ECE 3318
Applied Electricity and Magnetism
Spring 2023
Homework #7

Date assigned: Thursday, March 9, 2023
Date due: Thursday, March 23, 2023

Do Probs. 1-3 and 7-13. (You are welcome to do the other problems as well for extra practice, but only these should be turned in.)

1) Find the curl of the following vector functions:

$$\underline{V} = \underline{\hat{x}}\left(5x^2yz\right) - \underline{\hat{y}}\left(e^{-y}x\right) + \underline{\hat{z}}\left(3xyz^2\right)$$

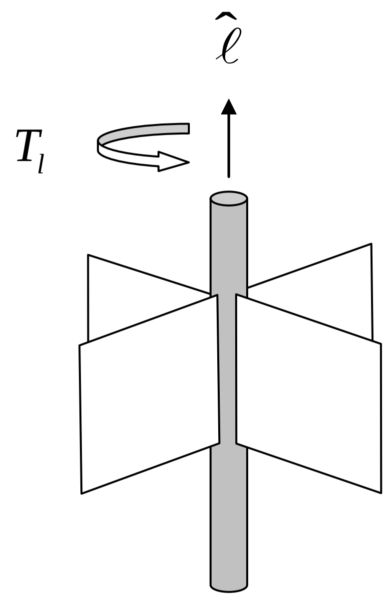
$$\underline{V} = \underline{\hat{\rho}}\left(\cos\phi\right) - \underline{\hat{\phi}}\left(3z\sin\phi\right) + \underline{\hat{z}}\left(z\rho\phi\right)$$

$$\underline{V} = \underline{\hat{r}}\left(\sin\theta\cos\phi\right) - \underline{\hat{\theta}}\left(3r\sin\phi\right) + \underline{\hat{\phi}}\left(r^3\cos\theta\right).$$

2) A simple "curl meter" is a paddle wheel that consists of a set of vanes attached to a spindle (the axis of rotation) as shown below. It is used to determine the curl of a vector function that describes the velocity vector of water. In order to determine the component of the curl of a vector in the direction $\hat{\ell}$, at a given point in space, the curl meter is placed at that point in space and the spindle axis is pointed in direction $\hat{\ell}$. A positive torque component T_l indicates that the meter wants to rotate in the direction defined by the right-hand rule, as shown in the figure below.

   Assume that our curl meter has the property that the magnitude of the torque is equal to the square of the curl component. (This is because when water hits an object, the force is proportional to the square of the velocity.) The torque is therefore given by

$$T_l = \pm\left(\hat{\ell}\cdot\operatorname{curl}\underline{V}\right)^2,$$
$$+ \quad \text{if} \quad \hat{\ell}\cdot\operatorname{curl}\underline{V} > 0\ ,$$
$$- \quad \text{if} \quad \hat{\ell}\cdot\operatorname{curl}\underline{V} < 0.$$



Part 1)

Find the torque component T_l on the curl meter for the vector function

$$\underline{V} = \underline{\hat{x}}\left(2x^2 y\right) + \underline{\hat{y}}\left(3zy^3\right) + \underline{\hat{z}}\left(2xyz\right)$$

at the following points (in rectangular coordinates in meters) and directions:

a) $\underline{r} = (1,1,1); \quad \underline{\hat{\ell}} = \underline{\hat{x}}$

b) $\underline{r} = (1,1,1); \quad \underline{\hat{\ell}} = \underline{\hat{y}}$

c) $\underline{r} = (1,1,1); \quad \underline{\hat{\ell}} = \underline{\hat{z}}$

d) $\underline{r} = (1,1,1); \quad \underline{\hat{\ell}} = \left(\underline{\hat{z}} + 2\underline{\hat{x}} + 4\underline{\hat{y}}\right)/\sqrt{21}$

e) $\underline{r} = (1,2,3); \quad \underline{\hat{\ell}} = \left(\underline{\hat{x}} + \underline{\hat{y}} + \underline{\hat{z}}\right)/\sqrt{3}$.

Part 2)

Find a unit vector $\underline{\hat{\ell}}$ that gives the direction to point the curl meter so that the torque component T_l will be maximum in magnitude and positive, assuming the same vector function $\underline{V}$ as above and assuming that we are at the point (1,1,1) in rectangular coordinates. Repeat for the case where we want the torque component T_l to be maximum in magnitude but negative at this same point.

