BODE PLOTS & THE STRAIGHT-LINE APPROXIMATION

It is convenient and useful to show the properties of the transfer function $T(\omega)$ as a function of frequency. The properties of interest are the gain $|T(\omega)|$ expressed in dB, and the phase $\angle T(\omega)$ expressed in degrees.

Gain (or "magnitude") and phase Bode plots show these quantities as functions of $\log \omega$, typically.

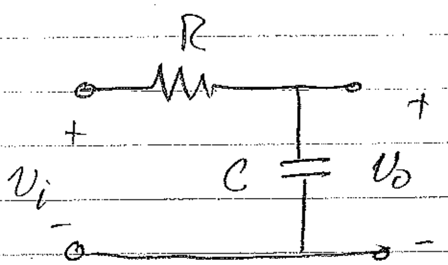
We can, of course, use mathematics packages to construct computer plots of $|T(\omega)|$ and $\angle T(\omega)$ vs. $\log \omega$. However, it is easier and more instructive to construct simple approximations, called Straight-Line Approximations, to Bode plots.

In the next few pages we will see how to do this by investigating a simple example.

ANALYSIS OF BODE PLOTS

We begin by looking at the behavior of Bode plots for a simple example.

LOW PASS FILTER



Tranfer Function

$$T(\omega) = \frac{1/j\omega C}{R + 1/j\omega C}$$

$$= \frac{1}{1 + j\omega CR}$$

Let's consider extreme values of ω:

- $\omega \to 0 \Rightarrow T(\omega) \simeq \frac{1}{1+0}$

  $|T(\omega)| \to 1 \Rightarrow 0\ dB \qquad \angle T(\omega) \to 0^\circ$

- $\omega \to \infty \Rightarrow T(\omega) \simeq \frac{1}{j\omega CR}$

  $|T(\omega)| \to 0 \Rightarrow -\infty\ dB \qquad \angle T(\omega) \to -90^\circ$

An important intermediate value to consider is

$$\omega_o = 1/RC$$

Note that this is frequency at which the real and imaginary parts of $1 + j\omega CR$

are equal (they are both 1). At ω_o,

$$T(\omega = \omega_o) = \frac{1}{1+j}$$

$$|T(\omega)| = \frac{1}{\sqrt{2}} \Rightarrow -3\,dB \qquad \angle T(\omega) = -45^\circ$$

For arbitrary ω,

$$|T(\omega)| = \frac{1}{\sqrt{1+\omega^2 C^2 R^2}} \qquad \angle T(\omega) = -\tan^{-1}(\omega C R)$$

We could simply plot this using a mathematics package but we want to develop the intuition of the straight-line approximation.

We now have $|T(\omega)|$ and $\angle T(\omega)$ at the extremes, and at the point $\omega_o = 1/RC$. Let's see what happens elsewhere.

$$T(\omega) = \frac{1}{1 + j\omega C R}$$

It is convenient to look at $T(\omega)$ for ω values on either side of ω_o, that is

$$\omega \ll \omega_o \quad \text{and} \quad \omega \gg \omega_o.$$

$$\omega \ll \omega_0 \quad \Rightarrow \quad T(\omega) \simeq \frac{1}{1+0}$$

So for $\omega \ll \omega_0$, $T(\omega)$ is approximately constant (and equal to 1).

$$\omega \gg \omega_0 \quad \Rightarrow \quad T(\omega) \simeq \frac{1}{j\omega CR}$$

In this range, $T(\omega)$ decreases as $1/\omega$. What if ω were to increase by a factor of 10?

$$\omega \rightarrow 10\,\omega \quad \Rightarrow \quad T(\omega) \rightarrow 0.1\,T(\omega)$$

This is a decrease of 20 dB : $20 \log(0.1) = -20$.

Some numbers: say $C = 2 \times 10^{-6}$ F
$R = 1.5 \times 10^{4}\ \Omega$

$RC = 3 \times 10^{-2}$ s $\Rightarrow$ $\omega_0 = 1/RC = 33.3$ rad/s

We want $\omega \gg \omega_0$; say $\omega = 500$ rad/s.

$$T(1000) = \frac{1}{j(500)(0.03)} = \frac{1}{j15}$$

$|T(\omega)| = -23.52$ dB $\quad \angle T(\omega) = -90^\circ$

Now $\omega \rightarrow 10\,\omega = 5000$ rad/s

$T(\omega) = \dfrac{1}{j150}$

$|T(\omega)| = -43.52\ dB \quad \angle T(\omega) = -90^\circ$

So as predicted, the change in $|T(\omega)|$ was -20 dB.

