0} \qquad \text{for } \rho<a$$

a<ρ<b:

$$\oint_C \mathbf{H}\cdot\mathbf{dl} = \int_S \mathbf{J}\cdot\mathbf{ds}$$

$$\Rightarrow \int_0^{2\pi} H_\phi\hat{\phi}\cdot\rho d\phi\hat{\phi} = \int_0^{2\pi}\int_a^{\rho}(2\rho)\rho d\rho d\phi$$

$$\Rightarrow \boxed{H_\phi = \frac{2}{3}\left[\frac{\rho^3 - a^3}{\rho}\right]} \qquad \text{for } a<\rho<b$$

ρ>b:

$$\oint_C \mathbf{H}\cdot\mathbf{dl} = \int_S \mathbf{J}\cdot\mathbf{ds}$$

$$\Rightarrow \int_0^{2\pi} H_\phi\hat{\phi}\cdot\rho d\phi\hat{\phi} = \int_0^{2\pi}\int_a^{b}(2\rho)\rho d\rho d\phi$$

$$\Rightarrow \boxed{H_\phi = \frac{2}{3}\left[\frac{b^3 - a^3}{\rho}\right]} \qquad \text{for } \rho>b$$

List of Identities:

$$\nabla^2 A = \frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} + \frac{\partial^2 A}{\partial z^2}$$

$$\nabla^2 A = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial A}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 A}{\partial \varphi^2} + \frac{\partial^2 A}{\partial z^2}$$

$$\nabla^2 A = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial A}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial A}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 A}{\partial \varphi^2}$$