Review: Numerical Methods for Differential Equations

ECE Generic and 2331

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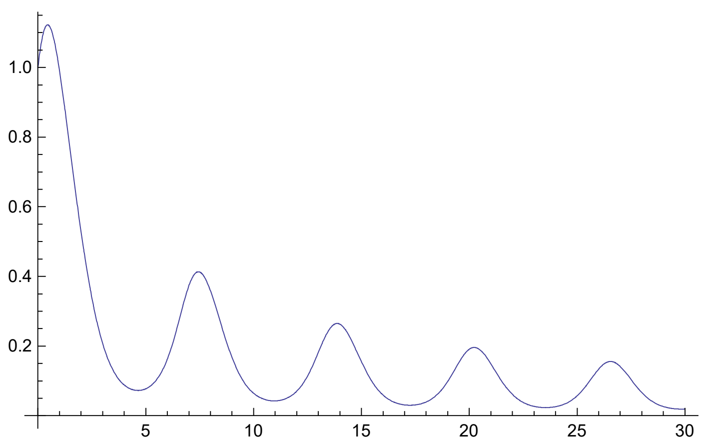
Segment 3 - Practice and examples - ODE

1. Basic use of NDSolve

Example how to use **NDSolve**

```
s = NDSolve[{y'[x] == y[x] Cos[x + y[x]], y[0] == 1}, y, {x, 0, 30}]
Plot[Evaluate[y[x] /. s], {x, 0, 30}, PlotRange → All]
```