So we have established that for $T(\omega) = \dfrac{1}{1+j\omega CR}$ and in the range $\omega \gg 1/RC$, the magnitude $|T(\omega)|$ decreases by 20 dB for an increase in ω of 10 times.

We can generalize this result:

- For a factor $\dfrac{1}{1+j\omega/\omega_0}$ appearing in the transfer function, the magnitude Bode plot decreases by 20 dB per decade in the range $\omega \gg \omega_0$.

It should not come as a surprise that:

- For a factor $(1+j\omega/\omega_0)$ appearing in the transfer function, the magnitude Bode plot increases by 20 dB per decade in the range $\omega \gg \omega_0$.

For $\omega \ll \omega_o$, $1 + j\omega/\omega_o \simeq 1$ so the transfer function is approximately constant at 0 dB.

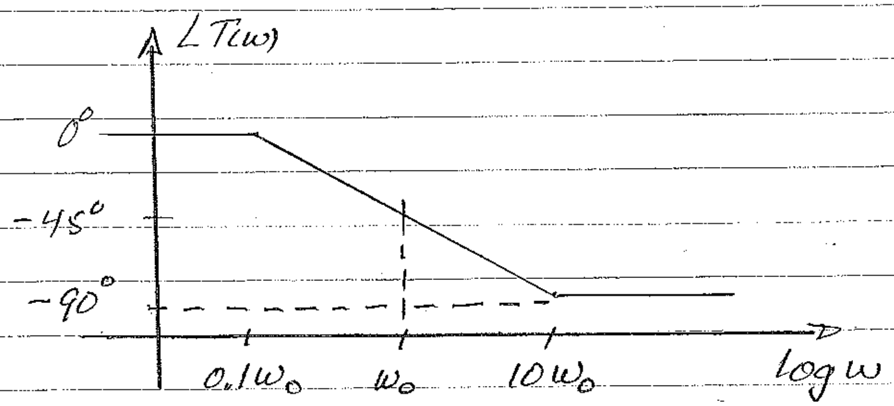
What about the phase?

For $T(\omega) = \dfrac{1}{1 + j\omega CR}$, we have seen that for $\omega = \omega_o = 1/RC$, $\angle T(\omega_o) = -45°$. For $\omega \gg \omega_o$, $\angle T(\omega) = -90°$.

For $\omega \ll \omega_o$, $1 + j\omega/\omega_o \simeq 1 \Rightarrow \angle T(\omega) \simeq 0°$. So $\angle T(\omega)$ is changing from 0 ($\omega \ll \omega_o$) to $-45°$ ($\omega = \omega_o$) to $-90°$ ($\omega \gg \omega_o$). By plugging in some numbers, we can show that...

- For a factor $\dfrac{1}{1 + j\omega/\omega_o}$ appearing in the transfer function the phase Bode plot decreases by 45° per decade beginning at $\omega = 0.1\omega_o$ and ending at $\omega = 10\omega_o$.

In graph form we have:



Also:

- For a factor $(1+j\,\omega/\omega_o)$ appearing in the transfer function the phase Bode plot increases by $45°$ beginning at $0.1\,\omega_o$ and ending at $10\,\omega_o$.

We can now state a set of rules by which to generate the straight-line approximation to the transfer function. We first provide some definitions and an algebraic note.

General Form of the Transfer Function

Transfer functions in this course will look like the following.

$$T(\omega) = K\,\frac{(j\omega + z_1)(j\omega + z_2)\cdots}{(j\omega + p_1)(j\omega + p_2)\cdots}$$

In other words, we will have products of the form $(j\omega + a)$, where 'a' is a number, in the numerator and denominator.

The 'K' accounts for the possibility of a multiplying factor which may also be present.

Terminology: The z's are called ZEROS.
The p's are called POLES

We can re-write $T(\omega)$ in another, algebraically equivalent format:

$$T(\omega) = K \frac{(1+j\omega/z_1)(1+j\omega/z_2)\ \cdots}{(1+j\omega/p_1)(1+j\omega/p_2)\ \cdots}$$

where now K will take a different value.

Note that if some of the z's or p's are 0 (which can happen), the equivalent form as written above will have terms like

$$\frac{j\omega/z}{j\omega/p} \quad \text{etc.}$$

It should be clear that the z's and p's are related to the ω_0's we considered earlier. For example, we had

$$T(\omega) = \frac{1}{1+j\omega CR} = \frac{1}{1+j\,\omega/\omega_0}$$

which is the same as $T(\omega) = \dfrac{1}{1+j\omega/p}$ for

$$p = \omega_0 = 1/RC.$$

A mathematical note: Strictly speaking, the z's and p's are not really zeroes and poles. We will leave the mathematical details to other courses, however.