3) Consider the vector function

$$\underline{V} = \underline{\hat{x}}\left(2x^2 y\right) + \underline{\hat{y}}\left(3zy^3\right) + \underline{\hat{z}}\left(2xyz\right).$$

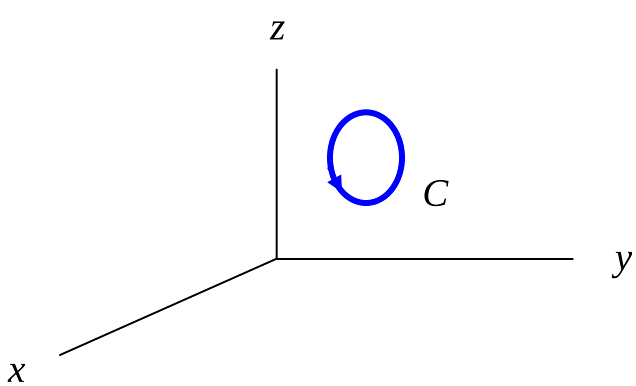
Calculate an approximate value for the integral

$$I = \oint_C \underline{V} \cdot d\underline{r} \,,$$

where the path C is a circular path of radius $a = 0.0001$ that is centered at the point (1, 2, 3) and lies in a plane perpendicular to the vector direction $\left(\underline{\hat{x}} + \underline{\hat{y}}\right)$. The direction of integration around the contour is counter-clockwise, as seen from an observer who is looking towards the loop from far away in the first octant, as shown below.

(Note: You may assume that the curl of the vector function $\underline{V}$ is approximately constant over the surface of the circular region inside the path C, since the radius is so small.)

Hint: Use Stokes's theorem.

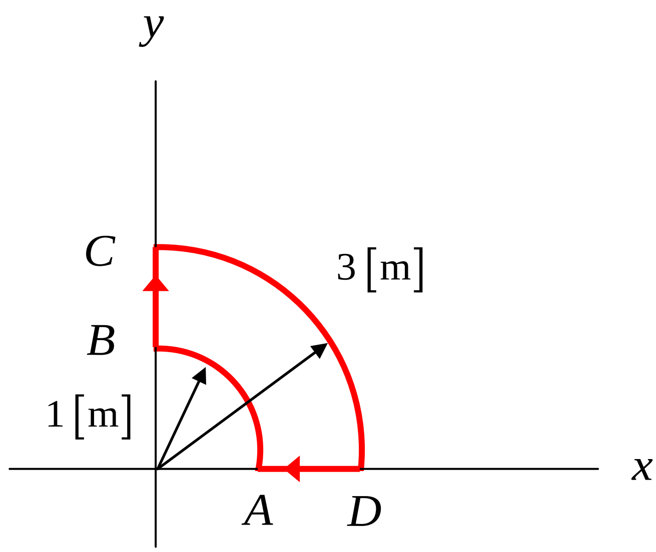


4) Assume a vector function $\underline{F} = \underline{\hat{\rho}}(5\rho \sin\phi) + \underline{\hat{\phi}}(\rho^2 \cos\phi)$.

a) Evaluate $\oint_C \underline{F} \cdot d\underline{r}$ around the contour ABCDA in the direction indicated in the figure.

b) Find $\nabla \times \underline{F}$.

c) Evaluate $\int_S \left(\nabla \times \underline{F}\right) \cdot \underline{\hat{n}}\, dS$ over the surface defined by the contour ABCDA and compare the result with that obtained in part (a), to see if you can verify Stokes's theorem.



5) Green's theorem states that if $P(x,y)$ and $Q(x,y)$ are two differentiable functions in a planar region S of the xy plane bounded by a closed curve C, then

$$\int_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dS = \oint_C P dx + Q dy \,,$$

where the integral around C is taken in a counterclockwise sense. Show that this is a special case of Stokes's theorem for a planar surface bounded by a closed contour lying in the xy plane,

$$\int_S \left(\nabla \times \underline{A} \right) \cdot \underline{\hat{z}} \; dS = \oint_C \underline{A} \cdot d\underline{r} \,,$$

in which $\underline{A}(x,y) = P(x,y)\underline{\hat{x}} + Q(x,y)\underline{\hat{y}}$.