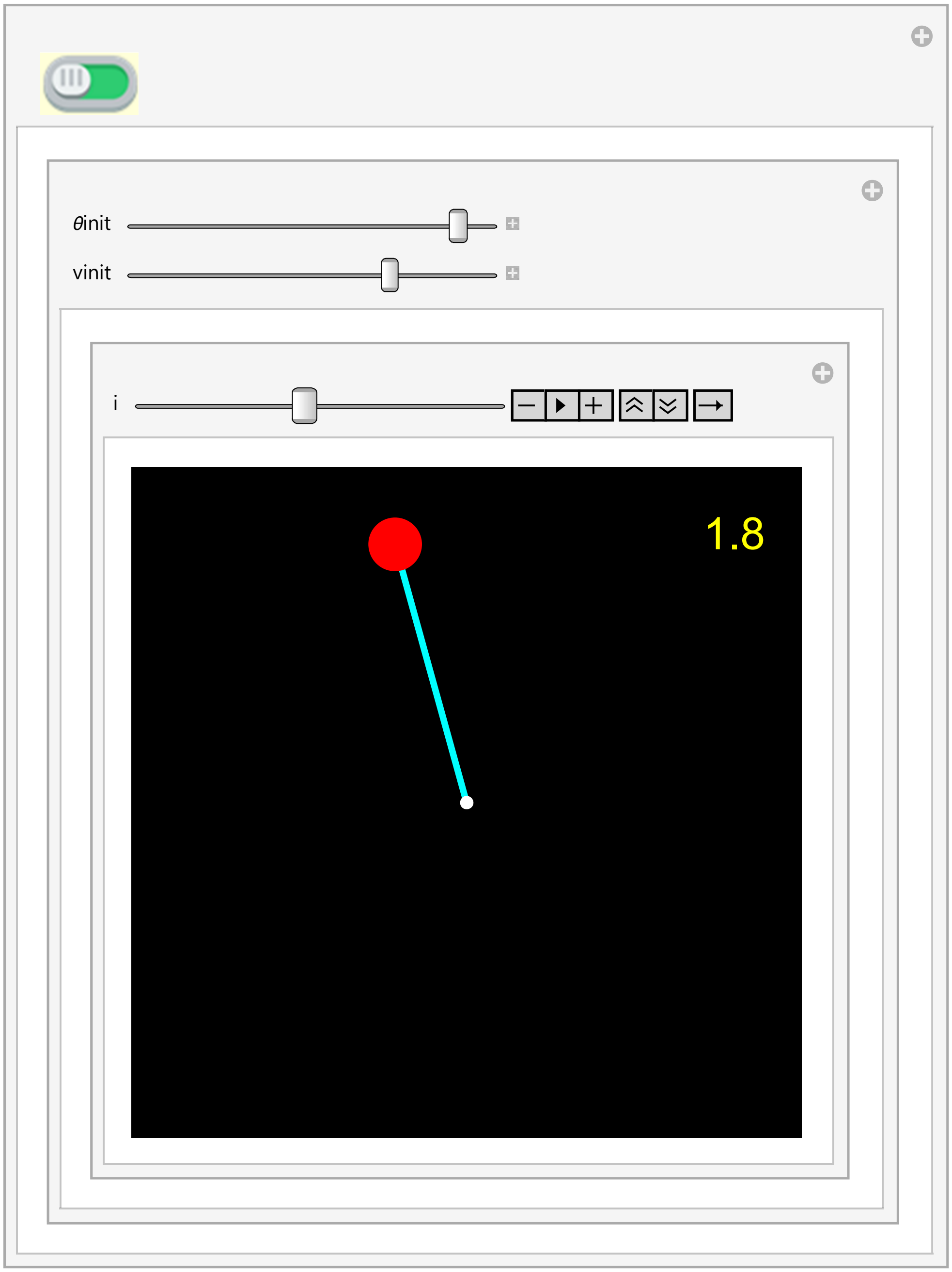
2. Stable pendulum

2.1 Single pendulum

Go to webpage to download the full app. Here shows only partial results to make the code simple to understand.