Also: our z's and p's will be positive and real.

We now set out the general rules to plot the...

STRAIGHT-LINE APPROXIMATION to the BODE PLOTS.

1. Find the transfer function $T(\omega)$.

   This can be done in any algebraic form you find convenient.

2. Identify the poles and zeroes (breakpoints)

   If your transfer function is written as products of things like $1 + j\omega/\omega_0$, your poles/zeroes are the ω_0's. If it is written as products of things like $(j\omega + a)$, they are the a's.

   More generally: a breakpoint occurs when the real and imaginary parts of the factors $(j\omega a + b)$ are equal, i.e. where

   $$\omega a = b \quad \Rightarrow \quad \omega = b/a$$

Breakpoint in the numerator ⇒ ZERO

Breakpoint in the denominator ⇒ POLE

3a Magnitude Bode Plot

As you progress from low to high frequency, the slope of the Bode plot changes by +20 dB/decade each time you reach a zero, and by -20 dB/decade each time you reach a pole.

The changes in slope are additive: if you are in a frequency range such that you have passed 1 pole and 3 zeroes, the net slope is -20 + 20 + 20 + 20 = 40 dB/dec.

3b Phase Bode Plot

As you progress from low to high frequency, the slope of the Bode plot changes by +45°/decade for each zero and by -45°/decade for each pole.

HOWEVER, unlike the case for the magnitude Bode plot, the effect of a breakpoint on the phase Bode plot extends for only two decades

$$0.1\,\omega_0 \le \omega \le 10\,\omega_0 \qquad \omega_0 \equiv \text{breakpoint}$$

slope change begins

slope change ends

Again the effect is additive: in a frequency range where 1 pole and 3 zeroes are effective, the net slope is $-45° + 45° + 45° + 45° = 90°$/decade.

But you have to be careful to be sure that any given pole or zero is not still having an effect on your plot if it is outside its frequency range.

4. Evaluate $|T(\omega)|$ and $\angle T(\omega)$ at a convenient frequency to fix the vertical scale.

The rules in 3a and 3b specify slopes only. To fix the vertical position we also need to evaluate $T(\omega)$ at a fixed frequency. Choose the frequency to be convenient: for example, evaluate $|T(\omega)|$ and $\angle T(\omega)$ at a place where the Bode plots are flat.

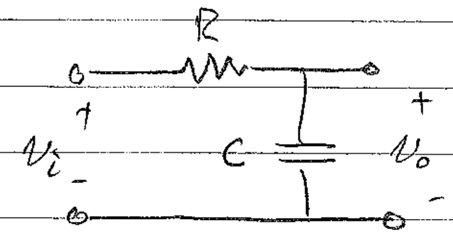
5. Check for plotting details.

Your Bode plot is not complete unless

- it is clear and easy to read
- all axes are labelled for units
- all axes have clear numerical labels so that slopes can be determined from them.

BODE PLOT: STRAIGHT LINE APPROXIMATION EXAMPLES

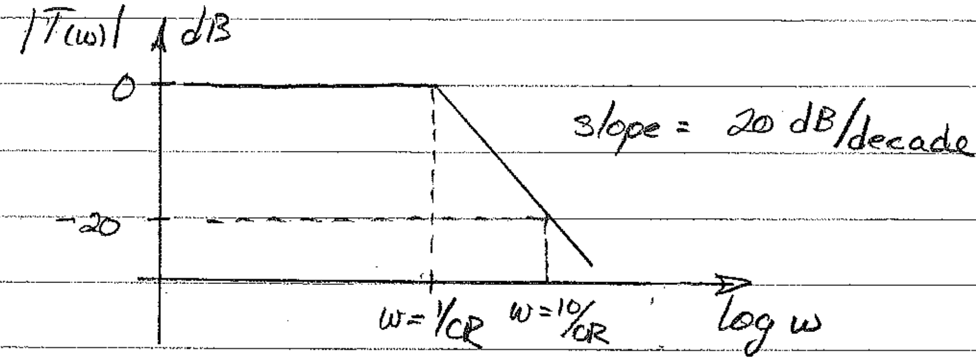
Consider previous RC filter:

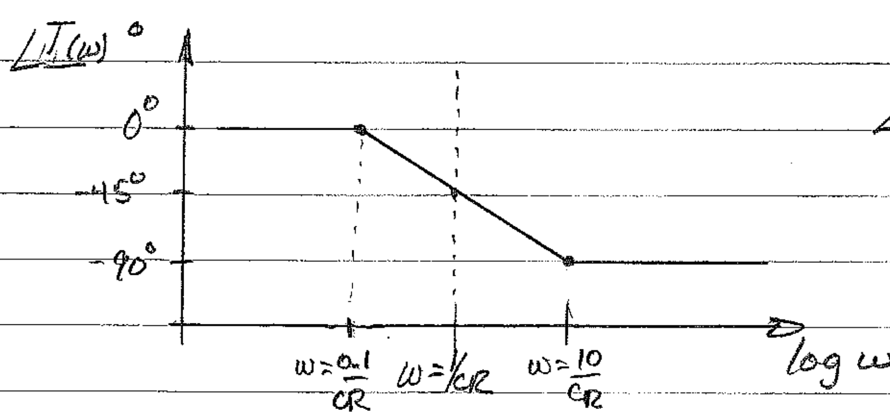


$$T(\omega) = \frac{\bar{V}_o}{\bar{V}_i} = K\frac{1}{1+j\omega CR} = K\frac{1/CR}{j\omega + 1/CR}$$

$\therefore$ we have a pole at $\omega = 1/CR$

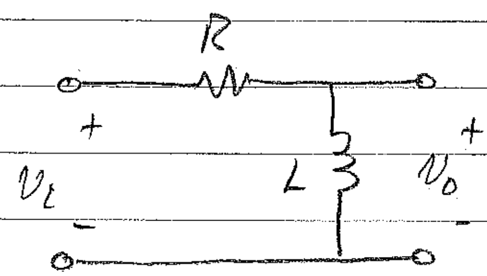
K? $T(\omega=0) = 1 \Rightarrow K = 1 \Rightarrow 0\text{ dB}$





$\angle T(\omega=0) = 0°$

EXAMPLE: HIGH-PASS FILTER



$$T(\omega) = \frac{j\omega L}{R + j\omega L} = \frac{j\omega L/R}{1 + j\omega L/R}$$

Simplified Analysis for Bode Plot:

$$\omega \to 0 \Rightarrow T(\omega) \to \frac{j\omega L/R}{1+0} \sim j\omega L/R$$

$$\therefore \; |T(\omega)| \to 0 \Rightarrow -\infty \, dB \quad (!)$$

$$\angle T(\omega) \to 90^\circ$$

$$\omega \to \infty \Rightarrow T(\omega) \to 1$$

$$\therefore \; |T(\omega)| \to 1 \Rightarrow 0 \, dB$$

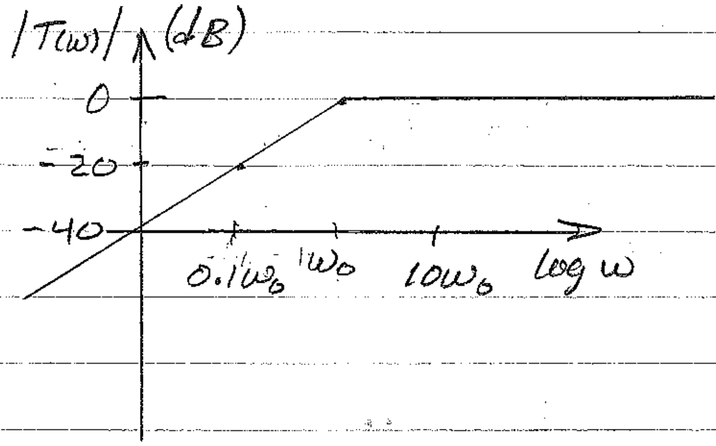
$$\angle T(\omega) \to 0$$

$$\omega = \omega_0 = R/L \Rightarrow T(\omega_0) = \frac{j}{1+j}$$

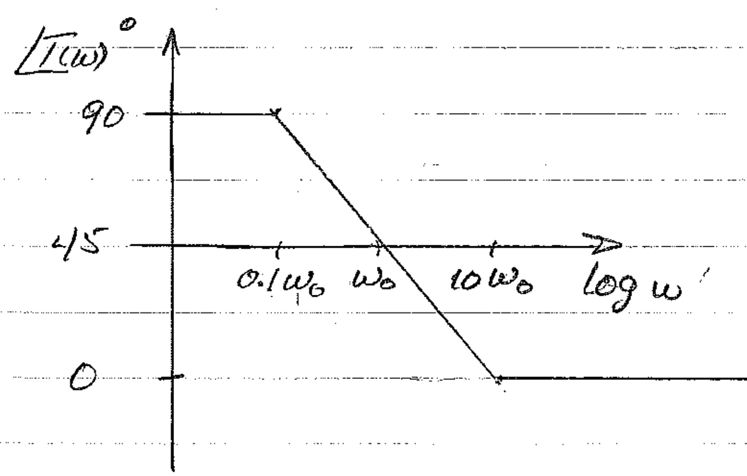
$$= \frac{j}{1+j} \cdot \frac{1-j}{1-j} = \frac{j+1}{2} = \frac{1}{2} + \frac{1}{2}j$$