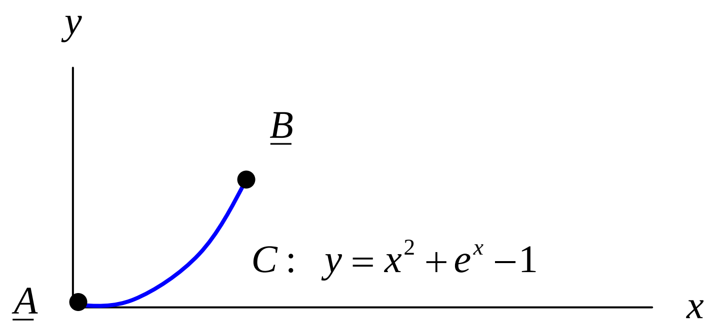
6) Calculate the voltage drop

$$V_{AB} = \int_A^B \underline{E} \cdot d\underline{r}$$

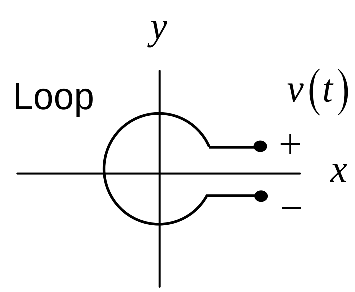
along the path C as shown below, for the electric field

$$\underline{E} = \underline{\hat{x}}(3x) + \underline{\hat{y}}\left(4y^3\right) + \underline{\hat{z}}(4z) \,.$$

You are free to choose a different path than the one specified, if you can <u>prove</u> that it makes no difference (explain clearly why). The point $\underline{A}$ is at the origin, and point $\underline{B}$ is at (1, e, 0) in rectangular coordinates. (Note: This may or may not be an electrostatic field – so the curl may or may not be zero – you should check.)



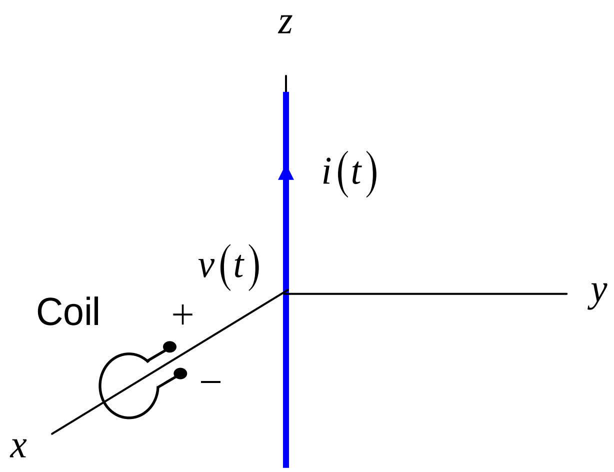
7) A loop of wire is shown below. The magnetic field in the z direction B_z through the loop is decreasing with time. Give a convincing argument, using Lenz's law, as to whether the output voltage $v(t)$ (with the polarity indicated) will be positive or negative.



8) A circular loop of wire has an area of 25 [cm^2]. A coil is made from 100 turns of these loops. The coil is held 50 [m] away from a 60 [Hz] power line. Find the output voltage $v(t)$ from the coil in the time domain and the corresponding output voltage V in the phasor domain. The power line can be assumed to run along the z axis, and is carrying a current in the z direction of $i(t)$ = 100 cos(ωt) [A], where $\omega = 2\pi f$. The coil is on the x axis, with the axis of the loop pointing in the y direction (see the figure below). The magnetic flux density from the power line (derived later in class) is given as

$$\underline{B}(t) = \underline{\hat{\phi}}\left(\frac{i(t)}{2\pi\rho}\right)\mu_0 \text{ [T]},$$

where $\mu_0 = 4\pi \times 10^{-7}$ [A/m].

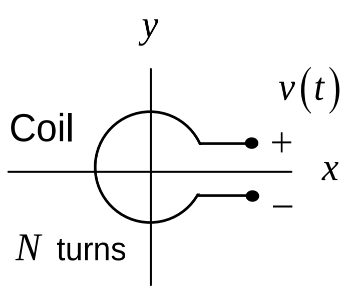


9) A coil of wire has N = 1000 turns (loops). The area of each loop is 100 [cm^2]. The total length of the wire in the coil has a resistance of 10 [Ω]. The loops lie in the z = 0 plane (see the figure below). A 60 [Hz] magnetic field is applied to the coil, having the form