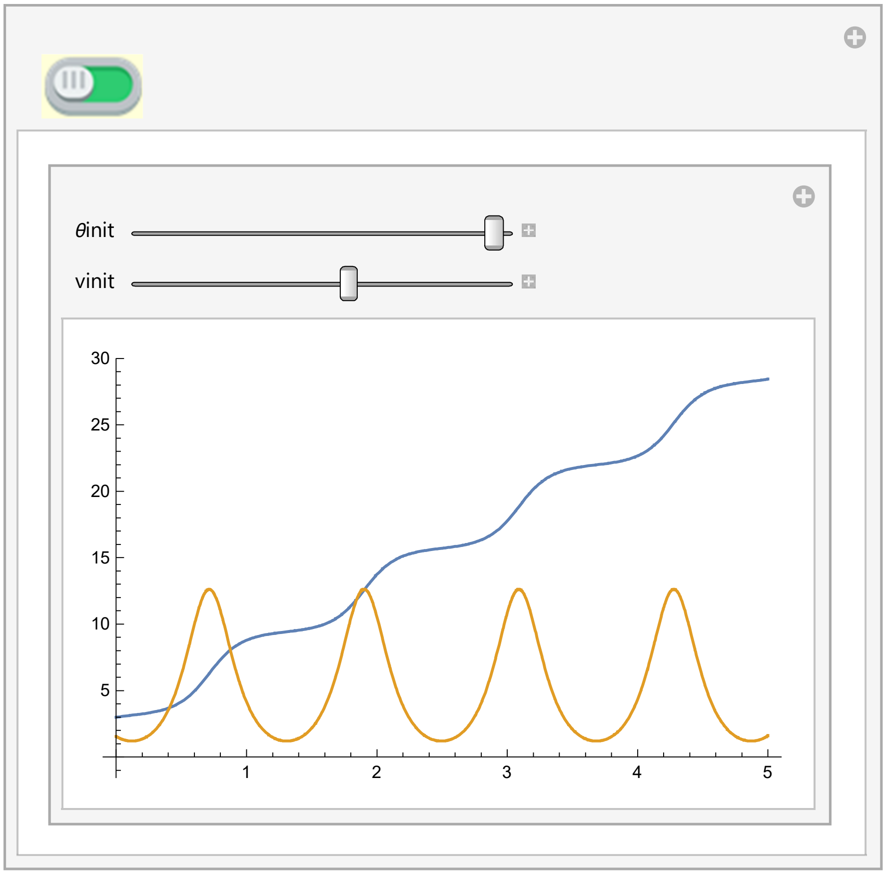
- code

Out[ ]=



- code

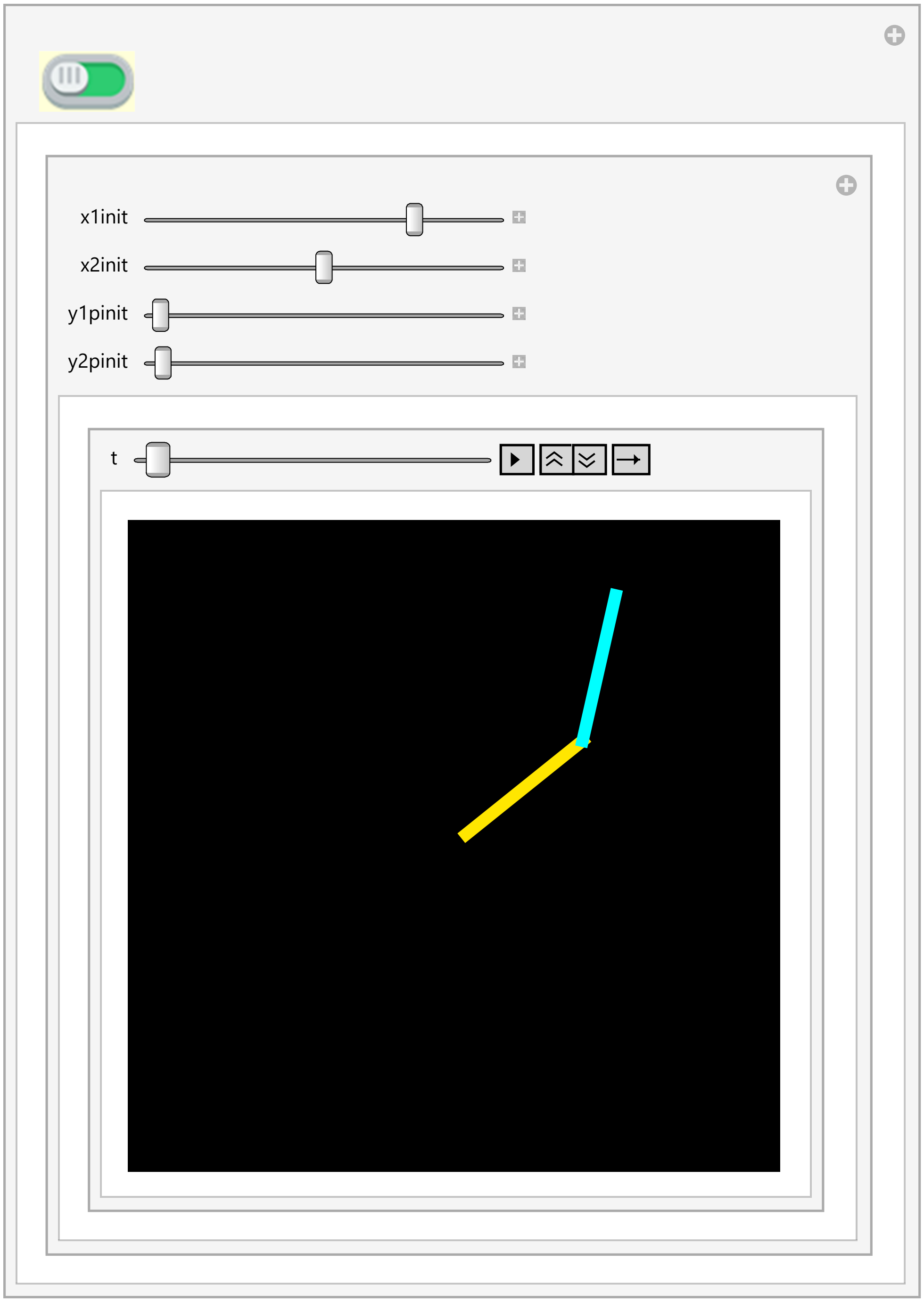
Out[ ]=



2.2 Double pendulum

- code

Out[ ]=



2.3 Chaos concept

Class discussion

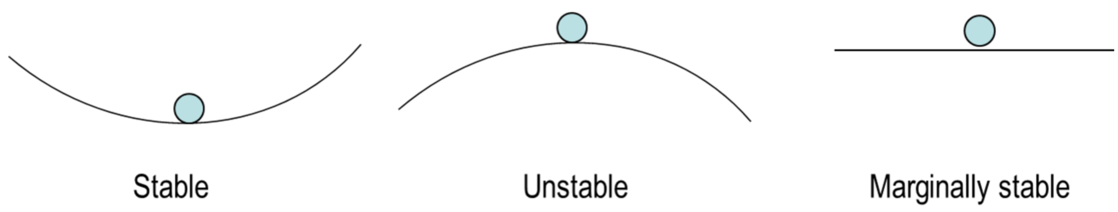
Additional demo

https://www.youtube.com/watch?v=QXf95_EKS6E

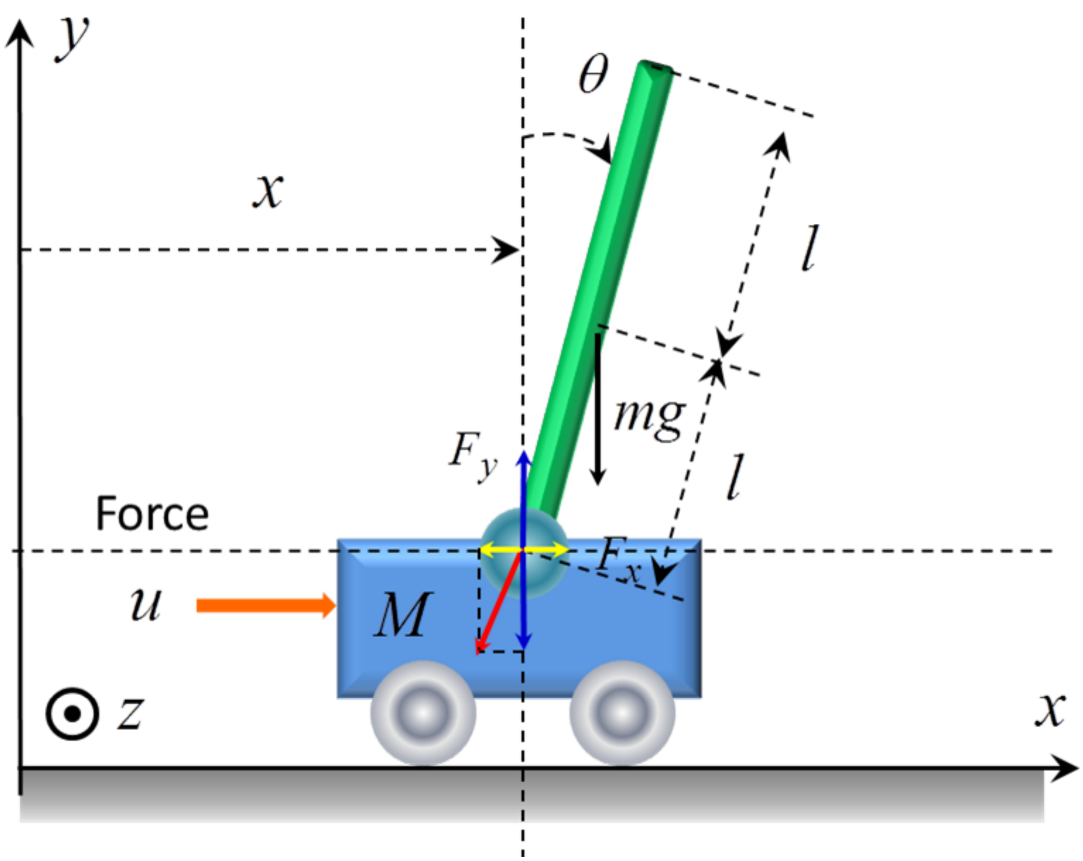
3. Inverted pendulum

3.1 Introduction

Concept of stability. See below.



We now consider a well-studied nonlinear problem of stabilization. But here, we will only investigate the linearized case. Consider an inverted pendulum problem below.

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There are only 2 degrees of freedom: x and θ as shown. But the pendulum moves relative to the cart. Newton 3rd law stipulates that the force F_x, F_y the cart exerts on the pendulum has to be equal and opposite to that of the pendulum on the cart. Hence, the equation of motion is:

For the cart center of mass: $M\ddot{x} = u - F_x$ (4.1.12)