$$\therefore \; |T(\omega_0)| = \frac{1}{\sqrt{2}} = -3 \, dB$$

$$\angle T(\omega_0) = 45^\circ$$



$\omega_0 = R/L$



$\omega_0 = R/L$

Connection w/ TIME DOMAIN:

$\omega \to 0 \Rightarrow Z_L \to 0 \Rightarrow V_o/V_i \to 0 \Rightarrow -\infty$ dB

$\omega \to \infty \Rightarrow Z_L \to \infty \Rightarrow V_o/V_i \to 1 \Rightarrow 0$ dB

NOTE: $L/R = 1/\omega_0$ is the TIME CONSTANT for a first order (single time const.) system

BODE PLOT

Let's go back to our amplifier for which we found

$$T(\omega) = \frac{270 R_L \cdot j\omega C_i R_i}{[1 + j\omega C_o(R_o + R_L)][1 + j\omega C_i R_i]}$$

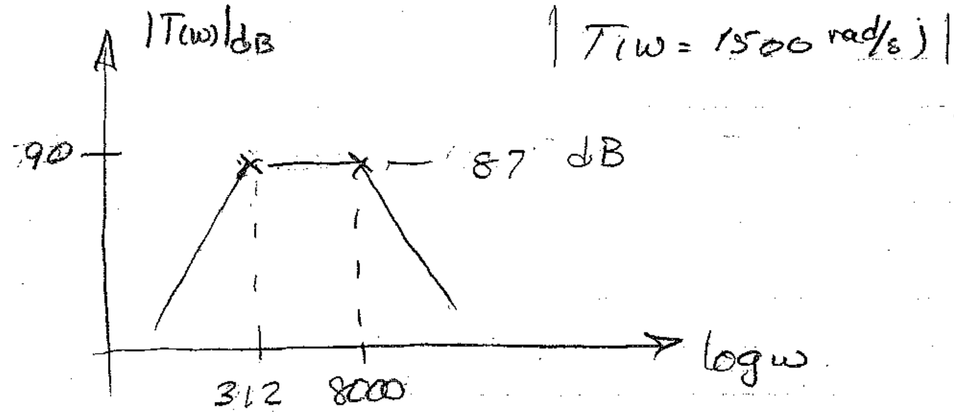
Analysis

$j\omega C_i R_i \rightarrow$ zero at 0 rad/s

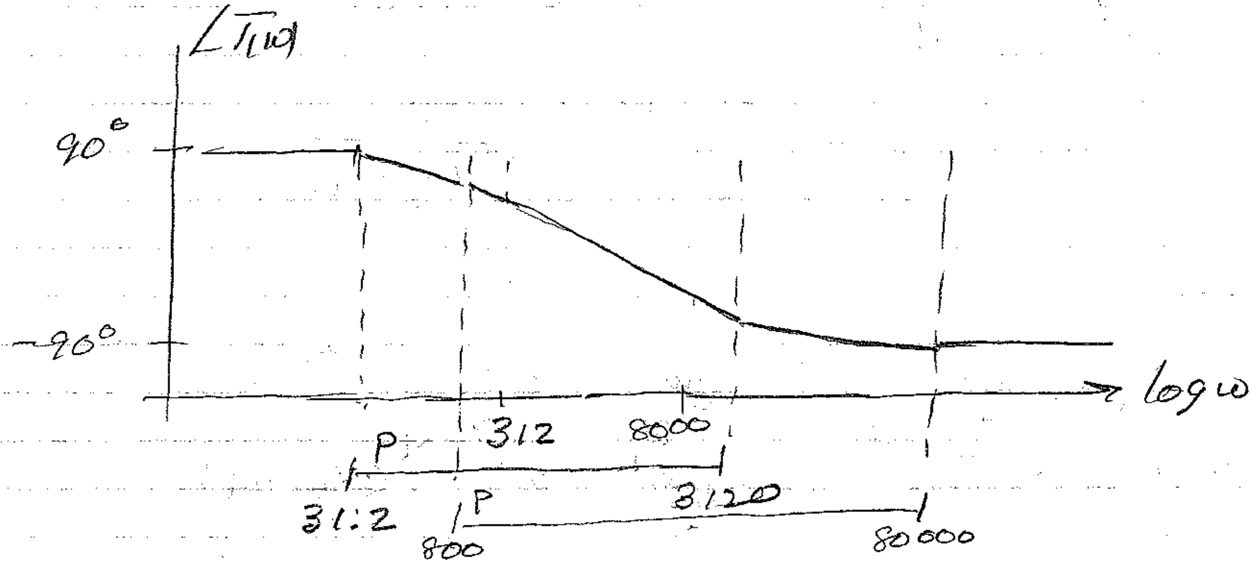
$(1 + j\omega C_o(R_o + R_L))^{-1} \Rightarrow$ pole at $1/C_o(R_o+R_L)$
$= 312.5$ rad/s