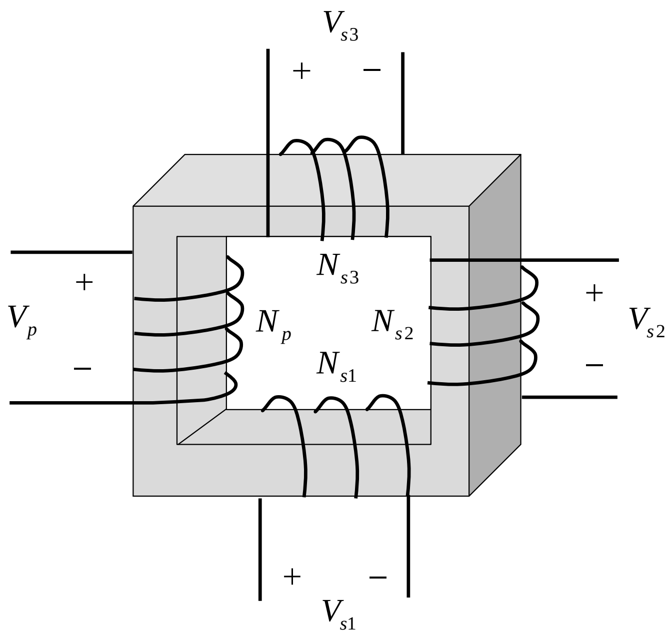
$$\underline{B} = \underline{\hat{z}}(0.1)\cos(\omega t) \quad [\mathrm{T}].$$

a) Find the Thévenin equivalent circuit of this coil system (which will be in the phasor domain), when it is used as an AC generator.

b) Find the maximum power that this AC generator can supply to a matched load.



10) An ideal transformer has one primary coil and three secondary coils, as shown below. The primary coil on the left has N_p turns. The secondary coil on the bottom has N_{s1} turns. The secondary coil on the right has N_{s2} turns. The secondary coil on the top has N_{s3} turns. Find the following voltage ratios (in the phasor domain): V_{s1}/V_p, V_{s2}/V_p, and V_{s3}/V_p. Note: These ratios could be positive or negative numbers. Use Lens's law to make sure that you get the signs correct, noting how the coils are wound. It might be helpful to imagine a magnetic flux circulating clockwise in the core, increasing with time, when you apply Lens's law.



11) Assume that three resistors of equal value R [Ω] are now added to each of the three secondary coils in the transformer above. What is the input resistance seen looking into the primary? Hint: Use conservation of energy, and the fact that this is a lossless transformer.

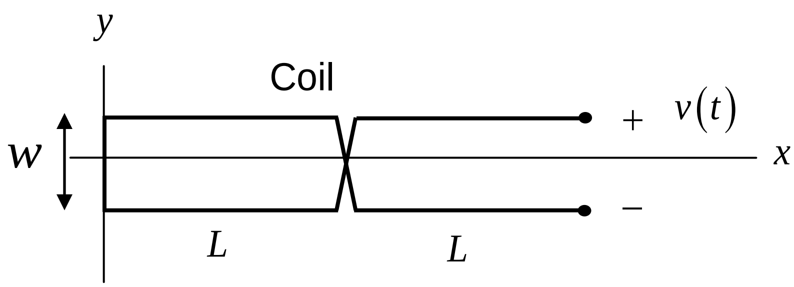
12) A voltmeter is being used to measure a 60 [Hz] voltage across a resistor in a circuit that is inside of a lab room. The two leads going from the voltmeter to the resistor are 5 [cm] apart and 2 [m] long, forming a rectangular loop with the voltmeter on one end and the resistor on the other end. Inside the lab there is a 60 [Hz] magnetic field coming from some equipment in the room. The magnetic flux density vector coming from this equipment has a peak magnitude of B_0 = 0.01 [T] and the direction of the magnetic flux density vector $\underline{B}$ from the equipment is at an angle of 45° from the normal to the rectangular loop formed by the leads. Calculate the peak magnitude of the extra (error) AC voltage that the voltmeter will read that is in addition to what the voltmeter is supposed to be reading. You can assume that the magnetic field vector is constant over the area of the rectangular loop.

13) A loop of wire is twisted in the middle as shown below. This means that the unit normal flips directions between the two sides of the loop. Each side (left and right) has a length of L and a width of w.

    (a) The loop is in the presence of a magnetic field, with the magnetic flux density vector given by

    $$\underline{B} = \underline{\hat{z}}\, B_0\, e^{-ax} \cos(\omega t) \quad \text{[T]}.$$

    Find the output voltage $v(t)$ of the loop as shown in the figure below.



(b) Repeat the problem assuming that the magnetic flux density vector is now given by

$\underline{B} = \underline{\hat{z}}\, B_0 \cos(\omega t)$ [T].

In this case the magnetic flux density vector is uniform (constant) in space.

Hint: The total flux through the loop is the sum of the fluxes through the two parts of the loop. But be mindful of the unit normal directions on the two different parts of the loop! Taking his into consideration, convince yourself that

$$v(t) = \frac{d\psi_z^{\text{right}}}{dt} - \frac{d\psi_z^{\text{left}}}{dt},$$

where ψ_z^{right} is the magnetic flux in the z direction going through the right part of the loop, and ψ_z^{left} is the magnetic flux in the z direction going through the right part of the loop.