For the pendulum center of mass: $m \frac{d^2}{dt^2}(x - l\,\mathrm{Sin}[\theta]) = F_x$ (4.1.13)

$m \frac{d^2}{dt^2}(l\,\mathrm{Cos}[\theta]) = -m\,g + F_y$ (4.1.14)

For the pendulum rotation motion: $J\ddot{\theta} = F_x\,l\,\mathrm{Cos}[\theta] + F_y\,l\,\mathrm{Sin}[\theta]$ (4.1.15)

Note: the polarity of θ is counter-clockwise (right hand rule).

We have 4 Eqs. for 4 unknowns. But we are not interested in F_x and F_y, hence we eliminate them: (Combine 4.1.13 into 4.1.12):

$$M\ddot{x} = u - F_x = u - m \frac{d^2}{dt^2}(x - l\,\mathrm{Sin}[\theta]) \quad (4.1.16)$$

and combine both (4.1.13) and (4.1.15) in (4.1.16):

$$J\ddot{\theta} = F_x\,l\,\mathrm{Cos}[\theta] + F_y\,l\,\mathrm{Sin}[\theta] \quad (4.1.15)$$

$$J\ddot{\theta} = m \frac{d^2}{dt^2}(x - l\,\mathrm{Sin}[\theta])\,l\,\mathrm{Cos}[\theta] \quad (4.1.17)$$
$$+\left(m\,g + m \frac{d^2}{dt^2}(l\,\mathrm{Cos}[\theta])\right) l\,\mathrm{Sin}[\theta]$$