$(1 + j\omega C_i R_i)^{-1} \Rightarrow$ pole at $1/C_i R_i$
$= 8000$ rad/s

Rough sketch: Magnitude

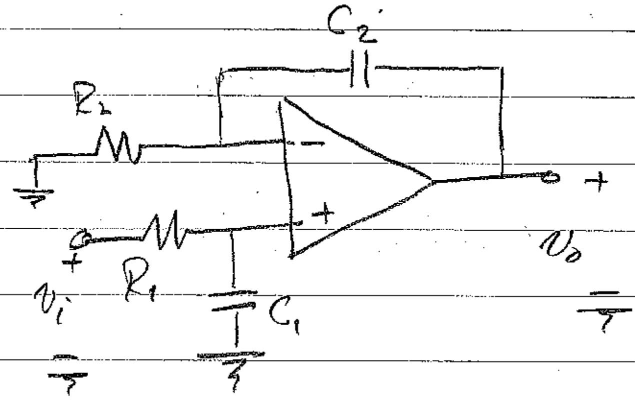


Rough Sketch: Phase



EXAMPLE: BODE PLOT STRAIGHT LINE APPROXIMATIONS

CIRCUIT:



$C_1R_1 = 10^{-5}$ s

$C_2R_2 = 2\times10^{-4}$ s

GIVEN:

$$T(\omega) = \left(\frac{1/j\omega C_1}{R_1 + 1/j\omega C_1}\right)\left(1 + \frac{1/j\omega C_2}{R_2}\right)$$

$$= \left(\frac{1}{1+j\omega C_1R_1}\right)\left(\frac{1+j\omega C_2R_2}{j\omega C_2R_2}\right)$$

Re-write this in the form shown previously.

$$T(\omega) = \frac{(1/C_1R_1)\,(j\omega + 1/C_2R_2)}{(j\omega + 1/C_1R_1)\,(j\omega + 0)}$$

This now has the form

$$T(\omega) = K\cdot\frac{1}{(j\omega + p_1)}\,\frac{(j\omega + z_1)}{(j\omega + p_2)}$$

$\Rightarrow$ $\underline{z_1 = 1/C_2R_2}$, $\underline{p_1 = 1/C_1R_1}$, $\underline{p_2 = 0}$

NOTE: We might also choose to work with $T(\omega)$ in the form given originally:

$$T(\omega) = \frac{1}{1+j\omega C_1 R_1} \frac{1+j\omega C_2 R_2}{j\omega C_2 R_2}$$

This has the form

$$T(\omega) = K \frac{1}{(1+j\omega b_1)(j\omega b_2)} (1+j\omega a_1)$$

Now remember that at a "zero", the real and imaginary parts of $(1+j\omega a)$ are equal, and at a "pole", the real and imaginary parts of $(1+j\omega b)$ are equal.