The final eqs are: $M\ddot{x} = u - m\frac{d^2}{dt^2}(x - l\,\mathrm{Sin}[\theta])$ (4.1.16)

$$J\ddot{\theta} = m\frac{d^2}{dt^2}(x - l\,\mathrm{Sin}[\theta])\,l\,\mathrm{Cos}[\theta] + \left(m\,g + m\frac{d^2}{dt^2}(l\,\mathrm{Cos}[\theta])\right)l\,\mathrm{Sin}[\theta] \quad (4.1.17)$$

Or: $\ddot{x} = \frac{u}{M} - \frac{m}{M}\frac{d^2}{dt^2}(x - l\,\mathrm{Sin}[\theta])$ (4.1.16)

$$\ddot{\theta} = \frac{m\,l^2}{J}\frac{d^2}{dt^2}\left(\frac{x}{l} - \mathrm{Sin}[\theta]\right)\mathrm{Cos}[\theta] + \frac{m\,l^2}{J}\left(\frac{g}{l} + \frac{d^2}{dt^2}\mathrm{Cos}[\theta]\right)\mathrm{Sin}[\theta]$$

Or: $\frac{\ddot{x}}{l} = \frac{s[t]}{\tau^2} - \mu\frac{d^2}{dt^2}\left(\frac{x}{l} - \mathrm{Sin}[\theta]\right)$ (4.1.16)

$$\ddot{\theta} = \nu\left(\frac{d^2}{dt^2}\left(\frac{x}{l} - \mathrm{Sin}[\theta]\right)\mathrm{Cos}[\theta] + \left(\omega_0{}^2 + \frac{d^2}{dt^2}\mathrm{Cos}[\theta]\right)\mathrm{Sin}[\theta]\right)$$

let $J = \frac{m\,l^2}{12}$ then ν=3.

3.2 Linearization

3.3 Numerical calculation

Additional info and demo

http://www.youtube.com/watch?v=MWJHcI7UcuE
http://www.youtube.com/watch?v=AZhQt7HOSWo
http://www.youtube.com/watch?v=a4c7AwHFkT8
http://www.youtube.com/watch?v=cyN-CRNrb3E
https://youtu.be/15DIidigArA
http://www.youtube.com/watch?v=bcr3cG2AKBs
http://www.youtube.com/watch?v=h-oZoNC_SIo
http://www.youtube.com/watch?v=LTyGdRG8y9Q

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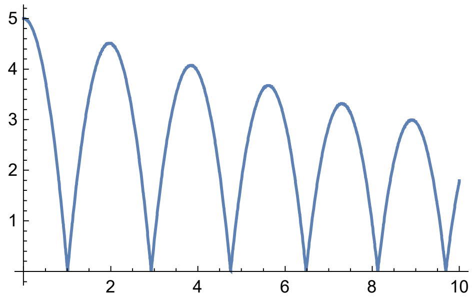
4. Demo from Mathematica: bouncing ball (energy attenuation)

4.1 Bouncing ball

Hybrid Differential Equation.
Simulate a bouncing ball that retains 95% of its velocity in each bounce:

```
In[ ]:= NDSolve[{y''[t] == -9.81, y[0] == 5, y'[0] == 0,
           WhenEvent[y[t] == 0, y'[t] -> -0.95 y'[t]]}, y, {t, 0, 10}];
```

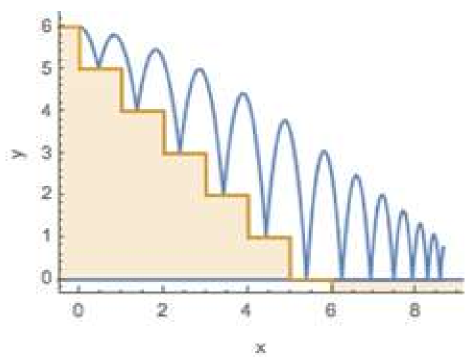
```
Plot[y[t] /. %, {t, 0, 10}]
```



Model a ball bouncing down steps:

```
ballsteps = NDSolve[{x''[t] == 0, y''[t] == -9.8, y[0] == 6, y'[0] == 0, x[0] == 0, x'[0] == 1,
    a[0] == 5, WhenEvent[Mod[x[t], 1] == 0, If[a[t] > 0, a[t] → a[t] - 1, "RemoveEvent"]],
    WhenEvent[y[t] == a[t], {x'[t], y'[t]} → .9 {x'[t], -y'[t]}]},
   {x, y, a}, {t, 0, 15}, DiscreteVariables → {a}];
```

```
Show[ParametricPlot[Evaluate[{{x[t], y[t]}, {x[t], a[t]}} /. ballsteps], {t, 0, 15}],
 Plot[{0, Floor[6 - x]}, {x, -1, 15}, Filling → {2 → 0}, Exclusions → None],
 Frame → {{True, False}, {True, False}}, FrameLabel → {"x", "y"}]
```