If we have a term $(0+j\omega a)$ or $(0+j\omega b)$, the zero/pole is at $\omega=0$. So we have

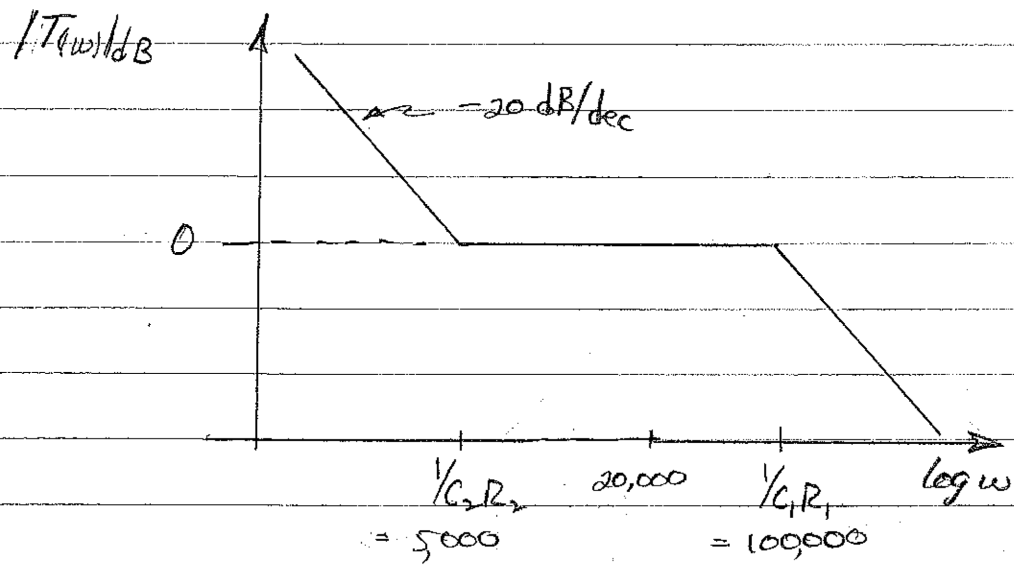
ZERO: $\omega = 1/a_1 = 1/C_2R_2$

POLE: $\omega = 0$, $\omega = 1/b_1 = 1/C_1R_1$

Of course this give the same result as the previous analysis.

MAGNITUDE BODE PLOT

- Pole at $0 \Rightarrow -20$ dB/dec.
- Zero at $1/C_2R_2 = 5\times10^3\ s^{-1} \Rightarrow -20+20 = 0$ dB/dec
- Pole at $1/C_1R_1 = 10^5\ s^{-1} \Rightarrow -20$ dB/dec



Note that at $\omega = 0$, $|T(\omega)| \to \infty$. This can be observed in the expression for $T(\omega)$, which blows up at $\omega = 0$. Of course, the amplifier would saturate at the power supply value first.

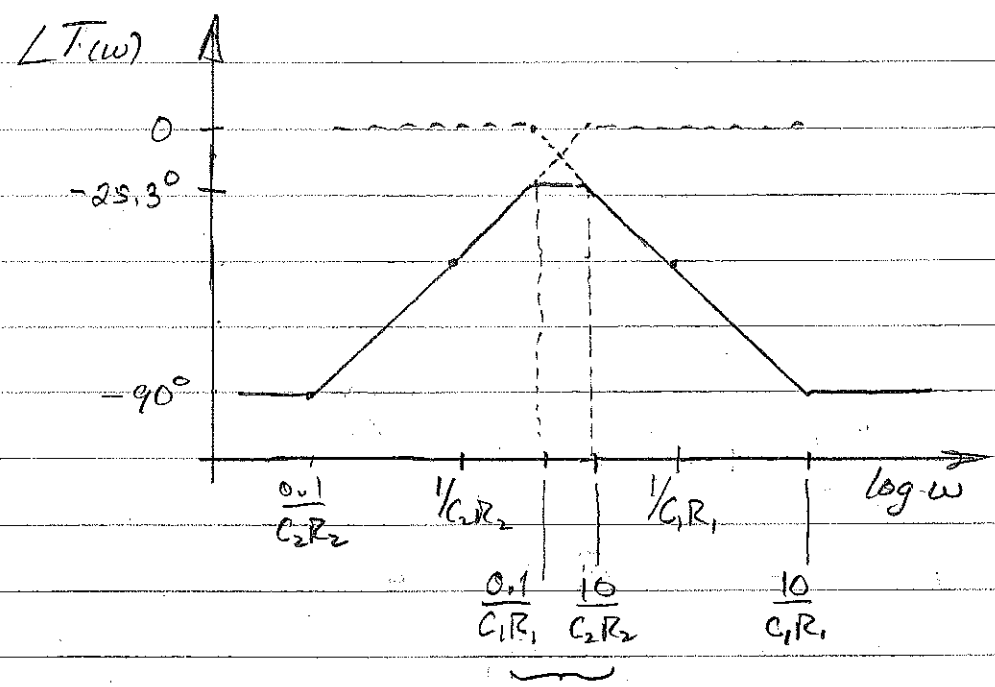
Evaluation: Take $1/C_2R_2 \ll \omega_0 \ll 1/C_1R_1$,
Say $\omega_0 = 20{,}000\ s^{-1}$