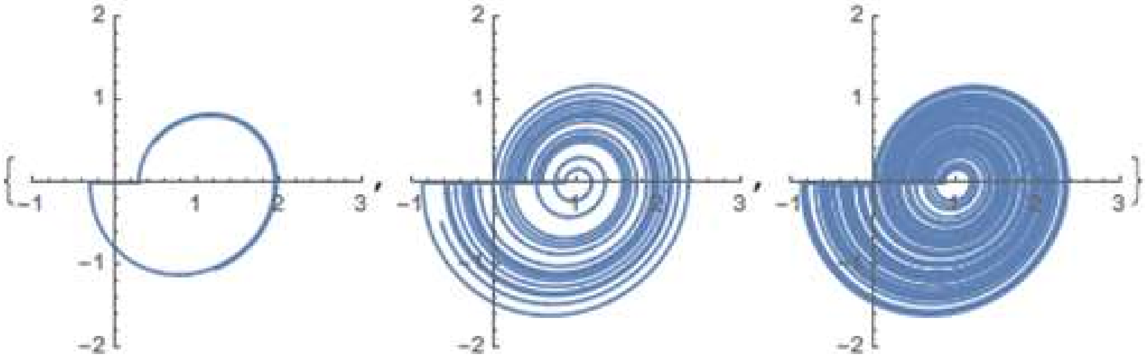
Each time a linear oscillator solution crosses the negative x axis, reflect it across the y axis:

```
de[δ_] := {x1'[t] == x2[t], x2'[t] == -x1[t] + 2 δ x2[t] + 1};
ic = {x1[0] == .5, x2[0] == .5};
```

```
{sol, {points}} = NDSolve[
    {de[0.1], ic, WhenEvent[And[x2[t] == 0, x1[t] < 0], {x1[t] → -x1[t], Sow[-x1[t]]}]},
    {x1, x2}, {t, 0, 1000}] // Reap;
```

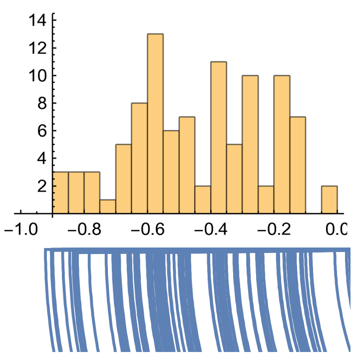
The solution of this reset oscillator exhibits chaotic behavior:

```
Map[ParametricPlot[{x₁[t], x₂[t]} /. sol, {t, 0, #},
   PlotPoints → 500, PlotRange → {{-1, 3}, {-2, 2}}] &, {10, 100, 400}]
```

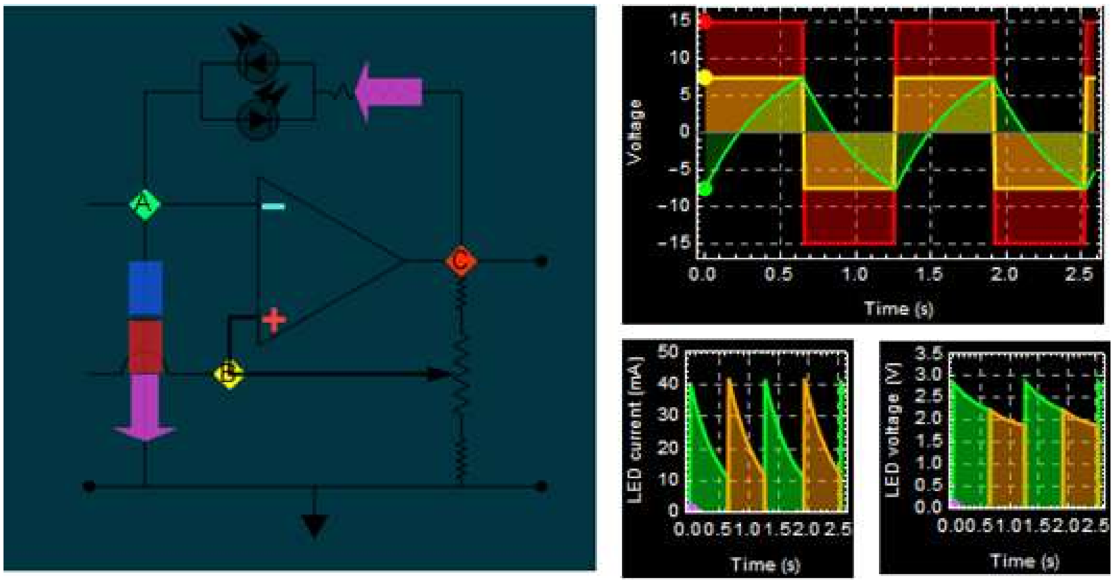


Plot the solution on the negative *x* axis with a histogram of the reflection points:

```
Grid[{{Histogram[points, {-1, 0, .05}]},
  {ParametricPlot[{x₁[t], x₂[t]} /. sol, {t, 0, 1000},
    PlotRange → {{-1, 0}, {-0.3, 0}}, PlotPoints → 2000, Axes → None]}}]
```



5. Non-linear response - Op amp oscillator



This problem is in your HW. Please go to this web site for details:
http://courses.egr.uh.edu/ECE/ECE3340/Class%20 Notes2100/Lab_5/ECE2100_lab%20 V_PID.htm
(scroll down to see the above circuit)

You can download the app to check for your answer. You are NOT required to do animation, only to solve the circuit and plot the time response for a few periods.

6. Non-linear response - amplifier

Segment 4 - PDE

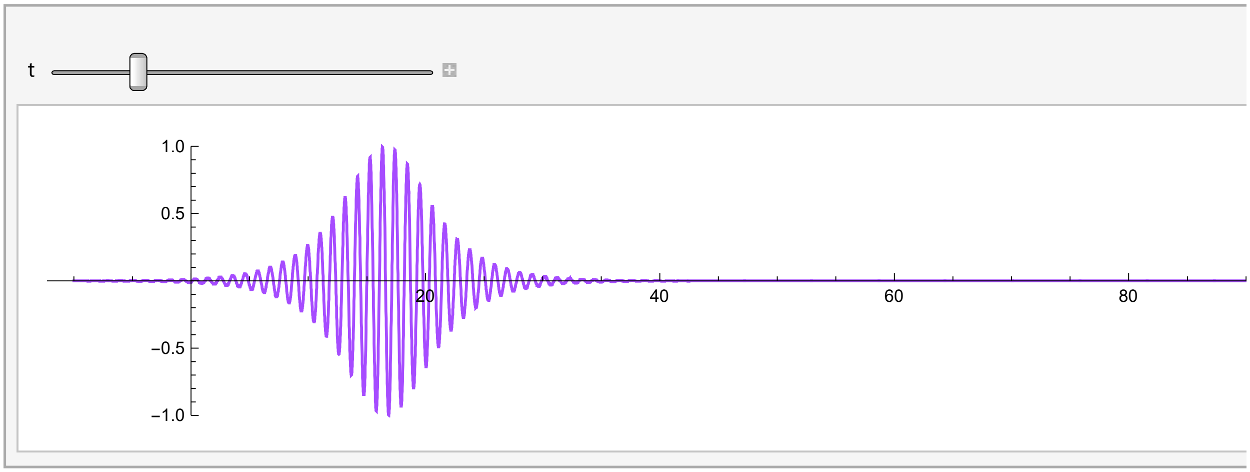
6. FDTD of EM wave in 1-dimensional air-dielectric

7. Nonlinear wave: soliton

7.1 soliton

- code for soliton with analytic solution

Demo



more about soliton

http://www.youtube.com/watch?v=Ud7STKWNmQw
http://www.youtube.com/watch?v=MADng1fqECY

http://www.youtube.com/watch?v=SAbQ4MvDqEE

http://www.youtube.com/watch?v=wEbYELtGZwI

7.2 Numerical calculation for soliton

See HW.

8. Heat diffusion (not for ECE3340)