$$T(\omega_0) = \frac{1}{1+j(2\times10^4)(10^{-5})}\ \frac{1+j(2\times10^4)(2\times10^{-4})}{j(2\times10^4)(2\times10^{-4})}$$

$$|T(\omega_0)| = 1.01 \cong 0\ dB \qquad \angle T(\omega_0) = -25.3^\circ$$

PHASE BODE PLOT

- Pole at 0 ⇒ nothing happening here, since effect of pole ends at $10 \times 0 = 0$!
- Zero at $1/C_2R_2 = 5 \times 10^3\ s^{-1}$ ⇒ $+45°$/dec for 2 decades
- Pole at $1/C_1R_1 = 10^5\ s^{-1}$ ⇒ $-45°$/dec for 2 decades



overlap : $10{,}000 < \omega < 50{,}000$

Note that in the range $\frac{0.1}{C_1R_1} < \omega_0 < \frac{10}{C_2R_2}$, the effects of the zero and the pole add, and since they are equal and opposite, they add to 0.

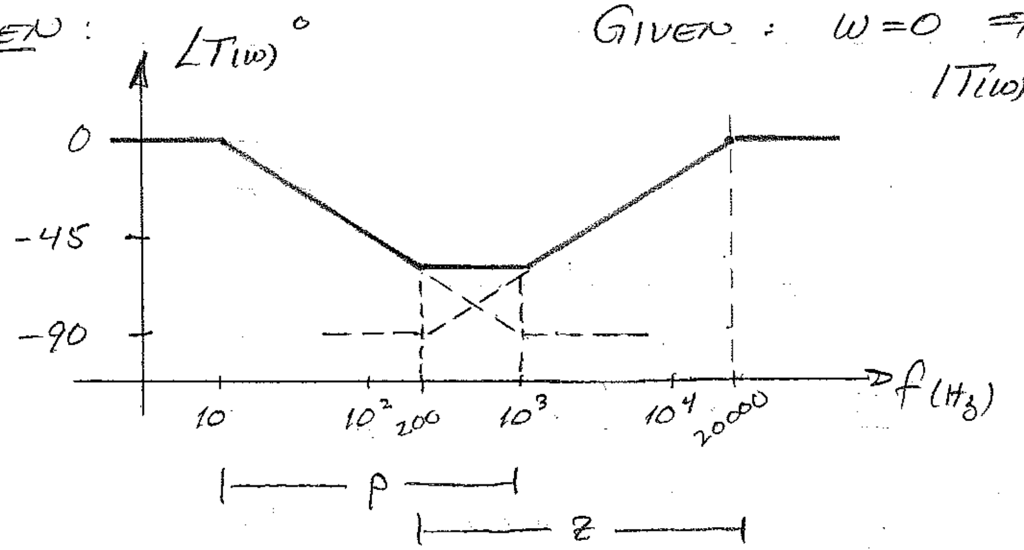
Evaluation: From previous page, $\omega_0 = 20{,}000$ is inside overlap region and $\angle T(\omega_0) = -25.3°$

Also, $\omega \to 0 \Rightarrow \angle \to -90°$

EXAMPLE from D. Shattuck's notes.

This problem turns things around a bit: we are given the phase Bode plot and asked to find the transfer function.

GIVEN:



GIVEN: $\omega = 0 \Rightarrow |T(i\omega)| = 10$

FIND: $T(i\omega)$; plot the magnitude Bode plot.

ANALYSIS: The solid lines represent the given Bode plot. The dashed lines were added as part of the analysis. They are used to indicate where each segment of the plot would have ended if it had been allowed to continue for two decades.

It should be clear that we have a $-45°$/decade drop between $f = 10$ Hz and $f = 1000$ Hz. We also have a $+45°$/decade increase between $f = 200$ Hz and $f = 20000$ Hz.

Therefore we have a zero z at 2000 Hz and a pole p at 100 Hz.

Thus

$$T(\omega) = K \; \frac{j\omega + 2\pi(2000)}{j\omega + 2\pi(100)}$$

The factor K accounts for a multiplication factor. K is found from the information given that

$$|T(\omega=0)| = 10 = K \; \frac{2\pi(2000)}{2\pi(100)}$$

$$\Rightarrow \quad K = 0.5$$

So the complete transfer function is

$$T(\omega) = 0.5 \; \frac{j\omega + 2\pi(2000)}{j\omega + 2\pi(100)}$$

Magnitude Bode plot: